Introduction

Mathematics is a unique discipline in that it deals with the intriguing concept of infinity. Fascinating as this may be to mathematicians, however, it may seem at first sight to have little relevance in the world of an average high school math student, whose tasks consist of proving formulas or finding solutions to problems. In fact, infinity plays a vital (if invidious) role in mathematics. More to the point, it is featured quite prominently in high school calculus. Calculus is the branch of mathematics which is based on real numbers and limits, both of which are intimately related to the notion of infinity. Therefore, one might say that calculus essentially is the study of infinity (even though when we try to find, say, the extrema of a cubic function using differentiation, we likely are not conscious of the fact that we are dealing with infinity). This lack of awareness has much to do with the fact that mathematics has been modified and has evolved over time into a tool that anyone can access. We now employ mathematics in virtually every aspect of science and technology. Paradoxically, it arguably is because of this omnipresence and "ease of access," as it were, that we may fail to realize that the concept of infinity is a foundation of the very "tools" with which we are so familiar.

In this chapter, we will discuss the history of a branch of mathematics which was developed by identifying the significance of a circle, a figure familiar to everyone. This branch of mathematics was developed around the 17th century, when calculus was still in its infancy.

When we see a full moon in the night sky, we cannot help but wondering at this mystical figure. Since ancient times, it has been suggested that part of that heavenly body's allure is its perfect shape (a disk). This captivating nature of the disk manifests itself *mathematically* as the ratio π of the length of a circle to its diameter. The length of half of the circle of radius one, is π. The peculiar allure of the circle is evidenced further by the centuries-long attempt to measure the exact length of the half-circle. Gradually, the value for π was approximated as 3.1, 3.14, 3.141, 3.1415, etc., as the values for the digits following the decimal point were further calculated. Eventually, people began to think that the journey to seek more values for the digits following the decimal point might, in fact, be endless. As the researchers continued to pursue their quest for the exact value of π, they eventually began to realize that the "end" of the journey was in the realm of infinity.

The seemingly endless sequence of numbers emanating from the "mystical" circle gave mankind the impetus to explore the concept of infinity.

Astronomers have long shown intense interest in the relationship between an arc and a chord of a circle. This relationship is expressed (in the study of functions) as the relationship between x and y in the sine function $y = \sin x$. The bridge between finite and infinite found in the relationship between the length of a chord and the length of an arc, in which the number π appears, is translated into the sine function $y = \sin x$. To study the properties of the sine function $y = \sin x$ in detail, it is necessary to introduce the concept of infinity. In turn, the concept of infinity gave rise to the concept of power expansion. Historically, mathematicians used the inverse sine function $y = \sin^{-1} x$, instead of the sine function $y = \sin x$, to study geometrically the relationship between a chord and an arc of a circle. Since the inverse sine function $y = \sin^{-1} x$ was easier to work with than the sine function $y = \sin x$, the infinite series for $y = \sin^{-1} x$ was obtained first. The successful use of the power series representation of $y = \sin^{-1} x$ was due to the effective application of **Newton's Generalized Binomial Theorem**. As you can see, trigonometric functions provided fertile soil to nourish seeds of mathematics.

Keep in mind that the functions we have just looked at deal with "perfect" circles. There also exist functions, developed by Gauss, Abel, and their contemporaries at the beginning of the 19th century, which deal with ellipses. These combined developments changed the mathematical landscape. With that background let us begin our discussion of trigonometric and elliptic functions.

LECTURE 1

Trigonometric Functions and Infinite Series

1.1. The Birth of Sine (sin), a Trigonometric Ratio

The phrase "Trigonometric Functions" may bring up images of right triangles (Figure 1.1) and trigonometric ratios such as sin, cos, and tan. Indeed, you likely learned about sin, cos, and tan as trigonometric ratios; later, you used them to study the relations between the angles and the sides of a triangle. Also, you probably studied certain formulas derived from addition theorems by regarding sin, cos, and tan as trigonometric functions. While you were studying trigonometric ratios or functions, you probably did not encounter anything spectacular. Despite that, it is not an exaggeration to say that trigonometric functions are a treasure chest of mathematics. Let us crack open the lid of this chest and take a peek at the glittering treasure inside.

