APPENDIX A

Introductory Lectures on Real Analysis

LECTURE 1: THE REAL NUMBERS

We assume without proof the usual properties of the integers: For example, that the integers are closed under addition and subtraction, that the principle of mathematical induction holds for the positive integers, and that 1 is the least positive integer.

We also assume the usual field and order properties for the real numbers $\mathbb{R}$. Thus, we accept without proof that the reals satisfy the *field axioms*, as follows:

F1 $a + b = b + a$, $ab = ba$ (commutativity)

F2 $a + (b + c) = (a + b) + c$, $a(bc) = (ab)c$ (associativity)

F3 $a(b + c) = ab + ac$ (distributive law)

F4 $\exists$ elements $0, 1$ with $0 \neq 1$ such that $0 + a = a$ and $1 \cdot a = a$ $\forall a \in \mathbb{R}$ (additive and multiplicative identities)

F5 $\forall a \in \mathbb{R}$, $\exists$ an element $-a$ such that $a + (-a) = 0$ (additive inverse)

F6 $\forall a \in \mathbb{R}$ with $a \neq 0$, $\exists$ an element a^{-1} such that $aa^{-1} = 1$ (multiplicative inverse).

Terminology: $a + (-b)$, ab^{-1} are usually written $a - b$, $\frac{a}{b}$, respectively. Notice that the latter makes sense only for $b \neq 0$.

Notice that all the other standard algebraic properties of the reals follow from these. (See Exercise 1.1 below.)

We here also accept without proof that the reals satisfy the following *order axioms:*

01 For each $a \in \mathbb{R}$, *exactly one* of the possibilities $a > 0$, $a = 0$, $-a > 0$ holds.

02 $a > 0$ and $b > 0 \Rightarrow ab > 0$ and $a + b > 0$.

Terminology: $a > b$ means $a - b > 0$, $a \geq b$ means that either $a > b$ or $a = b$, $a < b$ means $b > a$, and $a \leq b$ means $b \geq a$.

We claim that all the other standard properties of inequalities follow from these and from F1–F6. (See Problem 1.2 below.)

Notice also that the above properties (i.e., F1–F6, O1, O2) all hold with the *rational numbers* $Q \equiv \{\frac{p}{q} : p, q$ are integers, $q \neq 0\}$ in place of $\mathbb{R}$. F1–F6 also hold with the complex numbers $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$ in place of $\mathbb{R}$, but inequalities like $a > b$ make no sense for complex numbers.

In addition to F1–F6, O1, O2 there is one further key property of the real numbers. To discuss it we need first to introduce some terminology.

Terminology: If $S \subset \mathbb{R}$ we say:

(1) S is *bounded above* if $\exists$ a number $K \in \mathbb{R}$ such that $x \leq K \ \forall x \in S$. (Any such number K is called an *upper bound* for S.)

(2) S is *bounded below* if $\exists$ a number $k \in \mathbb{R}$ such that $x \geq k \ \forall x \in S$. (Any such number k is called a *lower bound* for S.)

(3) S is *bounded* if it is *both* bounded above *and* bounded below. (This is equivalent to the fact that $\exists$ a positive real number L such that $|x| \leq L \ \forall x \in S$.)

We can now introduce the final key property of the real numbers.

C. (**"Completeness property of the reals"**): If $S \subset \mathbb{R}$ is nonempty and bounded above, then S has a *least* upper bound.

Notice that the terminology "least upper bound" used here means exactly what it says: a number α is a least upper bound for a set $S \subset \mathbb{R}$ if

(i) $x \leq \alpha \ \forall \ x \in S$ i.e., α is *an* upper bound), and

(ii) if β is any other upper bound for S, then $\alpha \leq \beta$ i.e., α is $\leq$ any other upper bound for S).

Such a least upper bound is *unique*, because if α_1, α_2 are both least upper bounds for S, the property (ii) implies that both $\alpha_1 \leq \alpha_2$ *and* $\alpha_2 \leq \alpha_1$, so $\alpha_2 = \alpha_1$. It therefore makes sense to speak of *the* least upper bound of S (also known as "the supremum" of S). The least upper bound of S will henceforth be denoted sup S. Notice that property C guarantees sup S *exists* if S is nonempty and bounded above.

Remark: If S is nonempty and bounded below, then it has a greatest lower bound (or "infimum"), which we denote inf S. One can *prove* the existence of inf S (if S is bounded below and nonempty) by using property C on the set $-S = \{-x : x \in S\}$. (See Exercise 1.5 below.)

We should be careful to distinguish between the maximum element of a set S (if it exists) and the supremum of S. Recall that we say a number α is the maximum of S (denoted max S) if (i)′ $x \le \alpha \, \forall \, x \in S$ (i.e., α is an upper bound for S), and

(ii)′ $\alpha \in S$.

These two properties say exactly that α is a upper bound for S and also one of the elements of S. Thus, clearly a maximum α of S, if it exists, satisfies both (i), (ii) and hence must agree with sup S. But keep in mind that max S may not exist, even if the set S is nonempty and bounded above: for example, if $S = (0, 1)$ $(= \{x \in \mathbb{R} : 0 < x < 1\})$, then $\sup S = 1$, but $\max S$ does not exist, because $1 \notin S$.

Notice that of course any *finite* nonempty set $S \subset \mathbb{R}$ has a maximum. One can formally prove this by induction on the number of elements of the set S. (See Exercise 1.3 below.)

Using the above-mentioned properties of the integers and the reals it is now possible to give formal rigorous proofs of all the other properties of the reals which are used, even the ones which seem self-evident. For example, one can actually prove formally, using only the above properties, the fact that the set of positive integers are not bounded above. (Otherwise there would be a least upper bound α so, in particular, we would have $n \le \alpha$ for each positive integer n and hence, in particular, $n \le \alpha$ hence $n + 1 \le \alpha$ for each n, or in other words $n \le \alpha - 1$ for each positive integer n, contradicting the fact that α is the least upper bound!) Thus, we have shown rigorously, using only the axioms F1–F6, O1, O2, and C, that the positive integers are not bounded above. Thus,

$$\forall \text{ positive } a \in \mathbb{R}, \exists \text{ a positive integer } n \text{ with } n > a \quad \left(\text{i.e., } \frac{1}{n} < \frac{1}{a}\right).$$

This is referred to as "The Archimedean Property" of the reals.

Similarly, using only the axioms F1–F6, O1,2, and C, we can give a formal proof of all the basic properties of the real numbers—for example, in Problem 1.7 below you are asked to prove that square roots of positive numbers do indeed exist.

Final notes on the Reals: (1) We have *assumed* without proof all properties F1–F6, O1,O2 and C. In a more advanced course we could, starting only with the positive integers, give a rigorous *construction* of the real numbers, and *prove* all the properties F1–F6, O1, O2, and C. Furthermore, one can prove (in a sense that can be made precise) that the real number system is the *unique* field with all the above properties.

(2) You can of course freely use all the standard rules for algebraic manipulation of equations and inequalities involving the reals; normally you do not need to justify such things in terms of the axioms F1–F6, O1, O2 unless you are specifically asked to do so.

LECTURE 1 PROBLEMS

1.1 Using *only* properties F1–F6, prove

(i) $a \cdot 0 = 0 \, \forall \, a \in \mathbb{R}$

(ii) $ab = 0 \Rightarrow$ either $a = 0$ or $b = 0$

(iii) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ $\forall\, a, b, c, d \in \mathbb{R}$ with $b \neq 0, d \neq 0$.

Note: In each case present your argument in steps, stating which of F1–F6 is used at each step.

Hint for (iii): First show the "cancellation law" that $\frac{x}{y} = \frac{xz}{yz}$ for any $x, y, z \in \mathbb{R}$ with $y \neq 0, z \neq 0$.

1.2 Using only properties F1–F6 and O1,O2, (and the agreed terminology) prove the following:

(i) $a > 0 \Rightarrow 0 > -a$ i.e., $-a < 0$)

(ii) $a > 0 \Rightarrow \frac{1}{a} > 0$

(iii) $a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b}$

(iv) $a > b$ and $c > 0 \Rightarrow ac > bc$.

1.3 If S is a finite nonempty subset of $\mathbb{R}$, prove that max S exists. (Hint: Let n be the number of elements of S and try using induction on n.)

1.4 Given any number $x \in \mathbb{R}$, prove there is an integer n such that $n \leq x < n + 1$.

Hint: Start by proving there is a least integer $> x$.

Note: 1.4 Establishes rigorously the fact that every real number x can be written in the form $x =$ integer plus a remainder, with the remainder $\in [0, 1)$. The integer is often referred to as the "integer part of x." We emphasize once again that such properties are completely standard properties of the real numbers and can normally be used without comment; the point of the present exercise is to show that it is indeed possible to rigorously prove such standard properties by using the basic properties of the integers and the axioms F1–F6, O1,O2, C.

1.5 Given a set $S \subset \mathbb{R}$, $-S$ denotes $\{-x : x \in S\}$. Prove:

(i) S is bounded below if an only if $-S$ is bounded above.

(ii) If S is nonempty and bounded below, then inf S exists and $= -\sup(-S)$.

(Hint: Show that $\alpha = -\sup(-S)$ has the necessary 2 properties which establish it to be the greatest lower bound of S.)

1.6 If $S \subset \mathbb{R}$ is nonempty and bounded above, prove $\exists$ a sequence $a_1, a_2, \ldots$ of points in S with $\lim a_n = \sup S$.

Hint: In case $\sup S \notin S$, let $\alpha = \sup S$, and for each integer $j \geq 1$ prove there is at least one element $a_j \in S$ with $\alpha > a_j > \alpha - \frac{1}{j}$.

1.7 Prove that every positive real number has a positive square root. (That is, for any $a > 0$, prove there is a real number $\alpha > 0$ such that $\alpha^2 = a$.)

Hint: Begin by observing that $S = \{x \in \mathbb{R} : x > 0 \text{ and } x^2 < a\}$ is nonempty and bounded above, and then argue that $\sup S$ is the required square root.

