Evolution Processes and
Ordinary Differential Equations

V. I. Arnol'd

> *The equation was very complicated but the professor, being a tactful and modest man, used the word "ordinary" to describe it.*
>
> (From a newspaper interview with a mathematician.)

Differential equations are one of the basic tools of mathematics. They were first considered systematically by Sir Isaac Newton (1642–1727), although problems leading to differential equations had in fact arisen earlier. Before Newton, however, only such geniuses as Christiaan Huygens (1629–1695), President of the French Academy of Sciences, and Isaac Barrow (1630–1677), a mathematician and theologian, who was Newton's teacher, could solve them. Today, thanks to Newton, many differential equations are solvable by college students and even school children.

Newton distilled the essence of his ideas into an anagram,

$$6accdae13eff7i3l9n4o4qrr4s9t12vx.$$

In deciphered form, the anagram reads as follows: *Data aequatione quotcunque fluentes quantitates involvente fluxiones invenire et vice versa* — i.e., "given an equation which involves the derivatives of one or more functions, find the functions".[1] Newton's point was that differential equations are important because they express the laws of nature. Another, lengthier anagram of Newton's sums up his method for solving differential equations (and other types of equations, as well).

As we shall see, the theory of differential equations transforms scientific problems into geometric ones involving curves defined by *vector fields* (which are defined below) in much the same way as the method of Cartesian coordinates translates problems involving algebraic equations into ones having to do with lines and surfaces.

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[1]Newton's anagram is contained in his famous "second letter" of October 24, 1676, the purpose of which was to announce the invention of differential calculus. Newton sent his letter to Henry Oldenburg, Secretary of the Royal Society, and through him to G. W. Leibniz (1646–1716) in Germany. (Oldenburg was later imprisoned in the Tower of London for corresponding with foreigners!)

1. States of a Process.

To investigate a process we need a description of its permissible states. We begin with a few examples.

1.1. Motion of a point mass along a straight line. Common experience shows that in order to describe the motion of a point mass traveling along a straight line in the case when there are no external forces acting on the point mass, it is sufficient to specify its initial velocity and initial position on the line. Therefore the set of all possible states of a point mass is, mathematically speaking, in one-to-one correspondence with the set of all points of the coordinate plane: to each point $(s; v)$ in this plane there corresponds a point mass having velocity v and a position (coordinate) s, and conversely, to each state $(s; v)$ of the point mass, there is associated precisely one point in the plane (see Figure 1).
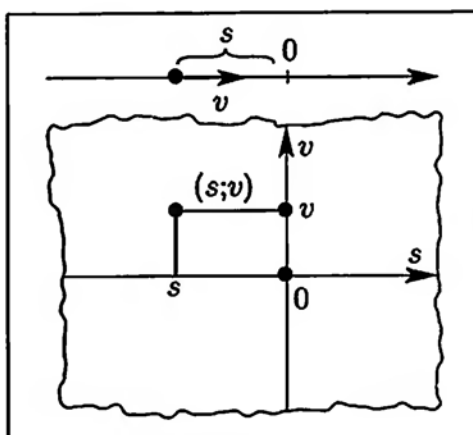

FIGURE 1. Motion in a straight line.

1.2. Oscillation of a pendulum. The state of a two-dimensional pendulum may also be specified by two parameters — for example, the pendulum's angle of displacement θ from the vertical line and its angular velocity ω (see Figure 2). Will the state space of the pendulum again correspond to the plane? Not quite, since the pendulum returns to its initial position once the angle θ has changed by the quantity 2π. Hence the state space in question is in fact a cylindrical surface (see Figure 2).

1.3. Positions of wagons on two roads. Towns A and B are joined by two nonintersecting roads, on each of which there is a wagon. We are interested in the positions of the wagons. The position of the wagon on the first road (see Figure 3) is specified by its relative distance x from A along the first road, by which we mean the distance of the wagon from A along the first road divided by the total distance between A and B along that road; and likewise, the position of the wagon on the second road is given by its relative distance y from A along the second road, which is defined similarly. Hence the state space in question may be represented by the points $(x; y)$ of the unit square $(0 \leq x \leq 1, 0 \leq y \leq 1)$.