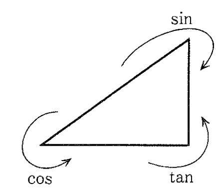

FIGURE 1.1

In the history of trigonometric functions **sine** (the trigonometric ratio) was the first to appear. This occurred about 2200 years ago. You might be surprised to know that the geometric figure which gave birth to sine was not the triangle but the circle. Hipparchus (approximately 190–125 B. C.), regarded by many as the father of Trigonometry, constructed a "table of chords" of a circle while he was attempting to verify numerically the results of many years of astronomical observations.

The "table of chords" of a circle is a table of the lengths of chords AA' corresponding to central angles α (see Figure 1.2). Since it is difficult to manually obtain the exact measurement of a chord, the length was approximated.

Hipparchus considered a circle with a diameter of 6875 units. From this circle, he obtained the lengths of chords corresponding to central angles

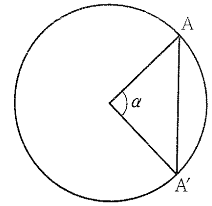

FIGURE 1.2

which are the integer multiples of 7.5 degrees. He then constructed the table of chords. Hipparchus' table of chords later was refined for additional accuracy and was a key element in the study of astronomical theory. Around 510 A. D., the Indian astronomer Aryabhata constructed a table consisting of half-chords. A table of half-chords is a table of values that represent the lengths of half-chords AB corresponding to the central angles α (see Figure 1.3).

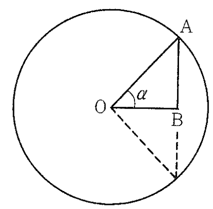

FIGURE 1.3

The "table of half-chords" is essentially the table of sines. Later we will explore the significance of the half-chord table, and discuss the relations between the degree measure and the radian measure of an angle.

In order to construct a table of half-chords, it is sufficient to use a single circle of a fixed radius. For example, if we construct a table of half-chords from a circle of radius 6 units, then according to the principle of similarity, the lengths of the corresponding half-chords of a circle of radius 12 units are twice those of a circle of radius 6. The lengths of the corresponding half-chords of a circle of radius 2 units are 1/3 unit. To simplify the usage and for standardization, let us construct a table of half-chords from a circle of radius 1 unit (the unit circle).

Next, we must determine how to measure the half of the central angle corresponding to a chord of a unit circle. If we refer to Figure 1.3, the measure of angle α corresponds to half-chord AB. The reader likely will recall measuring angles by the units of degrees with a protractor in grammar school. However, from the point of view of higher mathematics, the degree is

not an approriate unit for measuring angles. One reason is that 180 degrees is the measure of the angle corresponding to half of a circumference. For example, if we divide a half-circumference into n arcs, then the corresponding central angle for each arc is $\frac{180}{n}$ degrees. The number 180 is an inconvenient obstacle in any calculation. Also, the number 180 is an arbitrary number and, speaking in mathematical terms, has no relation to the circumference of a circle (the number 180 is related to the sexagesimal system used by ancient Mesopotamians some 3000 years ago). As a new unit of angle measure, we should change 180° on the protractor to a half-circumference π of the unit circle and convert degree measure by multiplying by $\frac{\pi}{180°}$ (see Figure 1.4). According to the conversion, 90° is $\frac{\pi}{2}$ and 60° is $\frac{\pi}{3}$. This improved method of measuring angles is called radian measure.