LECTURE 2: SEQUENCES OF REAL NUMBERS AND THE BOLZANO-WEIERSTRASS THEOREM

Let $a_1, a_2, \ldots$ be a sequence of real numbers; a_n is called the *n-th term* of the sequence. We sometimes use the abbreviation $\{a_n\}$ or $\{a_n\}_{n=1,2,\ldots}$

Technically, we should distinguish between *the sequence* $\{a_n\}$ and *the set of terms of the sequence*—i.e., the set $S = \{a_1, a_2 \ldots\}$. These are *not* the same: e.g., the sequence $1, 1, \ldots$ has infinitely many terms each equal to 1, whereas the set S is just the set $\{1\}$ containing one element.

Formally, a sequence is a *mapping* from the positive integers to the real numbers; the n^{th} term a_n of the sequence is just the value of this mapping at the integer n. From this point of view—i.e., thinking of a sequence as a mapping from the integers to the real numbers—a sequence has a graph consisting of discrete points in $\mathbb{R}^2$, one point of the *graph* on each of the vertical lines $x = n$. Thus, for example, the sequence $1, 1, \ldots$ (each term $= 1$) has graph consisting of the discrete points marked "$\otimes$" in the following figure:

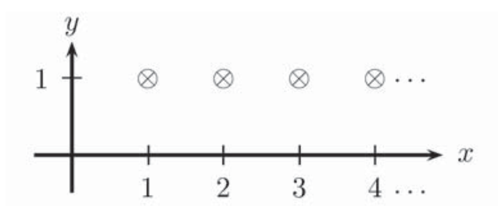

Figure A.1: Graph of the trivial sequence $\{a_n\}_{n=1,\ldots}$ where $a_n = 1 \; \forall n$.

Terminology: Recall the following terminology. A sequence $a_1, a_2, \ldots$ is:

(i) *bounded above* if $\exists$ a real number K such that $a_n \le K \;\; \forall$ integer $n \ge 1$.

(ii) *bounded below* if $\exists$ a real number k such that $a_n \ge k \;\; \forall$ integer $n \ge 1$.

(iii) *bounded* if it is *both* bounded above *and* bounded below. (This is equivalent to the fact that $\exists$ a real number L such that $|a_n| \le L \; \forall$ integer $n \ge 1$.)

(iv) *increasing* if $a_{n+1} \ge a_n \;\; \forall$ integer $n \ge 1$.

(v) *strictly increasing* if $a_{n+1} > a_n \; \forall$ integer $n \ge 1$.

(vi) *decreasing* if $a_{n+1} \le a_n \; \forall$ integer $n \ge 1$.

(vii) *strictly decreasing* if $a_{n+1} < a_n \; \forall$ integer $n \ge 1$.

(viii) *monotone* if *either* the sequence is increasing or the sequence is decreasing.

(ix) We say the sequence *has limit* ℓ (ℓ a given real number) if for each $\varepsilon > 0$ there is an integer $N \geq 1$ such that

$$(*) \qquad |a_n - \ell| < \varepsilon \ \forall \text{ integer } n \geq N .$$ [3]

(x) In case the sequence $\{a_n\}$ has limit ℓ we write

$$\lim a_n = \ell \text{ or } \lim_{n\to\infty} a_n = \ell \text{ or } a_n \to \ell .$$

(xi) We say the sequence $\{a_n\}$ *converges* (or "*is convergent*") if it has limit ℓ for some $\ell \in \mathbb{R}$.

Theorem 2.1. *If $\{a_n\}$ is monotone and bounded, then it is convergent. In fact, if $S = \{a_1, a_2 \ldots\}$ is the set of terms of the sequence, we have the following:*

(i) if $\{a_n\}$ is increasing and bounded then $\lim a_n = \sup S$.

(ii) if $\{a_n\}$ is decreasing and bounded then $\lim a_n = \inf S$.

Proof: See Exercise 2.2. (Exercise 2.2 proves part (i), but the proof of part (ii) is almost identical.)

Theorem 2.2. *If $\{a_n\}$ is convergent, then it is bounded.*

Proof: Let $l = \lim a_n$. Using the definition (ix) above with $\varepsilon = 1$, we see that there exists an integer $N \geq 1$ such that $|a_n - l| < 1$ whenever $n \geq N$. Thus, using the triangle inequality, we have $|a_n| \equiv |(a_n - l) + l| \leq |a_n - l| + |l| < 1 + |l| \ \forall$ integer $n \geq N$. Thus,

$$|a_n| \leq \max\{|a_1|, \ldots |a_N|, |l| + 1\} \ \forall \text{ integer } n \geq 1 .$$

Theorem 2.3. *If $\{a_n\}$, $\{b_n\}$ are convergent sequences, then the sequences $\{a_n + b_n\}$, $\{a_n b_n\}$ are also convergent and*

$$\text{(i)} \qquad \lim(a_n + b_n) = \lim a_n + \lim b_n$$

$$\text{(ii)} \qquad \lim(a_n b_n) = (\lim a_n) \cdot (\lim b_n) .$$

In addition, if $b_n \neq 0$ and $\lim b_n \neq 0$, *then*

$$\text{(iii)} \qquad \lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n} .$$

Proof: We prove (ii); the remaining parts are left as an exercise. First, since $\{a_n\}, \{b_n\}$ are convergent, the previous theorem tells us that there are constants $L, M > 0$ such that

$$(*) \qquad |a_n| \leq L \text{ and } |b_n| \leq M \ \forall \text{ positive integer } n .$$

[3] Notice that $(*)$ is equivalent to $\ell - \varepsilon < a_n < \ell + \varepsilon \ \forall$ integers $n \geq N$.

Now let $l = \lim a_n$, $m = \lim b_n$ and note that by the triangle inequality and $(*)$,

$$\begin{aligned} |a_n b_n - lm| &\equiv |a_n b_n - l b_n + l b_n - lm| \\ &\equiv |(a_n - l) b_n + l(b_n - m| \\ &\le |a_n - l||b_n| + |l||b_n - m| \\ (**) \qquad &\le M|a_n - l| + |l||b_n - m| \ \forall n \ge 1 \ . \end{aligned}$$

On the other hand, for any given $\varepsilon > 0$ we use the definition of convergence, i.e., (ix) above) to deduce that there exist integers $N_1, N_2 \ge 1$ such that

$$|a_n - l| < \frac{\varepsilon}{2(1 + M + |l|)} \ \forall \text{ integer } n \ge N_1$$

and

$$|b_n - m| < \frac{\varepsilon}{2(1 + M + |l|)} \ \forall \text{ integer } n \ge N_2 \ .$$

Thus, for each integer $n \ge \max\{N_1, N_2\}$, $(**)$ implies

$$\begin{aligned} |a_n b_n - lm| &< M \frac{\varepsilon}{2(1 + M + |l|)} + |l| \frac{\varepsilon}{2(1 + M + |l|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ , \end{aligned}$$

and the proof is complete because we have shown that the definition (ix) is satisfied for the sequence $a_n b_n$ with lm in place of l.

Remark: Notice that if we take $\{a_n\}$ to be the constant sequence $-1, -1, \ldots$ (so that trivially $\lim a_n = -1$) in part (ii) above, we get

$$\lim(-b_n) = -\lim b_n \ .$$

Hence, using part (i) with $\{-b_n\}$ in place of b_n we conclude

$$\lim(a_n - b_n) = \lim a_n - \lim b_n \ .$$

By similar argument (i), (ii), imply that

$$\lim(\alpha a_n + \beta b_n) = \alpha \lim a_n + \beta \lim b_n \ .$$

For any $\alpha, \beta \in \mathbb{R}$ (provided $\lim a_n, \lim b_n$ both exist).

The following definition of *subsequence* is important.

Definition: Given a sequence $a_1, a_2, \ldots$, we say $a_{n_1}, a_{n_2}, \ldots$ is a *subsequence* of $\{a_n\}$ if $n_1, n_2, \ldots$ are integers with $1 \le n_1 < n_2 < n_3 < \ldots$. (Note the *strict* inequalities.)

Thus, $\{b_n\}_{n=1,2,\ldots}$ is a subsequence of $\{a_n\}_{n=1,2,\ldots}$ if and only if *for each* $j \ge 1$ *both* the following hold:

(i) b_j is one of the terms of the sequence $a_1, a_2, \ldots$, and

(ii) the real number b_{j+1} appears later than the real number b_j in the sequence $a_1, a_2, \ldots$

Thus, $\{a_{2n}\}_{n=1,2,\ldots} = a_2, a_4, a_6 \ldots$ is a subsequence of $\{a_n\}_{n=1,2,\ldots}$ and $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots$ is a subsequence of $\{\frac{1}{n}\}_{n=1,2,\ldots}$ but $\frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \ldots$ is *not* a subsequence of $\{\frac{1}{n}\}$.

Theorem 2.4. (Bolzano-Weierstrass Theorem.) *Any bounded sequence $\{a_n\}$ has a convergent subsequence.*

Proof: ("Method of bisection") Since $\{a_n\}$ is bounded we can find upper and lower bounds, respectively. Thus, $\exists\, c < d$ such that

$$(*) \qquad c \le a_k \le d \ \forall \text{ integer } k \ge 1 \, .$$

Now subdivide the interval $[c, d]$ into the two half intervals $[c, \frac{c+d}{2}]$, $[\frac{c+d}{2}, d]$. By $(*)$, at least one of these (call it I_1 say) has the property that there are *infinitely many* integers k with $a_k \in I_1$. Similarly, we can divide I_1 into two equal subintervals (each of length $\frac{d-c}{4}$), at least one of which (call it I_2) has the property that $a_k \in I_2$ for infinitely many integers k. Proceeding in this way we get a sequence of intervals $\{I_n\}_{n=1,2\ldots}$ with $I_n = [c_n, d_n]$ and with the properties that, for each integer $n \ge 1$,

$$(1) \qquad \text{length } I_n \equiv d_n - c_n = \frac{d-c}{2^n}$$

$$(2) \qquad [c_{n+1}, d_{n+1}] \subset [c_n, d_n] \subset [c, d]$$

$$(3) \qquad a_k \in [c_n, d_n] \text{ for infinitely many integers } k \ge 1 \, .$$

(Notice that (3) says $c_n \le a_k \le d_n$ for infinitely many integers $k \ge 1$.)

Using properties (1), (2), (3), we proceed to prove the existence of a convergent subsequence as follows.

