MATH 140A - HW 5 SOLUTIONS

Problem 1 (WR Ch 3 #8)**.** If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Solution. Theorem 3.42 states that if

(a) the partial sums of $\sum a_n$ form a bounded sequence;

(b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;

(c) $\lim_{n\to\infty} b_n = 0$,

then $\sum a_n b_n$ converges.

First of all, since $\sum a_n$ converges, that means the sequence of partial sums $\{\sum_{n=1}^{k} a_n\}$ is a convergent sequence, so by Theorem 3.2(c) it is bounded, and thus part **(a)** is satisfied.

The problem with using this theorem with $\{b_n\}$ is that it doesn't necessarily converge to 0. However, we can create a new sequence $\{c_n\}$ based on $\{b_n\}$ which has the properties we need to apply the theorem. Since $\{b_n\}$ is monotonic and bounded, it converges by Theorem 3.14. Let $b = \lim b_n$. Now we have two cases:

$\boxed{b_0 \geq b}$ Then b_n must be decreasing, so define the sequence $c_n = b_n - b$, which is monotonic decreasing as well, so it satisfies **(b)**. Also, notice that

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} (b_n - b) = (\lim_{n\to\infty} b_n) - b = b - b = 0,$$

so it satisfies **(c)**. Therefore, $\sum a_n c_n$ converges. Then we note that

$$\sum a_n c_n = \sum a_n b_n - \sum a_n b \qquad \Longrightarrow \qquad \sum a_n b_n = \sum a_n c_n + b \sum a_n,$$

and the right side of the equation consists of two convergent sequences, so $\sum a_n b_n$ converges.

$\boxed{b_0 \leq b}$ Then b_n must be increasing, so let $c_n = b - b_n$, which is monotonic decreasing, so it satisfies **(b)**. Once again, $\lim_{n\to\infty} c_n = 0$, so it satisfies **(c)**, so $\sum a_n c_n$ converges. Then

$$\sum a_n c_n = \sum a_n b - \sum a_n b_n \qquad \Longrightarrow \qquad \sum a_n b_n = b \sum a_n - \sum a_n c_n,$$

so $\sum a_n b_n$ converges.

Problem 2 (WR Ch 3 #9)**.** Find the radius of convergence of each of the following power series:

(a) $\sum n^3 z^n$

Solution. By Theorem 3.39, the radius of convergence of the power series is given by $R = \frac{1}{\alpha}$ where α is defined as

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|n^3|} = \lim_{n\to\infty} n^{\frac{3}{n}} = \left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)^3 = 1^3 = 1,$$

so the radius of convergence is $R = \frac{1}{\alpha} = 1$.

(b) $\sum \frac{2^n}{n!} z^n$

Solution. Using the ratio test, the series converges if

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}|z|^{n+1}}{\frac{2^n}{n!}|z|^n} = \lim_{n\to\infty} \frac{2|z|}{n+1} < 1,$$

which is true for all $z \in \mathbb{C}$, so the radius of convergence is $R = \infty$.

(c) $\sum \frac{2^n}{n^2} z^n$

Solution. Using Theorem 3.39 again,

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{\left|\frac{2^n}{n^2}\right|} = \lim_{n\to\infty} \frac{2}{n^{\frac{2}{n}}} = \frac{2}{\left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)^2} = \frac{2}{1^2} = 2,$$

so the radius of convergence is $R = \frac{1}{\alpha} = \frac{1}{2}$.

(d) $\sum \frac{n^3}{3^n} z^n$

Solution. By Theorem 3.39,

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{\left|\frac{n^3}{3^n}\right|} = \lim_{n\to\infty} \frac{n^{\frac{3}{n}}}{3} = \frac{\left(\lim_{n\to\infty} n^{\frac{1}{n}}\right)^3}{3} = \frac{1^3}{3} = \frac{1}{3},$$

so the radius of convergence is $R = \frac{1}{\alpha} = 3$.

Problem 3 (WR Ch 3 #10)**.** Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Solution. To prove the radius of convergence is at most 1, we must show that if $|z| > 1$, then $\sum a_n z^n$ diverges. Notice that

$$|z| > 1 \qquad \Longrightarrow \qquad |a_n z^n| = |a_n||z|^n > |a_n|.$$

Next, note that if an integer a is nonzero, then $|a| \geq 1$. Therefore, since there are infinitely many $n \in \mathbb{N}$ such that $a_n \neq 0$, there are infinitely many $n \in \mathbb{N}$ such that

$$|a_n z^n| > |a_n| \geq 1,$$

so $\lim\limits_{n\to\infty} |a_n z^n| \neq 0$, and thus $\sum a_n z^n$ diverges by the divergence theorem (3.23).