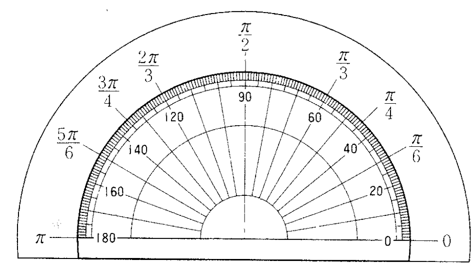

FIGURE 1.4

From now on, when we mention a circle, we will mean the unit circle (a circle of radius 1 unit). The measurement of angle α will be in radian measure. Therefore, if the central angle of the corresponding arc of the unit circle is α, then the length of the arc itself is also α (see Figure 1.5).

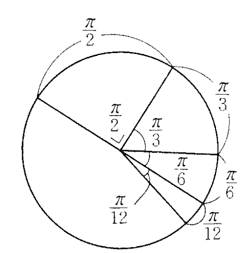

FIGURE 1.5

With this conversion of degree measure into radian measure, we now can consider the table of half-chords to be a table used to obtain the value y

(the length of a half-chord) when x is given (see Figure 1.6). This relation between y and x is the familiar sine function:

$$y = \sin x.$$

Therefore, a table of half-chords is the table of values of sine function.

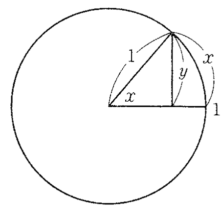

FIGURE 1.6

♣ Considering the relation between y and x in Figure 1.5, it is more suitable to call the trigonometric function $y = \sin x$ a "circular function". In fact, in recent years, the term "circular function" has sometimes replaced the term "trigonometric function" in mathematics textbooks. The definition of sine, cosine, and tangent as trigonometric ratios related to right triangles probably occurred in the 14th or 15th century, when Euclidean geometry was the central theme of mathematics.

1.2. The Trigonometric Function: $\sin x$

The trigonometric function $y = \sin x$ is the central theme of this section and we will restrict the angle x to the range $0 \le x < \frac{\pi}{2}$. In the expression of the function $y = \sin x$, x is a variable (precisely speaking, an independent variable). When x varies, the value of y varies accordingly. In that sense, y is a dependent variable. When the independent variable x is determined, then the dependent variable y is determined. From now on, we will regard the formula $y = \sin x$ as a functional relation between x and y, and we will study the properties of this relation.

The relation between x and y in the function $y = \sin x$ is given by the relation of central angles x, or the lengths of the corresponding arcs x, and the lengths of the corresponding half-chords y of the unit circle (see Figure 1.7, left). From this relation the additive formula

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

for sine is derived immediately using Figure 1.7.

CHALLENGE 1. Derive the additive formula for cos using similar arguments.

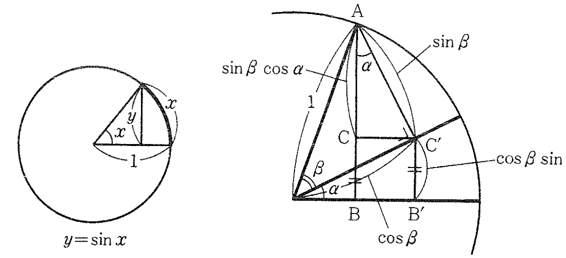

FIGURE 1.7

As you may have guessed by now, the main focus of the function sin is in the relation between the arc length x (measured in radians) and half-chord length y. However, the arc of a circle is curved and a half-chord is straight. We cannot measure a curved line with a straight edge ruler, nor can we measure a straight line with a curved measurement device. Curved lines and straight fines are like oil and water; they don't mix. For example, if we want to measure the length of an arc of a circle, we need to divide the arc into still finer arcs and measure the length of each chord of these finer arcs, then sum up the total measurements to obtain an approximation of the length of an arc of a circle (see Figure 1.8). Since ancient times, people tried to evaluate the half-circumference π using this approach.

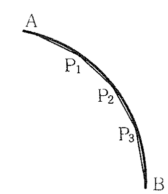

FIGURE 1.8

However, the half-circumference cannot be measured exactly. As you know, π is expressed as an infinite decimal fraction

$$\pi = 3.14159265358979323846\ldots.$$

The fact that π is an infinite decimal indicates that the half-circumference can never be measured by a finite number of operations. This shows that if we conceptualize a circle in terms of numbers, it will appear as "infinity".