Select any integer $k_1 \ge 1$ such that $a_{k_1} \in [c_1, d_1]$ (which we can do by (3)). Then select any integer $k_2 > k_1$ with $a_{k_2} \in [c_2, d_2]$. Such k_2 of course exists by (3). Continuing in this way we get integers $1 \le k_1 < k_2 \ldots$ such that $a_{k_n} \in [c_n, d_n]$ for each integer $n \ge 1$. That is,

$$(4) \qquad c_n \le a_{k_n} \le d_n \ \forall \text{ integer } n \ge 1 \, .$$

On the other hand, by (1), (2) we have

$$(5) \qquad c \le c_n \le c_{n+1} < d_{n+1} \le d_n \le d \ \forall \text{ integer } n \ge 1 \, .$$

Notice that (5), in particular, guarantees that $\{c_n\}$, $\{d_n\}$ are *bounded* sequences which are, respectively, *increasing* and *decreasing*, hence by Thm. 2.1 are *convergent*. On the other hand, (1) says $d_n - c_n = \frac{d-c}{2^n}$ ($\to 0$ as $n \to \infty$), hence $\lim c_n = \lim d_n (= \ell$ say). Then by (4) and the Sandwich Theorem (see Exercise 2.5 below) we see that $\{a_{k_n}\}_{n=1,2,\ldots}$ also has limit ℓ.

LECTURE 2 PROBLEMS

2.1 Use the Archimedean property of the reals (Lem. 1.1 of Lecture 1) to prove rigorously that $\lim \frac{1}{n} = 0$.

2.2 Prove part (i) of Thm. 2.1.

Hint: Let $\alpha = \sup\ S$, and show first that for each $\varepsilon > 0$, $\exists$ an integer $N \geq 1$ such that $a_N > \alpha - \varepsilon$.

2.3 Using the definition (ix) on p. 92 prove that a sequence $\{a_n\}$ cannot have more than one limit.

2.4 If $\{a_n\}$, $\{b_n\}$ are given convergent sequences and $a_n \leq b_n\ \forall\, n \geq 1$, prove $\lim a_n \leq \lim b_n$.

Hint: $\lim(a_n - b_n) = \lim a_n - \lim b_n$, so it suffices to prove that $\lim c_n \leq 0$ whenever $\{c_n\}$ is convergent with $c_n \leq 0\ \forall\, n$.

2.5 ("Sandwich Theorem.") If $\{a_n\}$, $\{b_n\}$ are given convergent sequences with $\lim a_n = \lim b_n$, and if $\{c_n\}$ is any sequence such that $a_n \leq c_n \leq b_n\ \forall\, n \geq 1$, prove that $\{c_n\}$ is convergent and $\lim c_n = \lim a_n (= \lim b_n)$ (Hint: Let $l = \lim a_n = \lim b_n$ and use the definition (ix) on p. 92.)

2.6 If k is a fixed positive integer and if $\{a_n\}$ is any sequence such that $\frac{1}{n^k} \leq a_n \leq n^k\ \forall\ n \geq 1$, prove that $\lim a_n^{1/n} = 1$. (Hint: use 2.5 and the standard limit result that $\lim n^{\frac{1}{n}} = 1$.)

LECTURE 3: CONTINUOUS FUNCTIONS

Here we shall mainly be interested in real valued functions for some closed interval $[a, b]$; thus $f : [a, b] \to \mathbb{R}$. (This is reasonable notation, since for each $x \in [a, b]$, f assigns a value $f(x) \in \mathbb{R}$.) First we recall the definition of *continuity* of such a function.

Definition 1. $f : [a, b] \to \mathbb{R}$ is said to be *continuous at the point* $c \in [a, b]$, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$x \in [a, b] \text{ with } |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon \ .$$

Definition 2. We say $f : [a, b] \to \mathbb{R}$ is continuous if f is continuous at *each* point $c \in [a, b]$.

We want to prove the important theorem that such a continuous function attains both its maximum and minimum values on $[a, b]$. We first make the terminology precise.

Terminology: If $f : [a, b] \to \mathbb{R}$, then:

(1) f is said to *attain its maximum* at a point $c \in [a, b]$ if $f(x) \leq f(c) \, \forall \, x \in [a, b]$;

(2) f is said to *attain its minimum* at a point $c \in [a, b]$ if $f(x) \geq f(c) \ \forall \, x \in [a, b]$.

We shall also need the following lemma, which is of independent importance.

Lemma 3.1. *If* $a_n \in [a, b] \, \forall n \geq 1$ *and if* $\lim a_n = c \in [a, b]$ *and if* $f : [a, b] \to \mathbb{R}$ *is continuous at* c, *then*

$$\lim f(a_n) = f(c) \ ,$$

i.e., the sequence $\{f(a_n)\}_{n=1,2...}$ *converges to* $f(c)$.

Proof: Let $\varepsilon > 0$. By Def. 1 above, $\exists \ \delta > 0$ such that

(∗) $$|f(x) - f(c)| < \varepsilon \text{ whenever } x \in [a, b] \text{ with } |x - c| < \delta \ .$$

On the other hand, by the definition of $\lim a_n = c$, with δ in place of ε (i.e., we use the definition (ix) on p. 92 of Lecture 2 with δ in place of ε) we can find an integer $N \geq 1$ such that

$$|a_n - c| < \delta \text{ whenever n} \geq \text{N} \ .$$

Then, (since $a_n \in [a, b] \, \forall \, n$) (∗) tells us that

$$|f(a_n) - f(c)| < \varepsilon \, \forall \, n \geq N \ .$$

Theorem 3.2. *If* $f : [a, b] \to \mathbb{R}$ *is continuous, then* f *is bounded and* $\exists$ *points* $c_1, c_2, \in [a, b]$ *such that* f *attains its maximum at the point* c_1 *and its minimum at the point* c_2; *that is,* $f(c) \leq f(x) \leq f(c_2) \, \forall \, x \in [a, b]$.

Proof: It is enough to prove that f bounded above and that there is a point $c_1 \in [a, b]$ such that f attains its maximum at c_1 , because we can get the rest of the theorem by applying this results to $-f$.

To prove f is bounded above we argue by contradiction. If f is not bounded above, then for each integer $n \geq 1$ we can find a point $x_n \in [a, b]$ such that $f(x) \geq n$. Since x_n is a bounded sequence, by the Bolzano-Weierstrass Theorem (Thm. 2.4 of Lecture 2) we can find a convergent subsequence $x_{n_1}, x_{n_2}, \ldots$ Let $c = \lim x_{n_j}$

Of course, since $a \leq x_{n_j} \leq b \,\forall j$, we must have $c \in [a, b]$. Also, since $1 \leq n_1 < n_2 < \ldots$ (and since $n_1, n_2 \ldots$ are integers), we must have $n_{j+1} \geq n_j + 1$, hence by induction on j

$$n_j \geq j \quad \forall \text{ integer } j \geq 1 \,. \tag{1}$$

Now since $x_{n_j} \in [a, b]$ and $\lim x_{n_j} = c \in [a, b]$ we have by Lem. 3.1 that $\lim f(x_{n_j}) = f(c)$. Thus, $f(x_{n_j})_{j=1,2,\ldots}$ is *convergent*, hence *bounded* by Thm. 2.2. But by construction $f(x_{n_j}) \geq n_j$ ($\geq j$ by (1)), hence $f(x_{n_j})_{j=1,2\ldots}$ is not bounded, a contradiction. This completes the proof that f is bounded above.

We now want to prove f attains its maximum value at some point $c_1 \in [a, b]$. Let $S = \{f(x) : x \in [a, b]\}$. We just proved above that S is bounded above, hence (since it is non empty by definition) S has a *least upper bound* which we denote by M. We claim that for each integer $n \geq 1$ there is a point $x_n \in [a, b]$ such that $f(x_n) > M - \frac{1}{n}$. Indeed, *otherwise* $M - \frac{1}{n}$ *would be an upper bound for* S, contradicting the fact that M was chosen to be the *least* upper bound. Again we can use the Bolzano-Weierstrass Theorem to find a convergent subsequence $x_{n_1}, x_{n_2} \ldots$ and again (1) holds. Let $c_1 = \lim x_{n_j}$. By Lem. 3.1 again we have

$$f(c_1) = \lim f(x_{n_j}) \,. \tag{2}$$

However, by construction we have

$$M \geq f(x_{n_j}) > M - \frac{1}{n_j} \geq M - \frac{1}{j} \quad \text{(by (1))} \,.$$

And hence by the Sandwich Theorem (Exercise 2.4 of Lecture 2) we have $\lim f(x_{n_j}) = M$. By (2) this gives $f(c_1) = M$. But M is an upper bound for $S = \{f(x) : x \in [a, b]\}$, hence we have $f(x) \leq f(c_1) \,\forall x \in [a, b]$, as required.

An important consequence of the above theorem is the following.

Lemma (Rolle's Theorem): *If* $f : [a, b] \to \mathbb{R}$ *is continuous, if* $f(a) = f(b) = 0$ *and if* f *is differentiable at each point of* (a, b), *then there is a point* $c \in (a, b)$ *with* $f'(c) = 0$.

Proof: If f is identically zero then $f'(c) = 0$ for every point $c \in (a, b)$, so assume f is not identically zero. Without loss of generality we may assume $f(x) > 0$ for some $x \in (a, b)$ (otherwise this property holds with $-f$ in place of f). Then $\max f$ (which exists by Thm. 3.2) is positive and is attained at some point $c \in (a, b)$. We claim that $f'(c) = 0$. Indeed, $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c_+} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c_-} \frac{f(x)-f(c)}{x-c}$ and the latter 2 limits are, respectively, ≤ 0 and ≥ 0. But they are equal, hence they must both be zero.

Corollary (Mean Value Theorem): *If $f : [a, b] \to \mathbb{R}$ is continuous and f is differentiable at each point of (a, b), then there is some point $c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$.*

Proof: Apply Rolle's Theorem to the function $\widetilde{f}(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$.

LECTURE 3 PROBLEMS

3.1 Give an example of *a* bounded function $f : [0, 1] \to \mathbb{R}$ such that f is continuous at each point $c \in [0, 1]$ except at $c = 0$, and such that f attains neither a maximum nor a minimum value.