In the theory of differential equations, the state space of a process is called its *phase space*. Each point of the phase space (a *phase point*) determines a state of the process. The evolution of a particular process is represented by a path in the phase space; such paths are called *phase trajectories*. The introduction of the
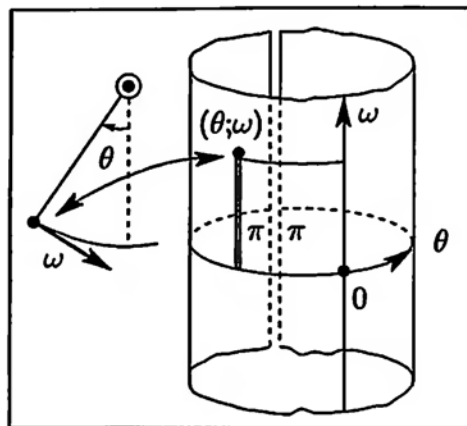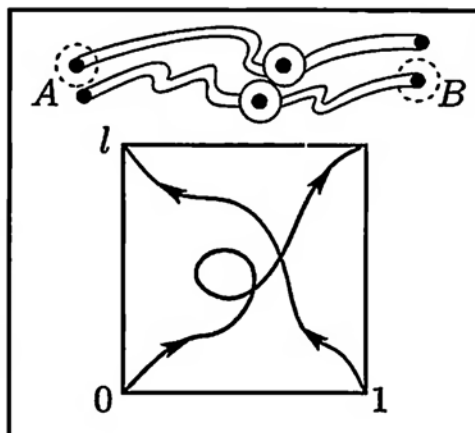
FIGURE 2. Oscillation of a pendulum.

FIGURE 3. Two wagons on two roads.

phase space of the process transforms the study of the process into the study of the behavior of phase trajectories — i.e., into a geometric problem. The resulting geometric problem can be complicated. However, it sometimes turns out that the mere introduction of the phase space leads to an easy solution of what otherwise would be a difficult problem.

(N. N. Konstantinov) *Towns A and B are joined by two nonintersecting roads. Given that two motorbikes taking different roads in the same direction from town A to town B and linked by a rope whose length is less than 2l were both able to reach town B without tearing the rope, will two wagons with circular loads of hay of radius l be able to pass one another without touching if their centers are moving along the roads in opposite directions?*

The phase space here may be taken to be a unit square. The initial position of the motorbikes (at town A) corresponds to the bottom left vertex of the square, and the movement of the motorbikes from town A to town B is represented by a curve in the square leading to the opposite vertex (see Figure 3). Similarly, the initial position of the wagons (at the towns A and B) corresponds to the bottom right vertex of the square, and the path of the wagons is represented by a curve in the square leading to the opposite (upper left) vertex of the square. But any two curves in the square joining two different pairs of opposite vertices will intersect.

Thus, whatever the paths of the wagons, at some instant they will occupy the same positions as were occupied by the motorbikes at some previous time. At this instant the distance between the wagon centers will be less than $2l$. Therefore the hay wagons will not be able to pass one another without touching.

In the problem considered above we did not use differential equations, but the argument is essentially the same as in the treatment of the problems below, in that the representation of the evolution of the process by a curve in a suitable phase space turns out to be crucial.

The general outline of the theory of differential equations is the following. Since the initial state of the process under consideration determines the evolution of the process, the initial state will also determine its rate of change — i.e., the velocity of the phase point. According to Newton, the dependence of the velocity of the phase point on its position (the *local law of evolution* of the process) can be described by a differential equation. Geometrically, this equation is represented by vectors with origins at each phase point, showing the direction and the velocity of the trajectory at this point. The totality of such vectors constitutes the *vector field of phase velocities*, or the *velocity field*, of the equation (see, for example, Figures 5 and 11 below). The aim of the theory of differential equations is to analyze the evolution of the process in terms of its vector field representation, describing the trajectories of phase points and determining whether they are bounded and whether they return to the initial point. Such questions are of obvious importance for the study of the process.

2. Equation of the Mathematical Pendulum.

According to the laws of mechanics, the angular acceleration of a pendulum is proportional to its gravitational momentum (see Figure 4):

$$I\ddot{\theta} = -mgl\sin\theta,$$

where m is the mass, l the length, θ the displacement angle; the dot above the letter designates the derivative with respect to time t (here and below), the double dot the second derivative, and $I = ml^2$ the moment of inertia. The minus sign in the equation means that the moment acts so as to decrease the displacement. After cancellations we obtain $\ddot{\theta} = -k\sin\theta$, where $k = g/l$. The coefficient k can be set equal to unity by a suitable choice of the time scale (dividing t by $\sqrt{k}$). Accordingly, the equation of mathematical pendulum takes the form

$$\ddot{\theta} = -\sin\theta. \tag{1}$$

The phase space of the pendulum is represented by the cylinder (θ, ω), where $\omega = \dot{\theta}$ is the angular velocity. We can write equation (1) as a system

$$\dot{\theta} = \omega, \quad \dot{\omega} = -\sin\theta, \tag{2}$$

which contains only the first derivatives of the unknowns θ and ω. This system expresses the *local law of evolution* of the pendulum state: the rate of change is expressed in terms of the state itself.