♦ π is called an irrational number, i.e., a number that cannot be expressed as a fraction. Johann Heinrich Lambert (1728–1777) and Adrien Marie Legendre (1752–1833) proved this at

> the end of the 18th century. Therefore, if π is given as an infinite decimal fraction, it cannot be periodic.

The function $y = \sin x$ shows the relation between the arc lengths and the half-chord lengths in a circle. When we study this relation, we will encounter infinity. In fact, the importance of the function $y = \sin x$ is that the concept of infinity is hidden within the function. In studying $\sin x$ in detail, we will catch a glimpse of the shining source of mathematical creativity. This is why we earlier compared the collection of trigonometric functions to a treasure chest.

So far, we have been talking about the trigonometric function $y = \sin x$, with y being a dependent variable and x being an independent variable. Observe the two figures in Figure 1.9. Both of them show the relation between the arc length and half-chord length of a circle. Without thinking too deeply, which would you choose so that x will be an independent variable. In the lectures, a little less than 1/3 of the students who attended picked the left figure. The rest chose the figure on the right (although some of those students didn't really know why they chose thusly!).

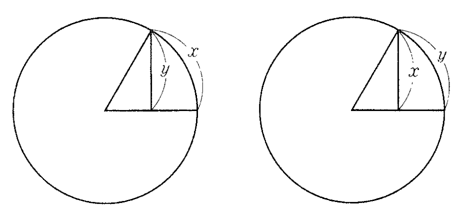

FIGURE 1.9

Whether by design or chance, the majority of the students have chosen the correct figure. When we study the relation between two variables, we normally choose x as the independent variable to represent a quantity which can be measured easily, and we choose y as the dependent variable to represent a quantity which is rather difficult to measure. For example, if we look at a train operation table, we will find that travel time, which is easy to measure, is given as the independent variable, and we will also find distance, which is not as easy to measure directly, as the dependent variable. However, if we consider an angular rotation or an equal division of an arc of a circle, then probably the figure on the left of Figure 1.9 would be a better choice. When we observe the figure on the right of Figure 1.9 in a geometric sense, the first thing we notice is the relation of lengths between the arc and the half-chord of the figure. It seems natural to choose the figure on the right, where x represents the length of a half-chord of a circle. Once we choose x to represent the length of a half-chord, it becomes possible to study geometrically the relation between x and y by pictorial representation. Doing just that, mathematicians in the 17th century interchanged the dependent and independent variables of the function $y = \sin x$, and a new avenue of research began.

The figure on the left of Figure 1.9 shows the relation between x and y as the familiar function

$$y = \sin x.$$

The figure on the right shows the relation between x and y as the function

$$y = \sin^{-1} x.$$

The function $y = \sin^{-1} x$ is called the **inverse sine function**, or the **arc sine function**. The relation between x and y is shown in Figure 1.10 as both a sine function and an inverse sine function.

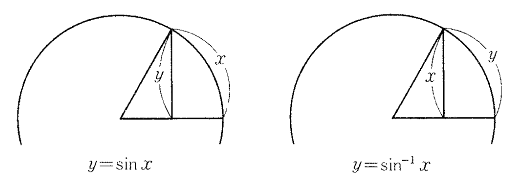

FIGURE 1.10

1.3. $\sin^{-1} x$ and $\tan^{-1} x$

Now we will get to the heart of the quest for infinity (hidden in trigonometric functions) by studying the function $y = \sin^{-1} x$. On the way, we will discuss the function $y = \tan^{-1} x$ (the inverse tangent function). The path to infinity via $y = \tan^{-1} x$ presents different scenery than that via $y = \sin^{-1} x$. The function $y = \tan^{-1} x$ is the inverse function, which is obtained by reversing the relation between dependent and independent variables in the function $y = \tan x$. Figure 1.11 might help to explain the relation between dependent and independent variables of the sine and inverse sine functions.