Problem 4 (WR Ch 3 #11)**.** Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

Solution. Assume (by way of contradiction) that $\sum \frac{a_n}{1+a_n}$ converges. Then $\frac{a_n}{1+a_n} \to 0$ by Theorem 3.23. Since $a_n \neq 0$, we can divide the top and bottom of this fraction by a_n to get $\frac{1}{\frac{1}{a_n}+1} \to 0$, which implies that $\frac{1}{a_n} \to \infty$, which again implies that $a_n \to 0$. This last result means that for any $\epsilon > 0$, there exists some $N_1 \in \mathbb{N}$ such that $|a_n - 0| < \epsilon$ for all $n \geq N_1$. Let $\epsilon = 1$, and choose $N_1 \in \mathbb{N}$ so that $|a_n| < 1$ for $n \geq N_1$.

Next, since we assumed that $\sum \frac{a_n}{1+a_n}$ converges, that means the sequence of partial sums of the series $\{\sum_{k=1}^{n} \frac{a_k}{1+a_k}\}$ converges in $\mathbb{R}$ as we increase n to ∞, so it is a Cauchy sequence. This is equivalent to the statement that for any $\epsilon > 0$ there exists some $N_2 \in \mathbb{N}$ such that

$$\left| \sum_{k=1}^{n} \frac{a_k}{1+a_k} - \sum_{k=1}^{m} \frac{a_k}{1+a_k} \right| < \epsilon \qquad \text{for all } n, m \geq N_2,$$

or, equivalently,

$$\frac{a_m}{1+a_m} + \cdots + \frac{a_n}{1+a_n} < \epsilon \qquad \text{for all } n, m \geq N_2.$$

Now, if we set $N = \max(N_1, N_2)$, then for any $k \geq N$ we have $a_k \leq 1$, so

$$\frac{a_m}{1+1} + \cdots + \frac{a_n}{1+1} < \frac{a_m}{1+a_m} + \cdots + \frac{a_n}{1+a_n} < \epsilon \qquad \text{for all } n, m \geq N,$$

but this proves that $\sum \frac{a_n}{2}$ is Cauchy, and thus $\sum a_n$ converges, contradicting our assumption that it diverges.

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

Solution. Since $s_{N+k} \geq s_{N+k-1} \geq s_{N+k-2} \geq \cdots \geq s_N$, we have

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1}}{s_{N+k}} + \cdots + \frac{a_{N+k}}{s_{N+k}} = \frac{a_{N+1} + \cdots + a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}.$$

Now assume (by way of contradiction) that $\sum \frac{a_n}{s_n}$ converges. Then it's Cauchy, so for any $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that

$$\frac{a_m}{s_m} + \cdots + \frac{a_n}{s_n} < \epsilon \qquad \text{for all } n, m \geq N.$$

Letting $\epsilon = \frac{1}{2}$, $m = N$, and $n = N + k$ for some positive integer k, we get

$$\frac{1}{2} > \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}},$$

and doing some arithmetic this means

$$\frac{s_N}{s_{N+k}} > \frac{1}{2} \qquad \text{for any positive integer } k.$$

However, since $\sum a_n$ diverges and $a_n > 0$, we have that $s_{N+k} \to \infty$ as $k \to \infty$, and this implies that $\frac{s_N}{s_{N+k}} \to 0$ as $k \to \infty$, contradicting the fact that $\frac{s_N}{s_{N+k}} > \frac{1}{2}$ for all $k \in \mathbb{N}$.

Problem 5 (WR Ch 3 #14)**.** If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + \cdots + s_n}{n+1} \qquad (n = 0, 1, 2, \ldots).$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

Solution. We want to show that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|\sigma_n - s| < \epsilon$ for all $n \geq N$. Since we already know that $\lim s_n = s$, then there is some $N_1 \in \mathbb{N}$ such that $|s_n - s| < \frac{\epsilon}{2}$ for all $n \geq N_1$. Also, since $\{s_n\}$ is a convergent sequence in a metric space ($\mathbb{C}$), it's bounded, so there exists some $M > 0$ such that $|s_n - s| < M$ for all n. Putting all this together, we have

$$\begin{aligned}
|\sigma_n - s| &= \left| \frac{s_0 + \cdots + s_n}{n+1} - s \right| \\
&= \left| \frac{s_0 + \cdots + s_n - (n+1)s}{n+1} \right| \\
&= \left| \frac{(s_0 - s) + \cdots + (s_n - s)}{n+1} \right| \\
&\leq \frac{|s_0 - s| + \cdots + |s_n - s|}{n+1} \\
&= \frac{|s_0 - s| + \cdots + |s_{N_1 - 1} - s|}{n+1} + \frac{|s_{N_1} - s| + \cdots + |s_n - s|}{n+1} \\
&< \frac{N_1 M}{n+1} + \frac{(n - N_1 + 1)\frac{\epsilon}{2}}{n+1} \\
&\leq \frac{N_1 M}{n+1} + \frac{\epsilon}{2}.
\end{aligned}$$

In the last step, we let choose N_2 to be the smallest positive integer such that $N_2 > \frac{2N_1 M}{\epsilon} - 1$, so that

$$\frac{N_1 M}{n+1} < \frac{\epsilon}{2} \qquad \text{for all } n \geq N_2.$$

Therefore, continuing from before, if we set $N = \max(N_1, N_2)$, then

$$|\sigma_n - s| < \frac{N_1 M}{n+1} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \text{for all } n \geq N,$$

finishing the proof that $\lim \sigma_n = s$.