Solving equation (1) (or system (2), for that matter) happens to be a difficult task. However, if we limit ourselves to small values of the angle θ (the case of *small oscillations*), then $\sin\theta \approx \theta$. Hence for small θ we can use the approximate equation $\ddot{\theta} = -\theta$ instead of (1). It is known as the *equation of small oscillations of the pendulum*. Its solution $\theta = C\sin(t+\varphi)$ is familiar to physics students: *for small*
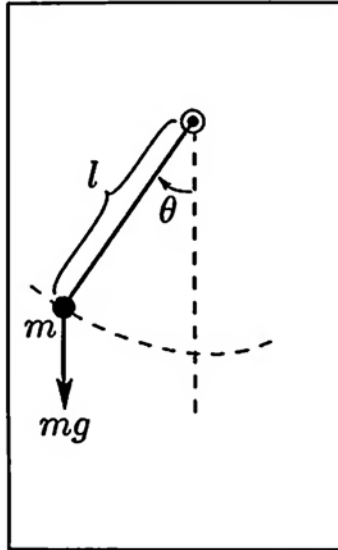
FIGURE 4. The mathematical pendulum.

angular displacements, a two-dimensional mathematical pendulum is in harmonic (sinusoidal) periodic motion. The question of whether the above conclusion is valid in the case of the original equation (1) calls for a special investigation.

We shall make use of the phase space of system (2), namely the cylinder $(\theta; \omega)$. At each point of the phase space we shall construct a *phase velocity vector* — i.e., a vector with coordinates $(\dot{\theta}; \dot{\omega}) = (\omega; -\sin\theta)$ which is a geometrical manifestation of the law of nature (1). The picture of the *velocity field* so obtained is shown in Figure 5. (For clarity, we have cut the cylinder along a generatrix and developed it.)
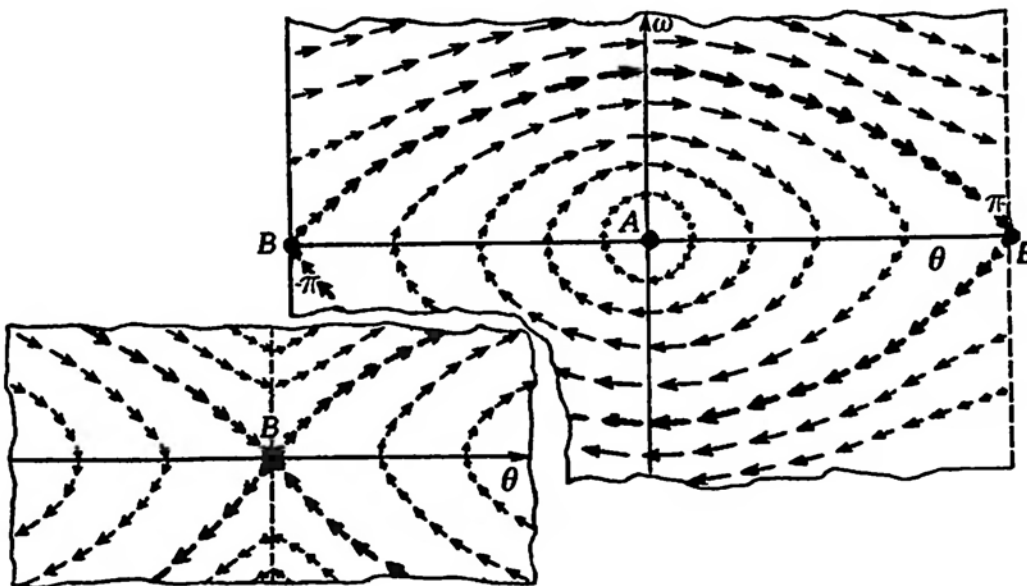

FIGURE 5. The velocity field of the mathematical pendulum.