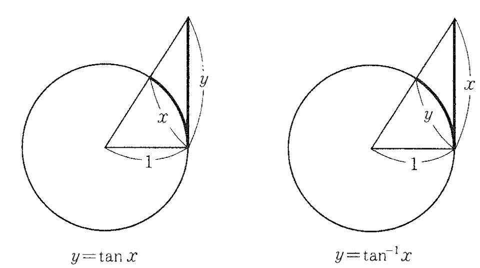

FIGURE 1.11

Let us now observe the effect on the functions $y = \sin^{-1} x$ and $y = \tan^{-1} x$ when we make small changes to the value of x. Let us change the independent variable x by a small amount and then observe the corresponding change in the dependent variable y. The change of x is denoted by Δx, and is called an **increment of** x. The corresponding change of y is denoted by Δy. Our goal is to approximate Δy in terms of Δx, that is, to find an approximate relation between Δx and Δy. We can best understand this relation by studying the geometric representation for Δx and Δy corresponding to the functions $y = \sin^{-1} x$ and $y = \tan^{-1} x$ (see Figure 1.12).

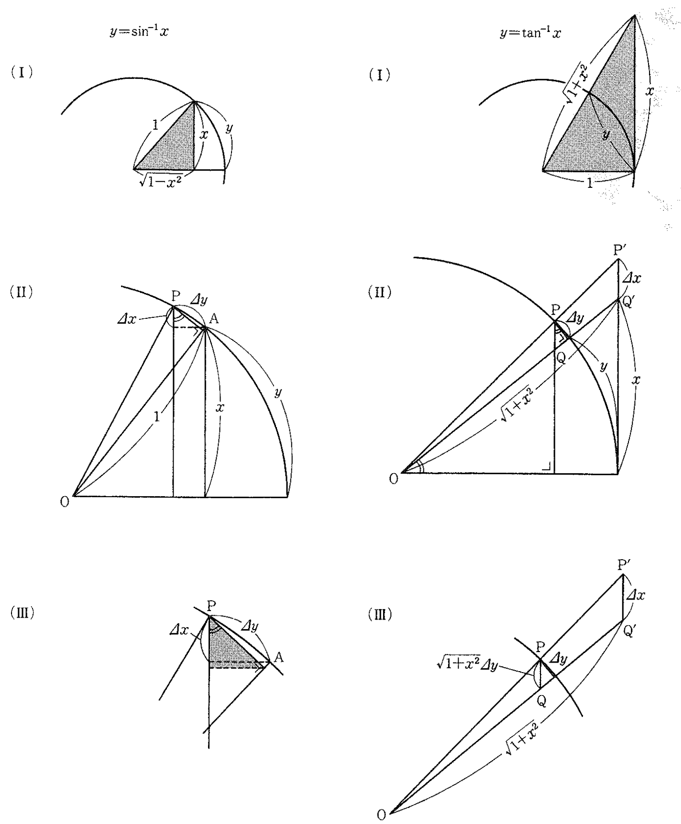

FIGURE 1.12

In Figure 1.12, the left column refers to the function $\sin^{-1}(x)$, and the right column refers to the function $\tan^{-1}(x)$. For the function $y = \sin^{-1}(x)$, relations in picture (I) hold by the Pythagorean theorem. In picture (II), the increment Δx and the corresponding increment Δy are shown. The increase $\widehat{AP}$ is shown in picture (III). In the shaded triangle the hypotenuse is $\approx \Delta y$ and the base is $\approx \Delta x$. This triangle is similar to the shaded triangle in picture (I). Therefore $\frac{\Delta y}{\Delta x} \approx \frac{1}{\sqrt{1-x^2}}$, hence $\Delta y \approx \frac{1}{\sqrt{1-x^2}}$.

