MATH 140A - HW 7 SOLUTIONS

Problem 1 (WR Ch 4 #14). Let $I = [0,1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Solution. Let $g(x) = x - f(x)$, which is also continuous. If $g(x) = 0$ for any $x \in I$, then we have proven the result, so assume by way of contradiction that $g(x) \neq 0$ for any $x \in I$. Since $f(0) \in [0,1]$, $g(0) = 0 - f(0) \leq 0$, and since we are assuming $g(0) \neq 0$, then $g(0) < 0$. Also, since $f(1) \in [0,1]$, $g(1) = 1 - f(1) \geq 0$, and since we are assuming $g(1) \neq 0$, then $g(1) > 0$. Then by the Intermediate Value Theorem, there is some $x \in (0,1)$ such that $g(x) = 0$, a contradiction.

Problem 2 (WR Ch 4 #15). Call a mapping of X into Y *open* if $f(V)$ is an open set in Y whenever V is an open set in X.

Prove that every continuous open mapping of $\mathbb{R}^1$ into $\mathbb{R}^1$ is monotonic.

Solution.

Claim (1). $a \neq b \implies f(a) \neq f(b)$.

Since f is a continuous function on a compact set, Theorem 4.16 says it must attain its maximum and its minimum, so let

$$M = \sup_{x \in [a,b]} f(x) \qquad \text{and} \qquad m = \inf_{x \in [a,b]} f(x),$$

and there must be some $z_1, z_2 \in [a,b]$ such that $f(z_1) = m$ and $f(z_2) = M$.

If $M = m$, then f is constant on $[a,b]$, but then $f((a,b)) = \{M\} = \{m\}$ is just a point, which is not an open set, contradicting the openness of f. Therefore, we can assume $M > m$. There are four cases left:

Case 1: $f(z_1) = m$ for some $z_1 \in (a,b)$.
$f((a,b))$ contains $z_1 \in \mathbb{R}$ but does not contain any real numbers less than z_1. Then any neighborhood N of z_1 will contain real numbers less than z_1, and N thus cannot be contained in $f((a,b))$, so it is not open, a contradiction.

Case 2: $f(z_2) = M$ for some $z_2 \in (a,b)$.
$f((a,b))$ contains $z_2 \in \mathbb{R}$ but does not contain any real numbers more than z_2. Then any neighborhood N of z_2 will contain real numbers more than z_2, and N thus cannot be contained in $f((a,b))$, so it is not open, a contradiction.

Case 3: $f(a) = m$ and $f(b) = M$
Then $f(a) \neq f(b)$.

Case 4: $f(a) = M$ and $f(b) = m$

Then $f(a) \neq f(b)$.

Claim (2)**.** If $a < b < c$ and $f(a) < f(b)$ then $f(b) < f(c)$.

Case 1: $f(c) = f(a)$.

Since $c \neq a$ this case contradicts claim 1.

Case 2: $f(c) = f(b)$.

Since $c \neq b$ this case contradicts claim 1.

Case 3: $f(c) < f(a)$.

Then $f(c) < f(a) < f(b)$, and by the Intermediate Value Theorem there is some $d \in (b, c)$ such that $f(d) = f(a)$, but $a \notin (b, c)$, so $a \neq d$, contradicting claim 1.

Case 4: $f(a) < f(c) < f(b)$.

Then $f(a) < f(c) < f(b)$, and by the Intermediate Value Theorem there is some $d \in (a, b)$ such that $f(d) = f(c)$, but $c \notin (a, b)$, so $c \neq d$, contradicting claim 1.

The only case left is $f(c) > f(b)$, and since all the other cases lead to contradictions, this is the only possible one.

Finally, if we assume f is not monotonic, then either there are some $a < b < c$ such that $f(a) < f(b)$ and $f(c) < f(b)$ or there are some $a < b < c$ such that $f(a) > f(b)$ and $f(c) > f(b)$. The first case contradicts claim 2. The second case turns into the first case if we replace $f(x)$ by $-f(x)$, which is still a continuous, open function.