In this picture we can see the following. There are two points, A and B, where the phase velocity vanishes. The point A corresponds to the lower (stable) equilibrium position, and the point B to the upper (unstable) equilibrium position. For small values of θ and $\dot{\theta}$, it is readily seen that points in the phase space move along closed curves that look like circles (small oscillations), and for larger values of θ the closed curves expand into curves resembling ellipses (oscillations with a large amplitude).[2] Of particular importance are the trajectories leading from point P back to P: falling from the unstable equilibrium position the pendulum performs a full revolution and stops again at the upper position. This trajectory separates oscillation from (nonuniform) pendulum rotation which occurs if the velocity of the pendulum at $\theta = 0$ is greater than 2 (in our dimensionless units).

3. Carp in a Pond (Normal Population Growth).

A large pond is used for raising crucian carp (referred to below simply as carp). The carp do not interfere with one another: there is plenty of food. How will the number of carp $x(t)$ change with time? The rate of growth in the carp population will be proportional to the population size, whence

$$\dot{x} = kx, \quad k > 0. \tag{3}$$

This equation is called the *differential equation of normal or Malthusian population growth*.[3] (Recall that $\dot{x}$ denotes the derivative of x with respect to time.)

THE EXISTENCE THEOREM. *Any function of the form $x = Ce^{kt}$, where C is a constant, is a solution to equation (3).*

PROOF. The substitution in (3) of the function $x = Ce^{kt}$ yields the proof. $\square$

This theorem implies the theorem that follows.

THE UNIQUENESS THEOREM. *Equation (3) has no other solutions.*

PROOF. Any function may be written in the form $x = C(t)e^{kt}$. Hence $\dot{x} = \dot{C}e^{kt} + kx$. Thus to satisfy equation (3) it is necessary and sufficient that $\dot{C} = 0$ — i.e., that C be a constant. $\square$

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[2] The fact that phase trajectories are closed implies the periodicity of oscillations (friction is not taken into account). The closedness property follows from *Huygens' theorem* (*the law of conservation of energy*): the quantity $E = \dot{\theta}^2/2 - \cos\theta$ does not change with position.
Proof. $\dot{E} = \ddot{\theta}\dot{\theta} + \dot{\theta}\sin\theta = 0$. Huygens' own proof was, of course, different, since he did not know differential calculus. However, we should not underestimate the ingenuity of Newton's predecessors. Thus Huygens and Barrow could find, say, the value of the limit

$$\lim_{x \to 0} \frac{\sin\tan x - \tan\sin x}{\arcsin\arctan x - \arctan\arcsin x}$$

instantaneously from geometrical considerations (there are few contemporary mathematicians who could evaluate this limit within an hour).

[3] In some cases the population growth rate turns out to be proportional not to the population size but to the number of pairs. Such abnormal population growth occurs at a much slower rate in contrast to normal population growth when the population density is low. (For example, some species of the whale find it hard to form a pair.) On the contrary, for large x population growth at a rate varying directly with x^2 leads to an "explosion" occurring in a finite time: the graph of $x(t)$ will have a vertical asymptote (why?). This situation arises in the equations of chemical kinetics where the reaction rate is proportional to the mass of each reagent.

The graphs of solutions of a differential equation are called its *integral curves.* These curves lie in the *space-time plane* — i.e., in the plane with coordinates $(t; x)$. When drawing integral curves, it is helpful first to graph the *direction field of the equation* which consists of tiny line segments assigned to each point of the space-time plane and having slopes given by the right-hand member of the equation (that is, by kx in the case of equation (3); see Figure 6). The points of an integral curve are points of tangency of the corresponding line segments of the direction field and, conversely, any curve all of whose points have this properly will be an integral curve, hence the graph of a solution of the equation. (Therein lies the geometrical significance of a differential equation.)
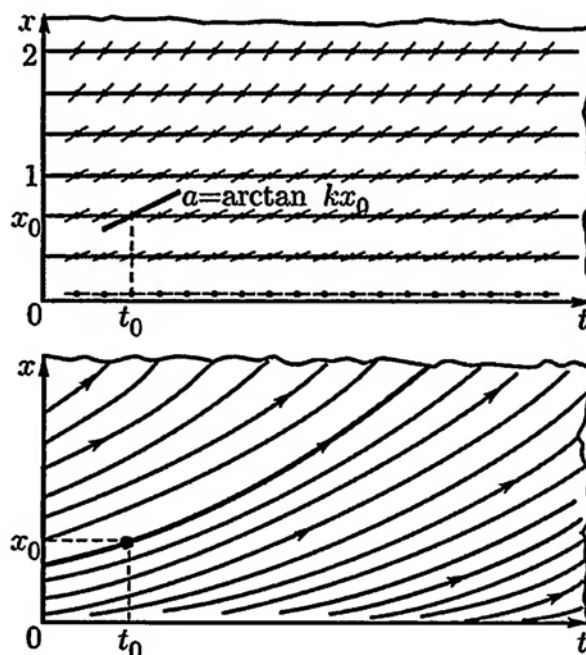