For the function $y = \tan^{-1} x$ relations in picture (I) hold by the Pithagorean theorem. In picture (II), the increment Δx and the corresponding increase Δy are shown. The small shaded triangle is similar to the shaded triangle in picture (I). Hence, $PQ \approx \sqrt{1+x^2}\,\Delta y$. In picture (III), the triangle OPQ is similar to the triangle $OP'Q'$. For small Δx, $OQ \approx 1$. Hence, $\frac{\sqrt{1+x^2}\,\Delta y}{1} \approx \frac{\Delta x}{\sqrt{1+x^2}}$. Therefore, $\Delta y \approx \frac{1}{1+x^2}\Delta x$.

Hence, the relation between Δx and Δy is given as follows:

$$\text{If the function is } y = \sin^{-1} x, \text{ then } \Delta y \approx \frac{1}{\sqrt{1-x^2}}\Delta x.$$

$$\text{If the function is } y = \tan^{-1} x, \text{ then } \Delta y \approx \frac{1}{1+x^2}\Delta x.$$

The two approximate expressions look differently due to the difference in the domains in which the independent variable x varies. In these expressions, as the increment Δx of the independent variable tends to zero, the approximation of the increment Δy of the dependent variable approaches the precise value. For both functions related to Figure 1.12 we obtain an approximate expression by estimating the length of a very small arc of a circle in terms of the line segment PR in Figure 1.13.

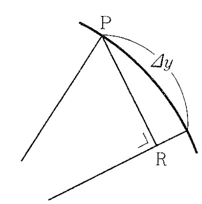

FIGURE 1.13

1.4. Integral Representations and Power Series Representations of Functions

If the independent variable x undergoes a small change Δx, then the corresponding change Δy in the dependent variable y is approximated as follows:

(1) if the function is $y = \sin^{-1} x$, then $\Delta y \approx \dfrac{1}{\sqrt{1-x^2}}\,\Delta x$;

(2) if the function is $y = \tan^{-1} x$, then $\Delta y \approx \dfrac{1}{1+x^2}\,\Delta x$.

(For the following paragraph, refer to Figure 1.14.)

Consider an arc with fixed length y. Taking n to be any positive integer, let us divide the arc equally into n smaller arcs by choosing points P_0, P_1, P_2, …, P_n, where $P_0 = E$ and P_n are the endpoints of the arc and P_k are the dividing points for $k = 1, 2, \ldots, n-1$. We then estimate the length of each arc $\widehat{P_kP_{k+1}}$ by using the above approximate expressions. Let R_1, R_2, …, R_n be the intersection points of the lines passing through P_1, P_2, …, P_n and perpendicular to the radii OE, OP_1, …, OP_{n-1}, respectively. Then the length of each perpendicular segment P_kR_k is approximated by Δx (see Figure 1.14). Hence the length of the arc $\widehat{EP_n}$ is approximated as follows:

(3) $$\begin{aligned} y &= (\text{length of } \widehat{EP_1}) + (\text{length of } \widehat{P_1P_2}) + \cdots + (\text{length of } \widehat{P_1P_2}) \\ &\approx (\text{length of } P_1R_1) + (\text{length of } P_2R_2) + \cdots + (\text{length of } P_nR_n). \end{aligned}$$

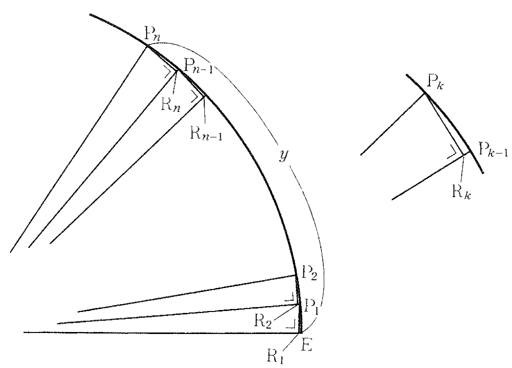

FIGURE 1.14

We have arrived at the approximate expression (3) by studying the preceding geometric figures. But to refine this expression, we must divide the arc into smaller and smaller arcs. Then the total length of the perpendicular segments gets closer and closer to the exact length of the arc. To replace the approximation with an equality, we must briefly depart the finite world by letting the integer n approach infinity, thus crossing over into the world of "infinity". As mentioned previously, to clarify the relation between x and y, we must walk the path to infinity. As the integer n tends to infinity,

which is written as $n \to \infty$, the limit of the total length of the perpendicular segments is given by the definite integral as follows:

(4) $$y = \int_0^x \frac{dt}{\sqrt{1-t^2}} \quad \text{(obtained by using expression (1) above),}$$

(5) $$y = \int_0^x \frac{dt}{1+t^2} \quad \text{(obtained by using expression (2) above).}$$

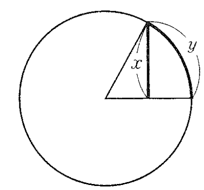

FIGURE 1.15. The relation between x and y in equation (4).

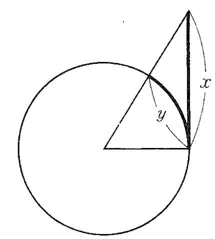

FIGURE 1.16. The relation between x and y in equation (5).

♣ For those of you who have not studied integral calculus, let us briefly discuss integrals. The notion of integral originated in the search for the areas of geometric figures. If a nonnegative function $y = f(x)$ is defined for real numbers between a and b, then we obtain the area under the graph of $y = f(x)$ from a to b using the definite integral. The area of the region bounded above by the graph of $y = f(x)$, below by the x-axis, and by vertical lines $x = a$ and $x = b$ is called the definite integral of $y = f(x)$ from a to b. It is denoted by

(*) $$\int_a^b f(x)\,dx.$$

The numbers a and b are called the **limits of integration**.

To obtain the definite integral of $y = f(x)$ from a to b (the area under the graph of $y = f(x)$), let us divide the interval $[a, b]$ into n equal subintervals by choosing numbers $x_0, x_1, \dots, x_n$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and $x_k - x_{k-1} = \frac{b-a}{n}$. If we denote $\frac{b-a}{n}$ by Δx, then the area under the graph of $y = f(x)$ from a to b is approximated by the sum of the areas of the n rectangles (shaded figures in Figure 1.17) as follows:

$$(**) \qquad f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x.$$

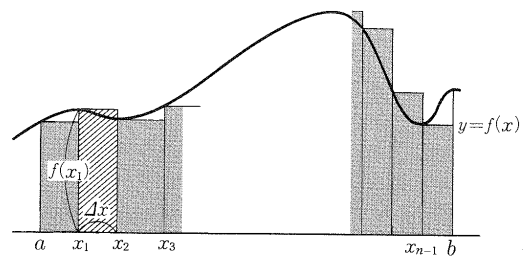

FIGURE 1.17. The area of the distinguished rectangle equals $f(x_1)\Delta x$.

If we let Δx approach zero ($\Delta x \to 0$) by further dividing the interval, then the sum $(**)$ approaches the area of the region below the graph of $y = f(x)$ from a to b. The definite integral $\int_a^b f(x)\,dx$ is, in a way, defined as the limit of the sum $(**)$.

Once we established the notion of the integral using this limit process, we can apply the limit process to any sum of the form $(**)$, even if the sum does not represent the approximation of the area. In fact, the definite integral of a function $y = f(x)$ from a to b is defined precisely as the limit of the so-called Riemann sums of the function. The area under the graph of $y = f(x)$ from a to b is given by the definite integral $(*)$ according to this precise definition.

In expressions (4) and (5), the limits of integration are zero and x. The expressions $\frac{1}{\sqrt{1-t^2}}$ and $\frac{1}{1+t^2}$ under the integral signs (called the integrands) came from $\frac{1}{\sqrt{1-x^2}}$ and $\frac{1}{1+x^2}$, respectively, by replacing x with t.

