Problem set 8 (modified) (The last one!)

Do for Monday, June 4.

Rudin p. 196: 13

Also:

1. Suppose $f : [0, \pi] \to \mathbb{R}$ is continuous and $\int_0^\pi f(x) \sin nx = 0$, $n = 1, \ldots$ Does it follow that $f \equiv 0$? Proof or counterexample.

2.

(a) Find a sequence of polynomials $\{P_n(x)\}$ and a continuous function f such that $P_n(x) \to f(x)$ pointwise on $[0, 1]$, but not uniformly.

(b) Suppose that N is fixed and $\{P_n(x)\}$ is a sequence of polynomials of degrees $\leq N$. Show that if there is a function f on $[0, 1]$ such that $P_n(x) \to f(x)$ pointwise on $[0, 1]$, then the convergence is uniform. You may assume that the coefficients of the polynomials $\{P_n(x)\}$ are uniformly bounded in n. (The hypothesis actually implies this assumption; see the notes for Weeks 9-10.). (Not so easy!)

3. If f is real analytic in a neighborhood of x_0 and $f(x_0) = 0$, show that $f(x)/(x - x_0)$ is real analytic in the same neighborhood.

4. Prove that if $f(x)$ is real analytic on (a, b) and $c \in (a, b)$, then $F(x) = \int_c^x f(t)dt$ is also real analytic on (a, b).

5. Prove that $x^n \to 0$ in the L^2 metric on $[0, 1]$, but not in the uniform metric.

6. Consider the subset $\mathcal{F} \subset \mathcal{C}([0, 1])$ given by

$$\mathcal{F} = \{f \in \mathcal{C}([0, 1]) : \|f\| \leq 1\}.$$

Show that $\mathcal{F}$ is closed, but not compact.