FIGURE 6. The direction field for the equation of normal population growth $\dot{x} = kx$ and the solution passing through the initial point (t_0, x_0).

We have just proved that through each space-time point there passes one and only one integral curve $x = Ce^{kt}$ of equation (3). This formula represents the *law of normal population growth*: it always takes the same time to double the carp population regardless of its size. (Currently, the Earth population doubles every 40 years.) Therefore the number of people presently alive is greater than the number of people who have died during the past 1000 years, and it would be greater than the number of people who have ever died if the birth rate had been constant. The lifetime of the human species would then be of the order of 2000 years; hence the birth rate of the human population in the past must have been less than it is today.

REMARKS. 1. Our argument was based on the solutions $x = Ce^{kt}$, which we obtained by inspection. We could have found these solutions systematically in the following way. The slope of an integral curve of the equation $\dot{x} = v(x)$ with respect to the t-axis is equal to $v(x)$. Hence the slope of the integral curve with respect to the x-axis will be equal to $1/v(x)$. We can express t as a function of x along the integral curve (unless it is parallel to the t axis). Using a prime to denote a

derivative with respect to x, we obtain $t' = 1/v(x)$. Thus t is an antiderivative of $1/v$. This makes it possible to express t in terms of x and then x in terms of t. (Of course, the straight lines in the space-time plane, where $v(x) = 0$, will have to be omitted to avoid division by zero.) For $v(x) = kx$ we obtain $t' = 1/kx$; hence $kt = \ln|x| + \text{constant}$ and $x = Ce^{kt}$.

2. The uniqueness theorem which states that noncoincident integral curves do not intersect is quite remarkable, since it seems to contradict both common sense and physical experiments. For example, the integral curves of equation (3) for $k = 1$ passing through the points $(0; 0)$ and $(0; 1)$ can no longer be distinguished by eye at $t = -5$, and in the case of $t = -30$ even atoms would not fit in between them. Nevertheless, from a mathematical viewpoint they are considered to be nonintersecting even for $t = -10^{10}$.

3. In the recent physics literature doubts have been expressed concerning Newton's deduction (in 1684) of Kepler's laws from the law of universal gravitation (which was communicated to Newton by Hooke in his letter of January 6, 1680): Newton's deduction implicitly used the uniqueness theorem (not proved) for the equation of motion. In reality, however, the uniqueness property follows from the existence of a solution whose dependence on the initial point can be specified by a differentiable function. The proof of the uniqueness property in this case is similar to the corresponding proof given above for the equation of normal population growth. Despite this gap, Newton actually produced a correct solution. In his letter to Halley describing his discussion with Hooke, Newton pointed out the difference in approach to natural science between a mathematician (Newton) and a physicist (Hooke) in these significant words: "Mathematicians who discover and establish all sorts of things and carry out all the work must be satisfied with their role of mere calculators and laborers, whereas others appropriate everything within their reach and lay claim to all inventions of both their disciples and their predecessors." As the curator of the Royal Society, at its weekly sessions Hooke had to prove by experiment two or three new laws of nature, which he did for nearly 40 years. The laws demonstrated by Hooke could well have been discovered by others, but Hooke laid claim to some 500 laws, asserting that he himself had discovered them. Of course, he did not have time to give mathematical justification for each law.

As to our carp, after sufficient time has elapsed, they will become so numerous that there will not be enough food for them; hence a further increase in population will be impossible, contradicting equation (3).