3.2 Prove carefully (using the definition of continuity on p. 96) that the function $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} +1 & \text{if } \quad 0 < x \leq 1 \\ 0 & \text{if } -1 \leq x \leq 0 \end{cases}$$

is not continuous at $x = 0$. (Hint: Show the definition fails with, e.g., $\varepsilon = \frac{1}{2}$.)

3.3 Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $|f| : [a, b] \to \mathbb{R}$ be defined by $|f|(x) = |f(x)|$. Prove that $|f|$ is continuous.

3.4 Suppose $f : [a, b] \to \mathbb{R}$ and $c \in [a, b]$ are given, and suppose that $\lim f(x_n) = f(c)$ for all sequences $\{x_n\}_{n=1,2,\ldots} \subset [a, b]$ with $\lim x_n = c$. Prove that f is continuous at c. (Hint: If not, $\exists\, \varepsilon > 0$ such that $(*)$ on p. 96 fails for each $\delta > 0$; in particular, $\exists\, \varepsilon > 0$ such that $(*)$ fails for $\delta = \frac{1}{n}$ $\forall$ integer $n \geq 1$.)

3.5 If $f : [0, 1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1 \text{ if } x \in [0, 1] \text{ is a rational number} \\ 0 \text{ if } x \in [0, 1] \text{ is not rational}\,. \end{cases}$$

Prove that f is continuous at no point of $[0, 1]$.

Hint: Recall that any interval $(c, d) \in \mathbb{R}$ (with $c < d$) contains both rational and irrational numbers.

3.6 Suppose $f : (0, 1) \to \mathbb{R}$ is defined by $f(x) = 0$ if $x \in (0, 1)$ is not a rational number, and $f(x) = 1/q$ if $x \in (0, 1)$ can be written in the form $x = \frac{p}{q}$ with p, q positive integers without common factors. Prove that f is continuous at each irrational value $x \in (0, 1)$.

Hint: First note that for a given $\varepsilon > 0$ there are at most finitely many positive integers q with $\frac{1}{q} \geq \varepsilon$.

3.7 Suppose $f : [0, 1] \to \mathbb{R}$ is continuous, and $f(x) = 0$ for each rational point $x \in [0, 1]$. Prove $f(x) = 0$ for all $x \in [0, 1]$.

3.8 If $f : \mathbb{R} \to \mathbb{R}$ is continuous at each point of $\mathbb{R}$, and if $f(x + y) = f(x) + f(y)\ \forall x, y \in \mathbb{R}$, prove $\exists$ a constant a such that $f(x) = ax\ \forall x \in \mathbb{R}$. Show by example that the result is false if we drop the requirement that f be continuous.

LECTURE 4: SERIES OF REAL NUMBERS

Consider the series

$$a_1 + a_2 + \cdots + a_n + \ldots$$

(usually written with summation notation as $\sum_{n=1}^{\infty} a_n$), where $a_1, a_2, \ldots$ is a given sequence of real numbers. a_n is called the n-th term of the series. The sum of the first n terms is

$$s_n = \sum_{k=1}^{n} a_k \; ;$$

s_n is called the n-th partial sum of the series. If

$$s_n \to s$$

(i.e., if $\lim s_n = s$) for some $s \in \mathbb{R}$, then we say the series *converges*, and *has sum* s. Also, in this case we write

$$s = \sum_{n=1}^{\infty} a_n \; .$$

If s_n does not converge, then we say the series *diverges.*

Example: If $a \in \mathbb{R}$ is given, then the series $1 + a + a^2 + \ldots$ (i.e., the geometric series) has nth partial sum

$$s_n = 1 + a + \cdots + a^{n-1} = \begin{cases} n & \text{if } a = 1 \\ \frac{1-a^n}{1-a} & \text{if } a \neq 1 \; . \end{cases}$$

Using the fact that $a^n \to 0$ if $|a| < 1$, we thus see that the series converges and has sum $\frac{1}{1-a}$ if $|a| < 1$, whereas the series diverges for $|a| \geq 1$. (Indeed $\{s_n\}$ is *unbounded* if $|a| > 1$ or $a = 1$, and if $a = -1$, $\{s_n\}_{n=1,2,\ldots} = 1, 0, 1, 0, \ldots$.)

The following simple lemma is of key importance.

Lemma 4.1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim a_n = 0$.

Note: The *converse* is not true. For example, we check below that $\sum_{n=1}^{\infty} \frac{1}{n}$ does *not* converge, but its nth term is $\frac{1}{n}$, which *does* converge to zero.

Proof of Lemma 4.1: Let $s = \lim s_n$. Then of course we also have $s = \lim s_{n+1}$. But, $s_{n+1} - s_n = (a_1 + a_2 + \cdots + a_{n+1}) - (a_1 + a_2 + \cdots a_n) = a_{n+1}$, hence (see the remark following Thm. 2.3 of Lecture 2) we have $\lim a_{n+1} = \lim s_{n+1} - \lim s_n = s - s = 0$. i.e., $\lim a_n = 0$.

The following lemma is of theoretical and practical importance.

Lemma 4.2. *If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ both converge, and have sum s, t, respectively, and if α, β are real numbers, then $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)$ also converges and has sum $\alpha s + \beta t$, i.e., $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{\infty} a_n + \beta \sum_{n=1}^{\infty} b_n$ if both $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ converge.*

Proof: Let $s_n = \sum_{k=1}^{n} a_n$, $t_n = \sum_{k=1}^{n} b_k$. We are given $s_n \to s$ and $t_n \to t$, then $\alpha s_n + \beta t_n \to \alpha s + \beta t$ (see the remarks following Thm. 2.3 of Lecture 2). But $\alpha s_n + \beta t_n = \alpha \sum_{k=1}^{n} a_n + \beta \sum_{k=1}^{n} b_k = \sum_{k=1}^{n}(\alpha a_k + \beta b_k)$, which is the nth partial sum of $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)$.

There is a very convenient criteria for checking convergence in case all the terms are *nonnegative*. Indeed, in this case

$$s_{n+1} - s_n = a_{n+1} \geq 0 \, \forall \, n \geq 1 \; ,$$

hence the sequence $\{s_n\}$ is *increasing* if $a_n \geq 0$. Thus, by Thm. 2.1(i) of Lecture 2 we see that $\{s_n\}$ converges if and only if it is *bounded*. That is, we have proved:

Lemma 4.3. *If each term of $\sum_{n=1}^{\infty} a_n$ is nonnegative (i.e., if $a_n \geq 0 \, \forall \, n$) then the series converges if and only if the sequence of partial sums (i.e., $\{s_n\}_{n=1,2,\ldots}$) is bounded.*

Example. Using the above criteria, we can discuss convergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where $p > 0$ is given. The nth partial sum in this case is

$$s_n = \sum_{k=1}^{n} \frac{1}{k^p} \; .$$

Since $\frac{1}{x^p}$ is a decreasing function of x for $x > 0$, we have, for each integer $k \geq 1$,

$$\frac{1}{(k+1)^p} \leq \frac{1}{x^p} \leq \frac{1}{k^p} \quad \forall \, x \in [k, k+1] \; .$$

Integrating, this gives

$$\frac{1}{(k+1)^p} \equiv \int_k^{k+1} \frac{1}{(k+1)^p} \leq \int_k^{k+1} \frac{1}{x^p}\, dx \leq \int_k^{k+1} \frac{1}{k^p}\, dx \equiv \frac{1}{k^p} \; ,$$

so if we sum from $k = 1$ to n, we get

$$\sum_{k=1}^{n} \frac{1}{(k+1)^p} \leq \int_1^{n+1} \frac{1}{x^p}\, dx \leq \sum_{k=1}^{n} \frac{1}{k^p} \; .$$

That is,

$$s_{n+1} - 1 \leq \int_1^{n+1} \frac{1}{x^p}\, dx \leq s_n \quad \forall \, n \geq 1 \; .$$

But

$$\int_1^{n+1} \frac{1}{x^p}\, dx = \begin{cases} \log(n+1) & \text{if } p = 1 \\ \frac{(n+1)^{1-p} - 1}{1-p} & \text{if } p \neq 1 \; . \end{cases}$$

Thus, we see that $\{s_n\}$ is *unbounded* if $p \leq 1$ and bounded for $p > 1$, hence from Lem. 4.2 we conclude

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \leq 1 \; . \end{cases}$$

Remark. The above method can be modified to discuss convergence of other series. See Exercise 4.6 below.

Theorem 4.4. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Terminology: If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent*. Thus, with this terminology, the above theorem just says "absolute convergence $\Rightarrow$ convergence."

Proof of Theorem 4.4: Let $s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} |a_k|$. Then we are given $t_n \to t$ for some $t \in \mathbb{R}$.

For each integer $n \geq 1$, let

$$p_n = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases}$$
$$q_n = \begin{cases} -a_n & \text{if } a_n \leq 0 \\ 0 & \text{if } a_n > 0\,, \end{cases}$$

and let $s_n^+ = \sum_{k=1}^{n} p_n$, $s_n^- = \sum_{k=1}^{n} q_n$. Notice that for each $n \geq 1$ we then have

$$\begin{aligned} a_n &= p_n - q_n, & s_n &= s_n^+ - s_n^- \\ |a_n| &= p_n + q_n, & t_n &= s_n^+ + s_n^-\,, \end{aligned}$$

and $p_n, q_n \geq 0$. Also,

$$0 \leq s_n^+ \leq t_n \leq t \text{ and } 0 \leq s_n^- \leq t_n \leq t\,.$$

Hence, we have shown that $\sum_{n=1}^{\infty} p_n$, $\sum_{n=1}^{\infty} q_n$ have bounded partial sums. Hence, by Lem. 4.3, both $\sum_{n=1}^{\infty} p_n$, $\sum_{n=1}^{\infty} q_n$ converge. But then (by Lem. 4.2) $\sum_{n=1}^{\infty}(p_n - q_n)$ converges, i.e., $\sum_{n=1}^{\infty} a_n$ converges.

Rearrangement of series: We want to show that the terms of an absolutely convergent series can be rearranged in an arbitrary way without changing the sum. First we make the definition clear.

Definition: Let $j_1, j_2, \ldots$ be any sequence of positive integers in which every positive integer appears once and only once (i.e., the mapping $n \to j_n$ is a $1:1$ mapping of the positive integers onto the positive integers). Then the series $\sum_{n=1}^{\infty} a_{j_n}$ is said to be a *rearrangement* of the series $\sum_{n=1}^{\infty} a_n$.