Problem 3 (WR Ch 4 #17)**.** Let f be a real function defined on (a, b). Prove that the set of points at which f has a simple discontinuity is at most countable.

Solution. A simple discontinuity is a point x where f is discontinuous but where $f(x+)$ and $f(x-)$ exist. There are three possible types of simple discontinuities we have to deal with:

Type 1: $f(x+) > f(x-)$.

For every simple discontinuity x of this type assign three rational numbers (p, q, r) such that

(a) $f(x-) < p < f(x+)$,

(b) $a < q < t < x \implies f(t) < p$,

(c) $x < t < r < b \implies f(t) > p$,

This is always possible because: for (a), $\mathbb{Q}$ is dense in $\mathbb{R}$ so there is a rational number in the open interval $(f(x-), f(x+))$; for (b), $f(x-)$ exists, so for $\epsilon = (p - f(x-))$ there exists some $\delta > 0$ so that $0 < x - t < \delta$ implies $f(t) - f(x-) < \epsilon = p - f(x-)$, implying that $f(t) < p$, and there is a rational number in the interval $(x-\delta, x)$ because $\mathbb{Q}$ is dense in $\mathbb{R}$; for (c), we use a similar argument as for (b).

So we have shown that such a triple of rational numbers always exists. Now we need to show that it is unique. Assume there exists another number $y \neq x$ such that

(a) $f(y-) < p < f(y+)$,

(b) $a < q < t < y \implies f(t) < p$,

(c) $y < t < r < b \implies f(t) > p$,

and assume without loss of generality that $x < y$. Then there exists some $t \in \mathbb{R}$ such that $x < t < y$, and thus

$$\left.\begin{array}{ll} \text{By property (c) for } x\text{: } x < t < r < b & \implies f(t) > p \\ \text{By property (b) for } y\text{: } a < q < t < y & \implies f(t) < p \end{array}\right\} \quad \text{a contradiction.}$$

Therefore our system assigns a unique triple of rational numbers to every simple discontinuity of this type, and since $\mathbb{Q}^3$ is countable, the simple discontinuities of this type are countable.

Type 2: $f(x+) < f(x-)$

For every simple discontinuity x of this type assign three rational numbers (p, q, r) such that

(a) $f(x+) < p < f(x-)$,

(b) $a < q < t < x \implies f(t) > p$,

(c) $x < t < r < b \implies f(t) < p$,

This is always possible because: for (a), $\mathbb{Q}$ is dense in $\mathbb{R}$ so there is a rational number in the open interval $(f(x+), f(x-))$; for (b), $f(x-)$ exists, so for $\epsilon = (f(x-) - p)$ there exists some $\delta > 0$ so that $0 < x - t < \delta$ implies $f(x-) - f(t) < \epsilon = f(x-) - p$, implying that $f(t) > p$, and there is a rational number in the interval $(x-\delta, x)$ because $\mathbb{Q}$ is dense in $\mathbb{R}$; for (c), we use a similar argument as for (b).

So we have shown that such a triple of rational numbers always exists. Now we need to show that it is unique. Assume there exists another number $y \neq x$ such that

(a) $f(y+) < p < f(y-)$,

(b) $a < q < t < y \implies f(t) > p$,

(c) $y < t < r < b \implies f(t) < p$,

and assume without loss of generality that $x < y$. Then there exists some $t \in \mathbb{R}$ such that $x < t < y$, and thus

$$\left.\begin{array}{ll} \text{By property (c) for } x\text{: } x < t < r < b & \implies f(t) < p \\ \text{By property (b) for } y\text{: } a < q < t < y & \implies f(t) > p \end{array}\right\} \quad \text{a contradiction.}$$

Therefore our system assigns a unique triple of rational numbers to every simple discontinuity of this type, and since $\mathbb{Q}^3$ is countable, the simple discontinuities of this type are countable.