4. Carp in the Case of Food Shortage (Logistic Curve).

If the carp pond is small, or if there has been a sharp increase in the size of the carp population, competition for food will lead to a decrease in the birth rate. The simplest assumption is that the coefficient k depends linearly on the number of the carp, i.e., $k = a - bx$ (for small values of x a smooth function of x can be approximated by a linear one). In this way we arrive at the *equation of population change under competition*: $\dot{x} = (a - bx)x$. The coefficients a and b may be assumed to equal unity by a suitable choice of units for t and x. This yields the so-called *logistic equation*

$$\dot{x} = (1 - x)x. \tag{4}$$

The direction field of equation (4) in the space-time plane is shown in Figure 7; the graphs of solutions are also shown there. The S-shaped integral curves in the strip $0 < x < 1$ are called logistic curves. We see that (1) the process has two

equilibrium states $x = 0$ and $x = 1$; (2) in the region bounded by the lines $x = 0$ and $x = 1$ the field is directed upwards (from 0 to 1), and for $x > 1$ it is directed downwards (towards 1).
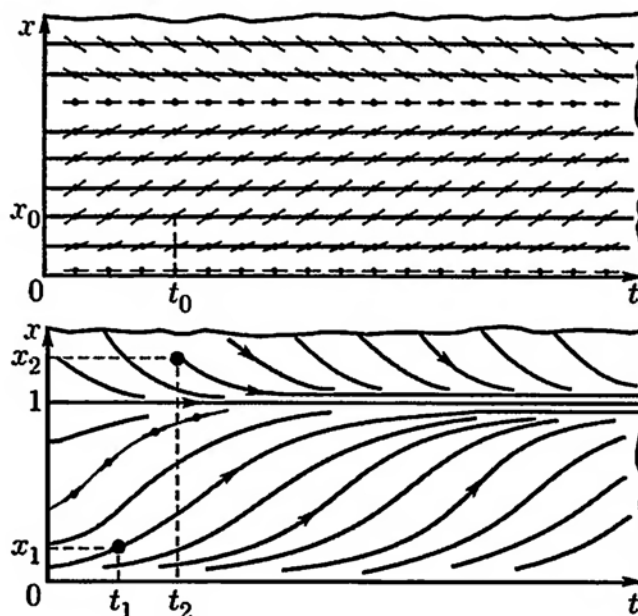

FIGURE 7. The direction field and the graphs of solutions for the logistic equation $\dot{x} = (1 - x)x$. The solutions $x = 1$ and $x = 0$ correspond to the positions of stable and unstable equilibrium, respectively.

Therefore the equilibrium state $x = 0$ is unstable (the carp population, once appearing on the scene, will start to grow in size), and the equilibrium state $x = 1$ is stable (a smaller population will grow, while a larger population will decrease). Whatever the initial number of carp $x > 0$, in time the process reaches the stable equilibrium state $x = 1$. Each pond, then, may be characterized by its own particular number of carp, and in time any carp population will tend to this number.

5. Carp Fishing.

So far we have considered a free carp population developing according to its own intrinsic laws. Assume now that the carp are regularly fished out (say, to provide a daily supply of fish to the local store) and that the fishing rate is a constant. We obtain the following *differential equation of fishing*:

$$\dot{x} = (1 - x)x - c. \tag{5}$$

The quantity c may be interpreted as the allowed fishing rate and is called the *quota*. Three versions of the solution of equation (5) corresponding to three distinct values of c are plotted in Figure 8.

As we see from the figure, when the fishing rate is not very large ($0 < c < 1/4$), there exist two equilibrium states ($x = A$ and $x = B$). The lower equilibrium state A is unstable. If at some time the population size x dips below A due, say, to overfishing or disease, then in a finite time the population will die out. The upper equilibrium state B is stable; this is, in fact, a steady-state solution for the
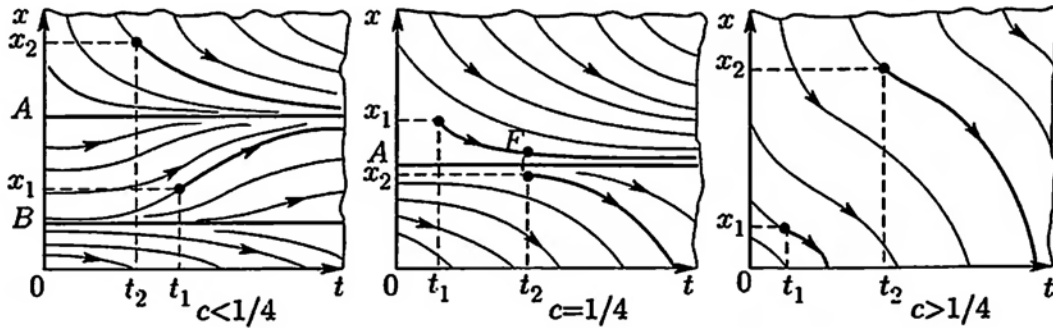
FIGURE 8. The graphs of solutions for the fishing equation $\dot{x} = (1 - x)x - c$ for different quota values c. For $c < 1/4$ the population tends to the stable equilibrium level $x = A$, for $c = 1/4$ the equilibrium position is unstable, and for $c > 1/4$ the population dies out.