Theorem 4.5. *If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then any rearrangement $\sum_{n=1}^{\infty} a_{j_n}$ converges, and has the same sum as $\sum_{n=1}^{\infty} a_n$.*

Proof: We give the proof when $a_n \geq 0\ \forall n$ (in which case "absolute convergence" just means "convergence"). This extension to the general case is left as a exercise. (See Exercise 4.8 below.)

Hence, assume $\sum_{n=1}^{\infty} a_n$ converges, and $a_n \geq 0\ \forall$ integer $n \geq 1$, and let $\sum_{n=1}^{\infty} a_{j_n}$ be any rearrangement. For each $n \geq 1$, let

$$P(n) = \max\{j_1, \ldots, j_n\}\,.$$

So that

$$\{j_1, \ldots, j_n\} \subset \{1, \ldots P(n)\} \text{ and hence (since } a_n \geq 0 \ \forall k)$$

$$a_{j_1} + a_{j_2} + \cdots + a_{j_n} \leq a_1 + \cdots + a_{P(n)} \leq s \ ,$$

where $s = \sum_{n=1}^{\infty} a_n$. Thus, we have shown that the partial sums of $\sum_{n=1}^{\infty} a_{j_n}$ are bounded above by s, hence by Lem. 4.3, $\sum_{n=1}^{\infty} a_{j_n}$ converges, and has sum t satisfying $t \leq s$. But $\sum_{n=1}^{\infty} a_n$ is a rearrangement of $\sum_{n=1}^{\infty} a_{j_n}$ (using the rearrangement given by the inverse mapping $j_n \to n$), and hence by the same argument we also have $s \leq t$. Hence, $s = t$ as required.

LECTURE 4 PROBLEMS

The first few problems give various criteria to test for *absolute convergence* (and also, in some cases, for testing for *divergence*).

4.1 (i) (Comparison test.) If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are given series and if $|a_n| \leq |b_n| \ \forall n \geq 1$, prove $\sum_{n=1}^{\infty} b_n$ absolutely convergent $\Rightarrow \sum_{n=1}^{\infty} a_n$ absolutely convergent.

(ii) Use this to discuss convergence of:

(a) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

(b) $\sum_{n=1}^{\infty} \frac{\sin(\frac{1}{n})}{n}$.

4.2 (Comparison test for *divergence*.) If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are given series with nonnegative terms, and if $a_n \geq b_n \ \forall n \geq 1$, prove $\sum_{n=1}^{\infty} b_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges.

4.3 (i) (Ratio Test.) If $a_n \neq 0 \ \forall n$, if $\lambda \in (0, 1)$ and if there is an integer $N \geq 1$ such that $\frac{|a_{n+1}|}{|a_n|} \leq \lambda \ \forall n \geq N$, prove that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. (Hint: First use induction to show that $|a_n| \leq \lambda^{n-N} \ \forall n \geq N$.)

(ii) (Ratio test for divergence.) If $a_n \neq 0 \ \forall n$ and if $\exists$ an integer $N \geq 1$ such that $\frac{|a_{n+1}|}{|a_n|} \geq 1 \ \forall n \geq N$ then prove $\sum_{n=1}^{\infty} a_n$ diverges.

4.4 (Cauchy root test.) (i) Suppose $\exists \lambda \in (0, 1)$ and an integer $N \geq 1$ such that $|a_n|^{\frac{1}{n}} \leq \lambda \ \forall n \geq N$. Prove that $\sum_{n=1}^{\infty} a_n$ converges.

(ii) Use the Cauchy root test to discuss convergence of $\sum_{n=1}^{\infty} n^2 x^n$. (Here $x \in \mathbb{R}$ is given—consider the possibilities $|x| < 1$, $|x| > 1$, $|x| = 1$.)

4.5 Suppose $a_n \geq 0 \ \forall n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ diverges. Prove

(i) $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges

(ii) $\sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n}$ converges.

4.6 (Integral Test.) If $f : [1, \infty) \to \mathbb{R}$ is positive and continuous at each point of $[1, \infty)$, and if f is *decreasing*, i.e., $x < y \Rightarrow f(y) \leq f(x)$, prove using a modification of the argument on pp. 100–102 that $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\{\int_1^n f(x)\, dx\}_{n=1,2,\ldots}$ is *bounded*.

4.7 Using the integral test (in Exercise 4.6 above) to discuss convergence of

(i) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$

(ii) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{1+\varepsilon}}$, where $\varepsilon > 0$ is a given constant.

4.8 Complete the proof of Thm. 4.5 (i.e., discuss the general case when $\sum |a_n|$ converges). (Hint: The theorem has already been established for series of *nonnegative* terms; use p_n, q_n as in Thm. 4.4.)

LECTURE 5: POWER SERIES

A power series is a series of the form $\sum_{n=0}^{\infty} a_n x^n$, where $a_0, a_1, \ldots$ are given real numbers and x is a real variable. Here we use the standard convention that $x^0 = 1$, so the first term $a_0 x^0$ just means a_0.

Notice that for $x = 0$ the series trivially converges and its sum is a_0.

The following lemma describes the key convergence property of such series.

Lemma 1. *If the series $\sum_{n=0}^{\infty} a_n x^n$ converges for some value $x = c$, then the series converges absolutely for every x with $|x| < |c|$.*

Proof: $\sum_{n=0}^{\infty} a_n c^n$ converges $\Rightarrow \lim a_n c^n = 0 \Rightarrow \{a_n c^n\}_{n=1,2,\ldots}$ is a bounded sequence. That is, there is a fixed constant $M > 0$ such that $|a_n c^n| \le M \ \forall\, n = 0, 1, \ldots$, and so $|x| < |c| \Rightarrow$, for any $j = 0, 1, \ldots$,

$$|a_j x^j| = \left|a_j c^j\right| \left|\frac{x^j}{c^j}\right| \le M \left|\frac{x^j}{c^j}\right| = M\left(\frac{|x|}{|c|}\right)^j = M r^j, \quad r = \frac{|x|}{|c|} < 1 ,$$

and hence

$$\textstyle\sum_{j=0}^{n-1} |a_j x^j| \le M \sum_{j=0}^{n-1} r^j = M\frac{1 - r^n}{1 - r} \le \frac{M}{1 - r}, \qquad n = 1, 2, \ldots .$$

Thus, the series $\sum_{n=0}^{\infty} |a_n x^n|$ is convergent, because it has nonnegative terms and we've shown its partial sums are bounded. This completes the proof.

We can now directly apply the above lemma to establish the following basic property of power series.

Theorem 5.1. *For any given power series $\sum_{n=0}^{\infty} a_n x^n$, exactly one of the following 3 possibilities holds:*

(i) *the series diverges* $\forall\, x \ne 0$, *or*

(ii) *the series converges absolutely* $\forall\, x \in \mathbb{R}$, *or*

(iii) $\exists\, \rho > 0$ *such that the series converges absolutely* $\forall\, x$ *with* $|x| < \rho$, *and diverges* $\forall\, x$ *with* $|x| > \rho$.

Terminology: If (iii) holds, the number ρ is called the *radius of convergence* and the interval $(-\rho, \rho)$ is called the *interval of convergence*. If (i) holds we say the radius of convergence is zero, and if (ii) holds we say the radius of convergence $=\infty$.

Note: The theorem says nothing about what happens at $x = \pm\rho$ in case (iii).

Proof of Theorem 5.1: Consider the set $S \subset \mathbb{R}$ defined by

$$S = \{|x| : \textstyle\sum_{n=0}^{\infty} a_n x^n \text{ converges}\} .$$

Notice that we always have $0 \in S$, so S is nonempty. If $S = \{0\}$ then case (i) holds, so we can assume S contains at least one c with $c \ne 0$. If S is not bounded then by Lem. 1 we have that $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent (A.C.) on $(-R, R)$ for each $R > 0$, and hence (ii) holds. If

$S \neq \{0\}$ is bounded then $R = \sup S$ exists and is positive. Now for any $x \in (-R, R)$ we have $c \in S$ with $|c| > |x|$ (otherwise $|c| \le |x|$ for each $c \in S$ meaning that $|x|$ would be an upper bound for S smaller than R, contradicting $R = \sup S$), and hence by Lem. 1 $\sum a_n x^n$ is A.C. So, in fact, $\sum_{n=0}^{\infty} a_n x^n$ is A.C. for each $x \in (-R, R)$. We must of course also have $\sum_{n=0}^{\infty} a_n x^n$ diverges for each x with $|x| > R$ because otherwise we would have x_0 with $|x_0| > R$ and $\sum_{n=0}^{\infty} a_n x_0^n$ convergent, hence $|x_0| \in S$, which contradicts the fact that R is an upper bound for S.

Suppose now that a given power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $\rho > 0$ (we include here the case $\rho = \infty$, which is to be interpreted to mean that the series converges absolutely for all $x \in \mathbb{R}$).

A reasonable and natural question is whether or not we can also *expand* $f(x)$ *in terms of powers of* $x - \alpha$ for given $\alpha \in (-\rho, \rho)$. The following theorem shows that we can do this for $|x - \alpha| < \rho - |\alpha|$ (and for all x in case $\rho = \infty$).

Theorem 5.2 (Change of Base-Point.) *If* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *has radius of convergence* $\rho > 0$ *or* $\rho = \infty$, *and if* $|\alpha| < \rho$ *(and* α *arbitrary in case* $\rho = \infty$*), then we can also write*

$(*)$ $\quad f(x) = \sum_{m=0}^{\infty} b_m (x - \alpha)^m \quad \forall x$ *with* $|x - \alpha| < \rho - |\alpha|$ *(*x, α *arbitrary if* $\rho = \infty$*)*,

where $b_m = \sum_{n=m}^{\infty} \binom{n}{m} a_n \alpha^{n-m}$ *(so, in particular,* $b_0 = f(\alpha)$*); part of the conclusion here is that the series for* b_m *converges, and the series* $\sum_{m=0}^{\infty} b_m (x - \alpha)^m$ *converges for the stated values of* x, α.