Type 3: $f(x+) = f(x-)$

Let $z = f(x+) = f(x-)$. For every simple discontinuity x of this type assign two rational numbers (q, r) such that

(a) $a < q < t < x \implies |f(t) - z| < |f(x) - z|$,

(b) $x < t < r < b \implies |f(t) - z| < |f(x) - z|$,

This is always possible because: for (a), $f(x-)$ exists, so for $\epsilon = |f(x) - z|$ there exists some $\delta > 0$ so that $0 < x - t < \delta$ implies $|f(t) - z| < \epsilon = |f(x) - z|$, and there is a rational number in the interval $(x - \delta, x)$ because $\mathbb{Q}$ is dense in $\mathbb{R}$; for (b), we use a similar argument.

So we have shown that such a triple of rational numbers always exists. Now we need to show that it is unique. Assume there exists another number $y \neq x$ such that

(a) $a < q < t < y \implies |f(t) - z| < |f(y) - z|$,

(b) $y < t < r < b \implies |f(t) - z| < |f(y) - z|$,

and assume without loss of generality that $x < y$.

$$\left.\begin{array}{ll} \text{By property (b) for } x\text{: } x < y < r < b & \implies |f(y) - z| < |f(x) - z| \\ \text{By property (a) for } y\text{: } a < q < x < y & \implies |f(x) - z| < |f(y) - z| \end{array}\right\} \text{ a contradiction.}$$

Therefore our system assigns a unique pair of rational numbers to every simple discontinuity of this type, and since $\mathbb{Q}^2$ is countable, the simple discontinuities of this type are countable.

Problem 4 (WR Ch 4 #21)**.** Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$, $q \in F$.

Show the conclusion may fail for two disjoint closed sets if neither is compact.

Solution. First we show that $\rho_F(x)$ is a continuous function, where $\rho_F(x)$ is defined by

$$\rho_F(x) = \inf_{y \in F} d(x, y).$$

Notice that if we pick any $z \in F$, then for any $x, y \in X$,

$$\rho_F(x) \leq d(x, z) \leq d(x, y) + d(y, z),$$

and taking an infimum of both sides with respect to all $z \in F$, we have

$$\rho_F(x) \leq d(x, y) + \inf_{z \in F} d(y, z) = d(x, y) + \rho_F(y) \qquad \implies \qquad \rho_F(x) - \rho_F(y) \leq d(x, y).$$

Repeating the same process but switching the x and y, we get $\rho_F(y) - \rho_F(x) \le d(y,x)$, and putting this together with the previous inequality, we finally have

$$|\rho_F(x) - \rho_F(y)| \le d(x,y).$$

So for any $\epsilon > 0$, if we let $\delta = \epsilon$, we have

$$d(x,y) < \delta \quad \Longrightarrow \quad |\rho_F(x) - \rho_F(y)| \le d(x,y) < \delta = \epsilon,$$

so $\rho_F(x)$ is uniformly continuous. Next, we want to show that $\rho_F(x) = 0$ iff $x \in F$. If $x \in F$, then $\rho_F(x) \le d(x,x) = 0$, so $\rho_F(x) = 0$. If $\rho_F(x) = 0$, then there is some sequence $\{y_n\}$ in F such that $d(y_n, x) \to 0$, but then $y_n \to x$, and since F is closed, $x \in F$.

Now we get down to the actual proof. Since K and F are disjoint, $\rho_F(x) \ne 0$ for any $x \in K$, so ρ_F is a continuous, positive function on a compact set K, so by Theorem 4.16, f must attain its minimum in K, so there is some $z \in K$ such that

$$\rho_F(x) > \rho_F(z) > 0 \quad \text{for all } x \in K.$$

Letting $\delta = \rho_F(z)/2 > 0$, we have $\rho_F(x) > \delta$ for all $x \in K$. This means for any $p \in K$, $q \in F$ we have

$$d(p,q) \ge \rho_F(p) > \delta.$$

To show the conclusion may fail if neither is compact, let $K = \{n + \frac{1}{2^n} : n \in \mathbb{N}\}$ and $F = \mathbb{N}$. Then $d(n + \frac{1}{2^n}, n) \to 0$, so the conclusion fails.