population size provided the fishing rate c does not change with time. Clearly, when fishing takes place, the pond equilibrium population size is less than in the case when there is no fishing. If $c > 1/4$, there are no equilibrium states, and all carp will be fished out in a finite time (an excessive supply of fish to the local store will lead to there being no more fish in the pond; the extermination of the Steller cow provides a historic example). If $c = 1/4$, there exists one unstable equilibrium state ($A = B = 1/2$). If the initial population size is sufficiently large, fishing at the indicated rate is theoretically possible for an arbitrary length of time; however an arbitrarily small decrease in the equilibrium population size (see the point F) will lead to a total fishing out of the carp in a finite time (thus fishing at the rate $c = 1/4$ is not a sensible thing to do). Therefore, though any quotas including the maximum one ($c \leq 1/4$) are theoretically feasible, the maximum quota $c = 1/4$ will result in a loss of stability and is impermissible.[4] More than that, even quotas close to $1/4$ are not acceptable, since the danger threshold A will then be near the equilibrium state B (a small deviation of the population size below the threshold A will lead to the disappearance of the fish). As is shown below, we may arrange fishing in such a way as to systematically catch fish at a rate of $c = 1/4$.

6. The Introduction of Feedback.

Instead of the absolute fishing rate c we shall now fix the relative fishing rate $p = c/x$ — i.e., the number of the carp fished per unit time divided by the total number of the carp. Then the differential equation of fishing takes the form

$$\dot{x} = (1 - x)x - px. \tag{6}$$

The space-time plane for equation (6) (with $p < 1$) and the graphs of several solutions are shown in Figure 9. Clearly, the lower unstable equilibrium state is at the point $x = 0$, while the second equilibrium state B is stable for any p (whenever $0 < p < 1$).

Upon the termination of the transient process the carp population size reaches the steady level $x = B$. The absolute fishing rate then becomes equal to $c = pB$.

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[4]Mathematical ecology yields the following result important for practical economy: The wish to maximize the profit or the output may lead to a loss of stability in the system.
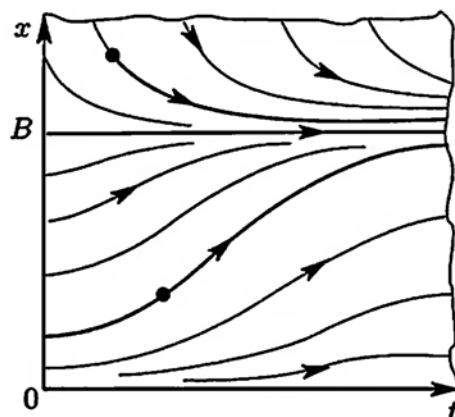
FIGURE 9. The graphs of solutions for the fishing equation $\dot{x} = (1-x)x - px$, with $p < 1$ (the relative quota being a fixed quantity). The population tends to a stable level $x = B$.

(This is the ordinate of the point of intersection of the graphs of the functions $v = (1 - x)x$ and $v = px$.) Investigate this quantity as a function of p. At low relative fishing rates (p is small) the steady absolute fishing rate is also small; as $p \to 1$ it also tends to zero (overfishing). The maximum value of the absolute rate c is equal to the maximum ordinate of the graph of the function $v = (1-x)x$. This value is $c = 1/4$; it can be obtained in the case in which the line $v = px$ passes through the apex of the parabola (i.e., for $p = 1/2$) as shown in Figure 10.
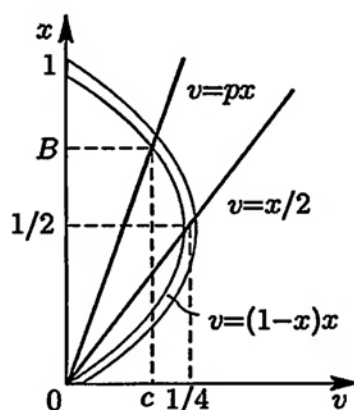

FIGURE 10. Determination of the optimum stable level of fishing with a fixed relative quota.