Note: The series on the right in $(*)$ is a power series in powers of $x - \alpha$, hence the fact that it converges for $|x - \alpha| < \rho - |\alpha|$ means that it has radius of convergence (as a power series in powers of $x - \alpha$) $\ge \rho - |\alpha|$ (and radius of convergence $= \infty$ in case $\rho = \infty$). Thus, in particular, the series on the right of $(*)$ automatically converges *absolutely* for $|x - \alpha| < \rho - |\alpha|$ by Lem. 1.

Proof of Theorem 5.2: We take any α with $|\alpha| < \rho$ and any x with $|x - \alpha| < \rho - |\alpha|$ (α, x are arbitrary if $\rho = \infty$), and we look at the partial sum $S_N = \sum_{n=0}^{N} a_n x^n$. Since the Binomial Theorem tells us that $x^n = (\alpha + (x - \alpha))^n = \sum_{m=0}^{n} \binom{n}{m} \alpha^{n-m} (x - \alpha)^m$, we see that S_N can be written

$$\sum_{n=0}^{N} a_n x^n = \sum_{n=0}^{N} a_n \sum_{m=0}^{n} \binom{n}{m} \alpha^{n-m} (x - \alpha)^m .$$

Using the interchange of sums formula (see Problem 6.3 below)

$$\sum_{n=0}^{N} \sum_{m=0}^{n} c_{nm} = \sum_{m=0}^{N} \sum_{n=m}^{N} c_{nm} ,$$

this then gives

$$(1) \qquad \sum_{n=0}^{N} a_n x^n = \sum_{m=0}^{N} \Big(\sum_{n=m}^{N} \binom{n}{m} a_n \alpha^{n-m} \Big) (x - \alpha)^m .$$

Now since $\binom{n}{m}^{1/n} \to 1$ as $n \to \infty$ for each fixed m, we see that for any $\varepsilon > 0$ we have N such that $\binom{n}{m} \le (1 + \varepsilon)^n$ for all $n \ge N$, hence $|a_n \binom{n}{m} x^n| \le |a_n ((1 + \varepsilon) x)^n| \ \forall n \ge N$ and hence by the comparison test $\sum_{n=0}^{\infty} \binom{n}{m} a_n x^n$ also converges absolutely for all $x \in (-\rho, \rho)$ (because $|x| < \rho \Rightarrow (1 +$

$\varepsilon)|x| < \rho$ for suitable $\varepsilon > 0$). Thus, since $|\alpha| < \rho$ we have, in particular, that $\sum\binom{n}{m}a_n\alpha^n$ is absolutely convergent and we can substitute $\sum_{n=m}^{N}\binom{n}{m}a_n\alpha^{n-m} = \sum_{n=m}^{\infty}\binom{n}{m}a_n\alpha^{n-m} - \sum_{n=N+1}^{\infty}\binom{n}{m}a_n\alpha^{n-m}$ in (1) above, whence (1) gives

$$\begin{aligned}\sum_{n=0}^{N} a_n x^n =&\sum_{m=0}^{N}\Big(\sum_{n=m}^{\infty}\binom{n}{m}a_n\alpha^{n-m}\Big)(x-\alpha)^m \\ &-\sum_{m=0}^{N}\Big(\sum_{n=N+1}^{\infty}\binom{n}{m}a_n\alpha^{n-m}\Big)(x-\alpha)^m\end{aligned}$$

and

$$\begin{aligned}\Big|\sum_{m=0}^{N}\Big(\sum_{n=N+1}^{\infty}\binom{n}{m}a_n\alpha^{n-m}\Big)(x-\alpha)^m\Big| &\le \sum_{m=0}^{N}\Big(\sum_{n=N+1}^{\infty}\binom{n}{m}|a_n|\,|\alpha|^{n-m}|x-\alpha|^m\Big) \\ &\equiv \sum_{n=N+1}^{\infty}\Big(\sum_{m=0}^{N}\binom{n}{m}|a_n|\,|\alpha|^{n-m}|x-\alpha|^m\Big) \\ &\le \sum_{n=N+1}^{\infty}\Big(\sum_{m=0}^{n}\binom{n}{m}|a_n|\,|\alpha|^{n-m}|x-\alpha|^m\Big) \\ &\equiv \sum_{n=N+1}^{\infty}|a_n|(|\alpha|+|x-\alpha|)^n, \quad |\alpha|+|x-\alpha|<\rho\ ,\end{aligned}$$

where we used the Binomial Theorem again in the last line. Now observe that we have

$$\begin{aligned}\sum_{n=N+1}^{\infty}|a_n|(|\alpha|+|x-\alpha|)^n =&\sum_{n=1}^{\infty}|a_n|(|\alpha|+|x-\alpha|)^n \\ &-\sum_{n=1}^{N}|a_n|(|\alpha|+|x-\alpha|)^n \to 0 \text{ as } N\to\infty\ ,\end{aligned}$$

so the above shows that $\lim_{N\to\infty}\sum_{m=0}^{N} b_m(x-\alpha)^m$ exists (and is real), i.e., the series $\sum_{m=0}^{\infty} b_m(x-\alpha)^m$ converges for $|x-\alpha| < \rho - |\alpha|$, and that the sum of the series agrees with $\sum_{n=0}^{\infty} a_n x^n$, so the proof of Thm. 5.2 is complete.

LECTURE 5 PROBLEMS

5.1 (i) Suppose the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is 1, and the radius of convergence of $\sum_{n=0}^{\infty} b_n x^n$ is 2. Prove that $\sum_{n=0}^{\infty}(a_n+b_n)x^n$ has radius of convergence 1.

(Hint: Lemma 1 guarantees that $\{a_n x^n\}_{n=1,2,\ldots}$ is unbounded if $|x|$ is greater than the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$.)

(ii) If *both* $\sum_{n=0}^{\infty} a_n x^n$, $\sum_{n=0}^{\infty} b_n x^n$ have radius of convergence = 1, show that

(a) The radius of convergence of $\sum_{n=0}^{\infty}(a_n+b_n)x^n$ is ≥ 1, and

(b) For any given number $R > 1$ you can construct examples with radius of convergence of $\sum_{n=0}^{\infty}(a_n+b_n)x^n = R$.

5.2 If $\exists$ constants $c, k > 0$ such that $c^{-1}n^{-k} \le |a_n| \le cn^k$ $\forall n = 1, 2, \ldots$, what can you say about the radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$.

LECTURE 6: TAYLOR SERIES REPRESENTATIONS

The change of base point theorem proved in Lecture 5 is actually quite strong; for example, it makes it almost trivial to check that a power series is differentiable arbitrarily many times inside its interval of convergence (i.e., a power series is "C^∞" in its interval of convergence), and furthermore all the derivatives can be correctly calculated simply by differentiating each term (i.e., "termwise" differentiation is a valid method for computing the derivatives of a power series in its interval of convergence). Specifically, we have:

Theorem 6.1. *Suppose the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $\rho > 0$ (or $\rho = \infty$), and let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < \rho$. Then all derivatives $f^{(m)}(x)$, $m = 1, 2, \ldots$, exist at every point x with $|x| < \rho$, and, in fact,*

$$f^{(m)}(x) = \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)a_n x^{n-m}, \quad x \in (-\rho, \rho),$$

which says precisely that the derivatives of f can be correctly computed simply by differentiating the series $\sum_{n=0}^{\infty} a_n x^n$ termwise (because $n(n-1)\cdots(n-m+1)a_n x^{n-m}$ is just the m^{th} derivative of $a_n x^n$); that is, $\frac{d^m}{dx^m}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d^m}{dx^m} a_n x^n$ for $|x| < \rho$.

Proof: It is enough to check the stated result $m = 1$, because then the general result follows directly by induction on m.

The proof for $m = 1$ is an easy consequence of Thm. 5.2 (change of base-point theorem), which tells us that we can write

$$(1) \qquad f(x) - f(\alpha) = \sum_{m=1}^{\infty} b_m (x-\alpha)^m \equiv (x-\alpha)b_1 + (x-\alpha)\sum_{m=2}^{\infty} b_m (x-\alpha)^{m-1}$$

for $|x - \alpha| < \rho - |\alpha|$, where $b_m = \sum_{n=m}^{\infty} \binom{n}{m} a_n \alpha^{n-m}$. That is,

$$(2) \qquad \frac{f(x) - f(\alpha) - b_1(x-\alpha)}{x-\alpha} = \sum_{m=2}^{\infty} b_m (x-\alpha)^{m-1}, \quad 0 < |x-\alpha| < \rho - |\alpha|$$

and the expression on the right is a convergent power series with radius of convergence at least $r = \rho - |\alpha| > 0$ and hence is A.C. if $x - \alpha$ is in the interval of convergence $(-r, r)$. In particular, it is A.C. when $|x - \alpha| = r/2$ and hence (2) shows that

$$(3) \qquad \left|\frac{f(x) - f(\alpha)}{x-\alpha} - b_1\right| \le \sum_{m=2}^{\infty} |b_m|\,|x-\alpha|^{m-1} \le |x-\alpha| \sum_{m=2}^{\infty} |b_m| (r/2)^{m-2}$$

for $0 < |x - \alpha| < r/2$ (where $r = \rho - |\alpha| > 0$). Since the right side in (3) $\to 0$ as $x \to \alpha$, this shows that $f'(\alpha)$ exists and is equal to $b_1 = \sum_{n=1}^{\infty} n a_n \alpha^{n-1}$.

We now turn to the important question of which functions f can be expressed as a power series on some interval. Since we have shown power series are differentiable to all orders in their interval of convergence, a *necessary* condition is clearly that f is differentiable to all orders; however, this is *not* sufficient; see Exercise 6.2 below. To get a reasonable sufficient condition, we need the following theorem.