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.

Solution. Let $s_n = (-1)^n$. Then

$$\sigma_n = \frac{s_0 + \cdots + s_n}{n+1} = \begin{cases} \frac{1}{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases},$$

so $\lim \sigma_n = 0$, but s_n does not converges.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?

Solution. Let

$$s_n = \begin{cases} 0 & \text{if } n = 0 \\ \frac{1}{n} + \sqrt[3]{n} & \text{if } n = k^3 \text{ for some integer } k \\ \frac{1}{n} & \text{otherwise} \end{cases}$$

Since the number of cubic numbers in $\{1, \ldots, n\}$ is given by $\lfloor \sqrt[3]{n} \rfloor$, which is the largest integer less than n, then we have

$$\sigma_n = \frac{s_0 + \cdots + s_n}{n+1} = \frac{1 + n\left(\frac{1}{n}\right) + \lfloor \sqrt[3]{n} \rfloor \left(\sqrt[3]{n}\right)}{n+1} \le \frac{2 + n^{\frac{2}{3}}}{n+1} \to 0$$

as $n \to \infty$. Therefore $\lim \sigma_n = 0$. Lastly, we check that

$$\limsup s_n \ge \lim_{n\to\infty} \left(\frac{1}{n} + \sqrt[3]{n} \right) = \infty.$$

(d) Put $a_n = s_n - s_{n-1}$, for $n \ge 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k. \tag{$*$}$$

Assume that $\lim(n a_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges.

Solution. We prove the first part by induction.

$$s_1 - \sigma_1 = s_1 - \frac{s_0 + s_1}{=} \frac{1}{2}(s_1 - s_0) = \frac{1}{2} a_1.$$

This establishes the base case. Next, assume that $s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k$, and we have

$$\begin{aligned}
\frac{1}{(n+1)+1} \sum_{k=1}^{n+1} k a_k &= \frac{1}{n+2} \left(\sum_{k=1}^{n} k a_k + (n+1)(s_{n+1} - s_n) \right) \\
&= \frac{1}{n+2} \left((n+1)(s_n - \sigma_n) + (n+1)(s_{n+1} - s_n) \right) \\
&= \frac{1}{n+2} (n+1)(s_{n+1} - \sigma_n) \\
&= \frac{(n+1)s_{n+1} - (s_0 + \cdots + s_n)}{n+2} \\
&= \frac{(n+2)s_{n+1} - (s_0 + \cdots + s_n + s_{n+1})}{n+2} \\
&= s_{n+1} - \frac{(s_0 + \cdots + s_{n+1})}{n+2} \\
&= s_{n+1} - \sigma_{n+1}.
\end{aligned}$$

This completes the induction. Now, since $\{n a_n\}$ is a complex sequence, and $\lim n a_n = 0$, then by part (a) the limit of the arithmetic means of $\{n a_n\}$ must also be 0, so to restate this,

$$\lim_{n\to\infty} \left(\frac{1}{n+1} \sum_{k=1}^{n} k a_k \right) = \lim_{n\to\infty} a_n = 0.$$

Therefore,

$$\lim(s_n - \sigma_n) = \lim_{n\to\infty} \left(\frac{1}{n+1} \sum_{k=1}^{n} k a_k \right) = 0,$$

by $(*)$, and thus $\lim s_n = \lim \sigma_n$ (and σ_n converges), so finally we have shown that $\{s_n\}$ converges.

Problem 6 (WR Ch 3 #16)**.** Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \ldots$ by the recursion formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n} \right).$$

(a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

Solution. If $a, b \in \mathbb{R}$, then

$$(a-b)^2 \geq 0 \qquad \Longrightarrow \qquad a^2 - 2ab + b^2 \geq 0 \qquad \Longrightarrow \qquad \frac{a^2+b^2}{2} \geq ab,$$

with equality holding iff $a = b$. We will use this last inequality to show that $\{x_n\}$ is bounded below by $\sqrt{\alpha}$. We do so by induction. $x_0 > \alpha$ is given. Assuming that $x_n > \alpha$, we have

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n} \right) > \sqrt{x_n} \cdot \frac{\sqrt{\alpha}}{\sqrt{x_n}} = \sqrt{\alpha}.$$

Now we who that $\{x_n\}$ is a decreasing sequence.