Select $p = 1/2$ (i.e., select the relative quota such that the steady-state population size constitutes one-half of the population size when there is no fishing). Then we have obtained the maximum steady fishing rate $c = 1/4$, and at the same time the system remains stable (it restores its equilibrium state provided the deviations of the population size from equilibrium are small).[5]

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[5]In order to attain this goal, we must have a flexible fishing policy: the fishing quota is decreased when the fish stock is getting low and increased when the conditions become favorable again.

7. Carp and Pike.

Suppose that pike are introduced into our carp pond. Without the pike, the carp would breed exponentially at a rate of $\dot{x} = kx$ (the pond is assumed to be large, so there is plenty of food for the carp). But now we have to take into account the carp eaten by the pike. We shall assume that the number of encounters between carp and pike is proportional to both the number of carp x and the number of pike y; then the rate of change for the carp population is $\dot{x} = kx - axy$. As to the pike, they will vanish without carp: $\dot{y} = -ly$; with carp available, they will breed at a rate proportional to the number of the carp eaten: $\dot{y} = -ly + bxy$. This may serve as an example of the simplest model of a *predator-prey system* and yields the following system of differential equations:

$$\begin{cases} \dot{x} = kx - axy, \\ \dot{y} = -ly + bxy. \end{cases} \tag{7}$$

This model is called the *Lotka-Volterra model*.

The phase space of the system is the first quadrant bounded by the semiaxes $x \geq 0$ and $y \geq 0$. Construct in it the velocity field for the system (see Figure 11) — i.e., construct the vector $(kx - axy; -ly + bxy)$ at each point $(x; y)$. This field has a critical point (a point where the corresponding vector is the zero vector), namely the point $(\frac{l}{b}; \frac{k}{a})$. It corresponds to equilibrium populations of both the carp and the pike; this situation arises when the potential growth of the carp population is offset by the activity of the pike, and the growth of the pike population by its natural mortality.
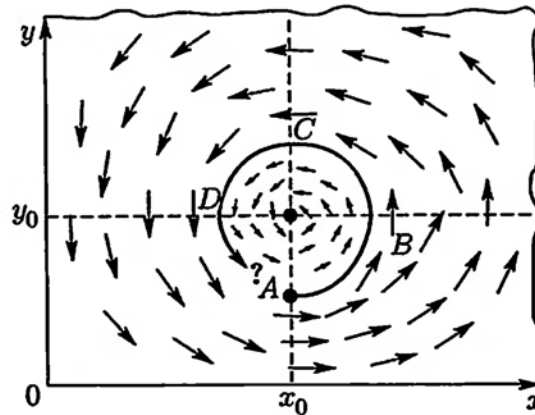

FIGURE 11. The velocity field for the predator-prey model.

If the initial number of pike is less than $y_0 = \frac{k}{a}$ (the point A in the figure), then the number of both carp and pike will grow until the newly spawned pike begin to eat more carp than there are newly spawned carp (the point B), at which time the number of carp will begin to dwindle while the pike population increases; this will go on until the resulting food shortage leads to an eventual decrease in the number of pike (the point C), which, in turn, will result in renewed carp spawning (the point D); in time this will lead to an increase in the number of pike that are spawned. But will we return to the point A? In other words, will the oscillation process be periodic? The answer to this question can depend on the choice of the initial point A, as well as on the values of the system parameters.

Problems. 1. Find the equation of the logistic curve.

2. Are the integral curves of equations (5) and (6) spread-out logistic curves?

3. Prove that the integral curves of equations (5) and (6) lying outside the strip between A and B have vertical asymptotes.

4. Can the noncoincident integral curves of the logistic equation intersect one another?

5. Prove that the period of oscillation of the pendulum (1) is proportional to the amplitude.

6. Calculate approximately the period of oscillation of a pendulum with a small amplitude a.

7. Find the limit of the period of oscillation of a pendulum as the amplitude tends to π.

8. Prove that the period of oscillation of a pendulum is equal to the derivative with respect to the energy E of the area bounded by the phase trajectory $\frac{\dot{\theta}^2}{2} - \cos\theta = E$.

9. Particles of unit mass, acted upon by a field of force $F = \sin x$, begin to move simultaneously with zero velocity from all points on the x axis. At what time do the particles begin to collide?

Answers to problems. 1. $x = e^t/(1 + e^t)$.

2. No.

4. No.

6. $T \approx 2\pi \left(1 + \frac{a^2}{16} + \cdots\right)$.

7. ∞.

9. $\pi/2$.

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