Theorem 6.2 (Taylor's Theorem.) *Suppose $r > 0$, $\alpha \in \mathbb{R}$ and f is differentiable to order $m+1$ on the interval $|x-\alpha| < r$. Then $\forall\, x$ with $|x-\alpha| < r$, $\exists\, c$ between α and x such that*

$$f(x) = \sum_{n=0}^{m} \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n + \frac{f^{(m+1)}(c)}{(m+1)!}(x-\alpha)^{m+1} .$$

Proof: Fix x with $0 < x-\alpha < r$ (a similar argument holds in case $-r < x-\alpha < 0$), and, for $|t-\alpha| < r$, define

(1) $$g(t) = f(t) - \sum_{n=0}^{m} \frac{f^{(n)}(\alpha)}{n!}(t-\alpha)^n - M(t-\alpha)^{m+1} ,$$

where M (constant) is chosen so that $g(x) = 0$, i.e.,

$$M = \frac{\left(f(x) - \sum_{n=0}^{m} \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n\right)}{(x-\alpha)^{m+1}} .$$

Notice that, by direct computation in (1),

(2) $$\begin{cases} g^{(n)}(\alpha) & = 0\ \forall\, n = 0, \dots, m \\ g^{(m+1)}(t) & = f^{m+1}(t) - M(m+1)!,\ |t-\alpha| < r. \end{cases}$$

In particular, since $g(\alpha) = g(x) = 0$, the mean value theorem tells us that there is $c_1 \in (\alpha, x)$ such that $g'(c_1) = 0$. But then $g'(\alpha) = g'(c_1) = 0$, and hence again by the mean value theorem there is a constant $c_2 \in (\alpha, c_1)$ such that $g''(c_2) = 0$.

After $(m+1)$ such steps we deduce that there is a constant $c_{m+1} \in (\alpha, x)$ such that $g^{(m+1)}(c_{m+1}) = 0$. However, by (2), $g^{(m+1)}(t) = f^{(m+1)}(t) - M(m+1)!$, hence this gives

$$M = \frac{f^{(m+1)}(c_{m+1})}{(m+1)!} .$$

In view of our definition of M, this proves the theorem with $c = c_{m+1}$.

Theorem 6.2 gives us a satisfactory sufficient condition for writing f in terms of a power series. Specifically we have:

Theorem 6.3. *If $f(x)$ is differentiable to all orders in $|x-\alpha| < r$, and if there is a constant $C > 0$ such that*

(∗) $$\left|\frac{f^{(n)}(x)}{n!}\right| r^n \le C\ \forall\, n \ge 0,\ \text{and}\ \forall\, x \text{ with } |x-\alpha| < r ,$$

then $\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n$ converges, and has sum $f(x)$, for every x with $|x-\alpha| < r$.

Note 1: Whether or not $(*)$ holds, and whether or not $\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n$ converges to $f(x)$, we call the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n$ the *Taylor series of* f about α.

Note 2: Even if the Taylor series converges in some interval $x-\alpha < r$, it may fail to have sum $f(x)$. (See Exercise 6.2 below). Of course the above theorem tells us that it *will* have sum $f(x)$ in case the additional condition $(*)$ holds.

Proof of Theorem 6.3: The condition $(*)$ guarantees that the term $\frac{f^{(m+1)}(c)}{(m+1)!}(x-\alpha)^{m+1}$ on the right in Thm. 6.2 has absolute value $\le C\big(\frac{|x-\alpha|}{r}\big)^{m+1}$ and hence Thm. 6.2, with $m = N$, ensures

$$\Big| f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n \Big| \le C\Big(\frac{|x-\alpha|}{r}\Big)^{N+1} \to 0 \text{ as } N \to \infty$$

if $|x-\alpha| < r$. Hence,

$$\lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n = f(x)$$

whenever $|x-\alpha| < r$, i.e.,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n = f(x), \quad |x-\alpha| < r\ .$$

LECTURE 6 PROBLEMS

6.1 Prove that the function f, defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\ , \end{cases}$$

is C^∞ on $\mathbb{R}$ and satisfies $f^{(m)}(0) = 0\ \forall\, m \ge 0$.

Note: This means the Taylor series of $f(x)$ about 0 is zero; i.e., it is an example where the Taylor series converges, but the sum is not $f(x)$.

6.2 If f is as in 6.1, prove that there does not exist any interval $(-\varepsilon, \varepsilon)$ on which f is represented by a power series; that is, there cannot be a power series $\sum_{n=0}^{\infty} a_n x^n$ such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for all $x \in (-\varepsilon, \varepsilon)$.

6.3 Let b_{nm} be arbitrary real numbers $0 \le n \le N$, $0 \le m \le n$. Prove

$$\sum_{n=0}^{N} \sum_{m=0}^{n} b_{nm} = \sum_{m=0}^{N} \sum_{n=m}^{N} b_{nm}\ .$$

Hint: Define $\tilde{b}_{nm} = \begin{cases} b_{nm} & \text{if } m \le n \\ 0 & \text{if } n < m \le N. \end{cases}$

6.4 Find the Taylor series about $x = 0$ of the following functions; in each case prove that the series converges to the function in the indicated interval.

(i) $\frac{1}{1-x^2}$, $|x| < 1$ (Hint: $\frac{1}{1-y} = 1 + y + y^2 \dots$, $|y| < 1$).

(ii) $\log(1 + x)$, $|x| < 1$

(iii) e^x, $x \in \mathbb{R}$

(iv) e^{x^2}, $x \in \mathbb{R}$ (Hint: set $y = x^2$).

6.5 (The analytic definition of the functions cos x, sin x, and the number π.)

Let $\sin x, \cos x$ be *defined* by $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$. For convenience of notation, write $C(x) = \cos x$, $S(x) = \sin x$. Prove:

(i) The series defining $S(x)$, $C(x)$ both have radius of convergence ∞, and $S'(x) \equiv C(x)$, $C'(x) \equiv -S(x)$ for all $x \in \mathbb{R}$.

(ii) $S^2(x) + C^2(x) \equiv 1$ for all $x \in \mathbb{R}$. Hint: Differentiate and use (i).

(iii) sin, cos (as defined above) are the unique functions S, C on $\mathbb{R}$ with the properties (a) $S(0) = 0$, $C(0) = 1$ and (b) $S'(x) = C(x)$, $C'(x) = -S(x)$ for all $x \in \mathbb{R}$. Hint: Thus, you have to show that $\widetilde{S} = S$ and $\widetilde{C} = C$ assuming that properties (a),(b) hold with $\widetilde{S}$, $\widetilde{C}$ in place of S, C, respectively; show that Thm. 6.3 is applicable.

(iv) $C(2) < 0$ and hence there is a $p \in (0, 2)$ such that $C(p) = 0$, $S(p) = 1$ and $C(x) > 0$ for all $x \in [0, p)$.

Hint: $C(x)$ can be written $1 - \frac{x^2}{2} + \frac{x^4}{24} - \sum_{k=1}^{\infty}\left(\frac{x^{4k+2}}{(4k+2)!} - \frac{x^{4k+4}}{(4k+4)!}\right)$.

(v) S, C are periodic on $\mathbb{R}$ with period $4p$. Hint: Start by defining $\widetilde{C}(x) = S(x + p)$ and $\widetilde{S}(x) = -C(x + p)$ and use the uniqueness result of (iii) above.

Note: The number $2p$, i.e., twice the value of the first positive zero of $\cos x$, is rather important, and we have a special name for it—it usually denoted by π. Calculation shows that $\pi = 3.14159\dots$.

(vi) $\gamma(x) = (C(x), S(x))$, $x \in [0, 2\pi]$ is a C^1 curve, the mapping $\gamma|[0, 2\pi)$ is a 1:1 map of $[0, 2\pi)$ onto the unit circle S^1 of $\mathbb{R}^2$, and the arc-length $S(t)$ of the part of the curve $\gamma|[0, t]$ is t for each $t \in [0, 2\pi]$. (See Figure A.2.)

Remark: Thus, we can geometrically think of the angle between $\underset{\sim}{e}_1$ and $(C(t), S(t))$ as t (that's t radians, meaning the arc on the unit circle going from $\underset{\sim}{e}_1$ to $P = (C(t), S(t))$ (counterclockwise) has length t as you are asked to prove in the above question) and we have the geometric interpretation that $C(t)(= \cos t)$ and $S(t)(= \sin t)$ are, respectively, the lengths of the adjacent and the opposite sides of the right triangle with vertices at $(0, 0)$, $(0, C(t))$, $(C(t), S(t))$ and angle t at the vertex $(0, 0)$, at least if $0 < t < p = \frac{\pi}{2}$. (Notice this is now a theorem concerning $\cos t$, $\sin t$ as distinct from a definition.)

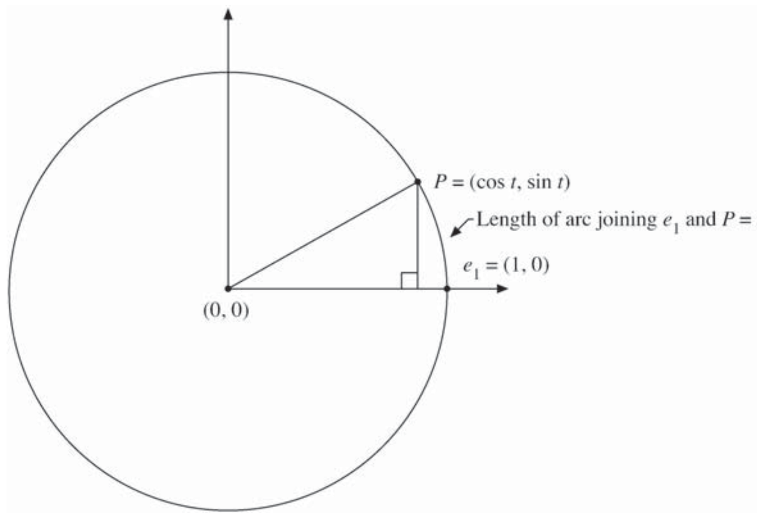

Figure A.2:

Note: Part of the conclusion of (vi) is that the length of the unit circle is 2π. (Again, this becomes a theorem; it is not a definition—π is *defined* to be $2p$, where p, as in (iv), is the first positive zero of the function $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$.)

LECTURE 7: COMPLEX SERIES, PRODUCTS OF SERIES, AND COMPLEX EXPONENTIAL SERIES

In Real Analysis Lecture 4 we discussed series $\sum_{n=1}^{\infty} a_n$ where $a_n \in \mathbb{R}$. One can analogously consider complex series, i.e., the case when $a_n \in \mathbb{C}$. The definition of convergence is exactly the same as in the real case. That is we say the series converges if the n^{th} partial sum (i.e., $\sum_{j=1}^{n} a_j$) converges; more precisely:

Definition: $\sum_{n=1}^{\infty} a_n$ converges if the sequence of partial sums $\{\sum_{j=1}^{n} a_j\}_{n=0,1,\ldots}$ is a convergent sequence in $\mathbb{C}$; that is, there is a complex number $s = u + iv$ (u, v real) such that $\lim_{n\to\infty} \sum_{j=1}^{n} a_j = s$.