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{\alpha}{x_n} \right) = \frac{1}{2}\left(x_n - \frac{\alpha}{x_n} \right) = \frac{1}{2}\left(\frac{x_n^2 - \alpha}{x_n} \right) > 0,$$

using the fact that $x_n^2 > \alpha$ in the last inequality, so we have proven that $x_n - x_{n+1} > 0$ or equivalently that $x_n > x_{n+1}$. Since $\{x_n\}$ is a strictly decreasing, bounded sequence, it must converge. Let $x = \lim x_n$. Notice that $x \geq \sqrt{a} > 0$, so

$$\begin{aligned} x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n} \right) \qquad &\Longrightarrow \qquad \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(x_n + \frac{\alpha}{x_n} \right) \\ &\Longrightarrow \qquad x = \frac{1}{2}\left(x + \frac{\alpha}{x} \right) \\ &\Longrightarrow \qquad \frac{x}{2} = \frac{\alpha}{2x} \\ &\Longrightarrow \qquad x^2 = \alpha \\ &\Longrightarrow \qquad x = \sqrt{\alpha}. \end{aligned}$$

(b) Put $\epsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\epsilon_{n+1} < \beta \left(\frac{\epsilon_1}{\beta} \right)^{2^n} \qquad (n = 1, 2, 3, \ldots).$$

Solution.

$$\epsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \tfrac{1}{2}(x_n - \tfrac{\alpha}{x_n}) - \sqrt{\alpha} = \frac{x_n^2 - 2x_n\sqrt{\alpha} + \alpha}{2x_n} = \frac{(x_n - \sqrt{\alpha})^2}{2x_n} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

since $x_n > \sqrt{\alpha}$. So $\epsilon_{n+1} < \frac{\epsilon_n^2}{\beta}$, and applying this n times we get

$$\epsilon_{n+1} < \frac{\epsilon_n^2}{\beta} < \frac{\left(\frac{\epsilon_{n-1}^2}{\beta}\right)^2}{\beta} = \beta\left(\frac{\epsilon_{n-1}}{\beta}\right)^{2^2} < \cdots < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}.$$

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\epsilon_1/\beta < \frac{1}{10}$ and that therefore

$$\epsilon_5 = 4 \cdot 10^{-16}, \qquad \epsilon_6 < 4 \cdot 10^{-32}.$$

Solution.

$$\frac{\epsilon_1}{\beta} = \frac{2-\sqrt{3}}{2\sqrt{3}} = \frac{1}{2\sqrt{3}(2+\sqrt{3})} = \frac{1}{6+4\sqrt{3}} < \frac{1}{10}.$$

$$\epsilon_5 < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^4} < 2\sqrt{3}\cdot 10^{-16} < 4\cdot 10^{-16}.$$

$$\epsilon_6 < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^5} < 2\sqrt{3}\cdot 10^{-32} < 4\cdot 10^{-32}.$$

Problem 7 (WR Ch 3 #18)**.** Replace the recursion formula of Exercise 16 by

$$x_{n+1} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1}$$

where p is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$.

Solution. If the limit exists, let $x = \lim x_n$. Then

$$\begin{aligned}
x_{n+1} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1} \qquad &\Longrightarrow \qquad \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1} \\
&\Longrightarrow \qquad x = \frac{p-1}{p}x + \frac{\alpha}{p}x^{-p+1} \\
&\Longrightarrow \qquad \frac{x}{p} = \frac{\alpha}{px} \\
&\Longrightarrow \qquad x^p = \alpha \\
&\Longrightarrow \qquad x = \sqrt[p]{\alpha}.
\end{aligned}$$

To show the limit exists, we'll show again the sequence is bounded and decreasing. First we show it's bounded. To do so, we'll have to use Young's Inequality. It says that if $\frac{1}{p} + \frac{1}{q} = 1$ and if $a, b > 0$, then $\frac{a}{q} + \frac{b}{p} \geq a^{\frac{1}{q}} b^{\frac{1}{p}}$. Thus, letting $q = \frac{p}{p-1}$, we have $\frac{1}{p} + \frac{1}{q} = 1$, and so

$$x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1} > x_n^{\frac{1}{q}} \frac{\alpha^{\frac{1}{p}}}{x_n^{\frac{p-1}{p}}} = \sqrt[p]{\alpha},$$

so $\{x_n\}$ is bounded. Next,

$$x_n - x_{n+1} = x_n - \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1} = \frac{x_n}{p} + \frac{\alpha}{p x_n^{p-1}} = \frac{x_n^p - \alpha}{p x_n^{p-1}} > 0,$$

so $\{x_n\}$ is decreasing.