Note: Of course here we are using the terminology that a sequence $\{z_n\}_{n=1,2,\ldots} \subset \mathbb{C}$ converges, with limit $a = \alpha + i\beta$, if the real sequence $|z_n - a|$ has limit zero, i.e., $\lim_{n\to\infty} |z_n - a| = 0$. In terms of "$\varepsilon, N$" this is the same as saying that for each $\varepsilon > 0$ there is N such that $|z_n - a| < \varepsilon$ for all $n \geq N$. Writing z_n in terms of its real and imaginary parts, $z_n = u_n + iv_n$, then this is the same as saying $u_n \to \alpha$ and $v_n \to \beta$, so applying this to the sequence of partial sums we see that the complex series $\sum_{n=1}^{\infty} a_n$ with $a_n = \alpha_n + i\beta_n$ is convergent if and only if both of the real series $\sum_{n=1}^{\infty} \alpha_n, \sum_{n=1}^{\infty} \beta_n$ are convergent, and in this case $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \alpha_n + i \sum_{n=1}^{\infty} \beta_n$.

Most of the theorems we proved for real series carry over, with basically the same proofs, to the complex case. For example, if $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ are convergent complex series then the series $\sum_{n=1}^{\infty}(a_n + b_n)$ is convergent and its sum (i.e., $\lim_{n\to\infty} \sum_{j=1}^{n}(a_j + b_j)$) is just $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$.

Also, again analogously to the real case, we say the complex series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent. We claim that just as in the real case absolute convergence implies convergence.

Lemma 1. *The complex series $\sum_{n=1}^{\infty} a_n$ is convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent.*

Proof: Let α_n, β_n denote the real and imaginary parts of a_n, so that $a_n = \alpha_n + i\beta_n$ and $|a_n| = \sqrt{\alpha_n^2 + \beta_n^2} \geq \max\{|\alpha_n|, |\beta_n|\}$, so $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \exists$ a fixed $M > 0$ such that $\sum_{j=1}^{n} |a_j| \leq M \ \forall n \Rightarrow \sum_{j=1}^{n} |\alpha_j| \leq M \ \forall n \Rightarrow \sum_{j=1}^{\infty} |\alpha_n|$ is convergent, so $\sum_{n=1}^{\infty} \alpha_n$ is absolutely convergent, hence convergent. Similarly, $\sum_{n=1}^{\infty} \beta_n$ is convergent. But $\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} \alpha_j + i \sum_{j=1}^{n} \beta_j$ and so $\lim_{n\to\infty} \sum_{j=1}^{n} a_n$ exists and equals $\lim_{n\to\infty} \sum_{j=1}^{n} \alpha_j + i \lim_{n\to\infty} \sum_{j=1}^{n} \beta_j = \sum_{n=1}^{\infty} \alpha_n + i \sum_{n=1}^{\infty} \beta_n$, which completes the proof.

We next want to discuss the important process of multiplying two series: If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are given complex series, we observe that the product of the partial sums, i.e., the product $(\sum_{n=0}^{N} a_n) \cdot (\sum_{n=0}^{N} b_n)$, is just the sum of all the elements in the rectangular array

$$\begin{pmatrix} a_N b_0 & a_N b_1 & a_N b_2 & \cdots & \cdots & \cdots & a_N b_N \\ a_{N-1} b_0 & a_{N-1} b_1 & a_{N-1} b_2 & \cdots & \cdots & \cdots & a_{N-1} b_N \\ \vdots & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ a_2 b_0 & a_2 b_1 & a_2 b_2 & \cdots & \cdots & \cdots & a_2 b_N \\ a_1 b_0 & a_1 b_1 & a_1 b_2 & \cdots & \cdots & \cdots & a_1 b_N \\ a_0 b_0 & a_0 b_1 & a_0 b_2 & \cdots & \cdots & \cdots & a_0 b_N \end{pmatrix}$$

$$\Big(\sum_{n=0}^{N} a_n\Big)\Big(\sum_{n=0}^{N} b_n\Big) = \sum_{n=0}^{2N}\Big(\sum_{0\le i,j\le N,\, i+j=n} a_i b_j\Big),$$

and observe that if $i, j \geq 0$ and $i + j \leq N$ we automatically have $i, j \leq N$, and so with $c_n = \sum_{i,j\geq 0,\, i+j=n} a_i b_j$ $(= \sum_{i=0}^{n} a_i b_{n-i})$ we see that the right side of the above identity can be written $\sum_{n=0}^{N} c_n + \sum_{0\leq i,j\leq N,\, i+j>N} a_i b_j$, and so we have the identity

$$(*) \qquad \Big(\sum_{n=0}^{N} a_n\Big)\Big(\sum_{n=0}^{N} b_n\Big) - \sum_{n=0}^{N} c_n = \sum_{0\le i,j\le N,\, i+j>N} a_i b_j$$

for each $N = 0, 1, \ldots$. (Geometrically, $\sum_{n=0}^{N} c_n$ is the sum of the lower triangular elements of the array, including the leading diagonal, and the term on the right of $(*)$ is the sum of the remaining, upper triangular, elements.) If the given series $\sum a_n, \sum b_n$ are absolutely convergent, we show below that the right side of $(*) \to 0$ as $N \to \infty$, so that $\sum_{n=0}^{\infty} c_n$ converges and has sum equal to $(\sum_{n=0}^{\infty} a_n)\,(\sum_{n=0}^{\infty} b_n)$. That is:

Lemma (Product Theorem.) *If $\sum a_n$ and $\sum b_n$ are absolutely convergent complex series, then $(\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n) = \sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ for each $n = 0, 1, 2, \ldots$; furthermore, the series $\sum c_n$ is absolutely convergent.*

Proof: By $(*)$ we have

$$\begin{aligned} \Big|\Big(\sum_{i=0}^{N} a_i\Big)\Big(\sum_{j=0}^{N} b_j\Big) - \sum_{n=0}^{N} c_n\Big| &= \Big|\sum_{i,j\le N,\, i+j>N} a_i b_j\Big| \le \sum_{i,j\le N,\, i+j>N} |a_i||b_j| \\ &\le \sum_{i,j\le N,\, i>N/2} |a_i||b_j| + \sum_{i,j\le N,\, j>N/2} |a_i||b_j| \\ &= \Big(\sum_{i=[N/2]+1}^{N} |a_i|\Big)\Big(\sum_{j=0}^{N} |b_j|\Big)\Big(\sum_{i=0}^{N} |a_i|\Big)\Big(\sum_{j=[N/2]+1}^{N} |b_j|\Big) \end{aligned}$$

$$\le \Big(\sum_{i=[N/2]+1}^{\infty} |a_i|\Big)\Big(\sum_{j=0}^{\infty} |b_j|\Big) + \Big(\sum_{i=0}^{\infty} |a_i|\Big)\Big(\sum_{j=[N/2]+1}^{\infty} |b_j|\Big)$$
$$\to 0 \text{ as } N \to \infty\,,$$

where $[N/2] = \frac{N}{2}$ if N is even and $[N/2] = \frac{N-1}{2}$ if N is odd. Notice that in the last line we used the fact that $\sum_{i=[N/2]+1}^{\infty} |a_i| = \sum_{i=0}^{\infty} |a_i| - \sum_{i=0}^{[N/2]} |a_i| \to 0$ as $N \to \infty$ because by definition $\sum_{n=0}^{\infty} |a_n| = \lim_{J\to\infty} \sum_{n=0}^{J} |a_n|$, and similarly, $\sum_{j=[N/2]+1}^{\infty} |b_j| = \sum_{j=0}^{\infty} |b_j| - \sum_{j=0}^{[N/2]} |b_j| \to 0$, because $\sum_{n=0}^{\infty} |b_n| = \lim_{J\to\infty} \sum_{n=0}^{J} |b_n|$.

This completes the proof that $\sum_{n=0}^{\infty} c_n$ converges, and $\sum_{n=0}^{\infty} c_n = (\sum_{n=0}^{\infty} a_n)\,(\sum_{n=0}^{\infty} b_n)$.

To prove that $\sum_{n=0}^{\infty} |c_n|$ converges, just note that for each $n \ge 0$ we have $|c_n| \equiv \left|\sum_{i+j=n} a_i b_j\right| \le \sum_{i+j=n} |a_i||b_j| \equiv C_n$ say, and the above argument, with $|a_i|$, $|b_j|$ in place of a_i, b_j, respectively, shows that $\sum_{n=0}^{\infty} C_n$ converges, so by the comparison test $\sum_{n=0}^{\infty} |c_n|$ also converges.

We now define the complex exponential series.

Definition: The complex exponential function $\exp z$ (also denoted e^z) is defined by $\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

Observe that this makes sense for all z because the series $\sum_{n=0}^{\infty} \frac{|z|^n}{n!}$ is the real exponential series, which we know is convergent on all of $\mathbb{R}$, so that the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is absolutely convergent (hence convergent by Lem. 1) for all z.

We can use the above product theorem to check the following facts, which explains why the notation e^z is sometimes used instead of $\exp z$:

(i) $$(\exp a)(\exp\ (b) = \exp(a+b), \qquad a, b \in \mathbb{C},$$
(ii) $$\exp ix = \cos x + i \sin x, \qquad x \in \mathbb{R}\,.$$

The proof is left as an exercise (Exercises 7.1 and 7.2 below).

Notice that it follows from (i) that $\exp z$ is never zero (because by (i) $(\exp z)\,(\exp -z) = \exp 0 = 1 \neq 0\ \forall z \in \mathbb{C}$).

LECTURE 7 PROBLEMS

7.1 Use the product theorem to show that $\exp a\ \exp b = \exp(a+b)$ for all $a, b \in \mathbb{C}$.

7.2 Justify the formula $\exp ix = \cos x + i \sin x$ for all $x \in \mathbb{R}$.

Note: cos, sin are defined by $\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$ and $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ for all real x.