The expressions (4) and (5) can be rewritten as follows:

$$(6) \qquad y = \sin^{-1} x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt,$$

$$(7) \qquad y = \tan^{-1} x = \int_0^x \frac{1}{1+t^2}\,dt.$$

We will now discuss the integral representations of the above two functions. The expressions (4) and (5) show that the "mother" of $\sin^{-1} x$ is $\frac{1}{\sqrt{1-t^2}}$ and that of $\tan^{-1} x$ is $\frac{1}{1+t^2}$. Referring to Figure 1.12, we see that the expression $\frac{1}{\sqrt{1-t^2}}$ is derived directly from the Pythagorean theorem. But the expression $\frac{1}{1+t^2}$ is derived rather indirectly because the relation between x and y in the function $y = \tan^{-1} x$ with respect to a unit circle is indirect, resulting in a form other than the Pythagorean theorem. But if we restrict ourselves to observing the end results of our geometric proofs, the equation (7) has a much simpler form than that of equation (6).

However, in the case of equation (6), the arc length y and the half-chord length x must be related to each other via an infinite expression, as the formula $\pi = 3.141592\ldots$ shows, and that infinite expression must somehow be incorporated in the integral notation on the right-hand side of equation (6). (At this point, the reader may feel that although the entrance door to the world of infinity is opening, yet the interior doors seem to remain tightly shut!)

We will see how x and y are related to each other by removing the integral signs from equations (6) and (7). At first glance, it appears that it will be easier to start with equation (7). The integrand is given by the infinite geometric series[1] as follows:

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 - \cdots \qquad (|t| < 1).$$

This equation shows that if the degree n of a polynomial increases to infinity, the resulting infinite series represents the fractional expression. Fortunately, since the expression $\frac{1}{1+t^2}$ is under the integral sign of the definite integral (7), the integration is rather simple. By utilizing the integral formula

$$\int_0^x t^n\,dt = \frac{1}{n+1}x^{n+1},$$

---

[1]The partial sum S_n of the geometric series is given by

$$(\#) \qquad a + ar + \cdots + ar^{n-1} = \frac{a(1-r^2)}{1-r} \quad \text{for } r \neq 1.$$

We obtain this formula by comparing S_n with the left-hand side of the equation and subtracting rS_n from S_n. We obtain $S_n - rS_n = (1-r)S_n = a - a^n$. Therefore $S_n = \frac{a(1-r^2)}{1-r}$. The sum of the infinite geometric series is given by $a + ar + \cdots + ar^n + \cdots = \frac{a}{1-r}$. We obtain this formula by applying the limit process to S_n. To accomplish this, we first rewrite formula (#) as $a + ar + \cdots + ar^{n-1} = \frac{a}{1-r} - \frac{ar^2}{1-r}$. The limit of S_n as n tends to infinity is equal to the limit of $\left(\frac{a}{1-r} - \frac{ar^2}{1-r}\right)$. If $|r| < 1$, then the limit of r^n as n tends to infinity is zero ($r^n \to 0$ as $n \to \infty$). Therefore, $a + ar + \cdots + ar^n + \cdots = \frac{a}{1-r}$.

and by using the termwise integration, we have

$$\begin{aligned} y = \tan^{-1} x &= \int_0^x \frac{1}{1+t^2}\,dt \\ &= \int_0^x (1 - t^2 + t^4 - t^6 + t^8 - \cdots)\,dt \\ &= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \cdots \quad (|x| < 1). \end{aligned}$$

When the "veil" of the integral sign is removed, as it were, from the definite integral of the function $y = \tan^{-1} x$, the relation between x and y is revealed clearly in the infinite series representation of y. This fact is stated as follows:

(8) $$y = \tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \cdots .$$

The right-hand side of the above equation is called the **Power Series Expansion** of the function $y = \tan^{-1} x$. Let us approximate $y = \tan^{-1} x$ by using the first five terms of the power series expansion for $x = 0.8$. The computation gives y as approximately 0.6798. Since $\tan^{-1} 0.8 = 0.67474\ldots$, the error is less than $\frac{5}{1000}$ (see Figure 1.18).

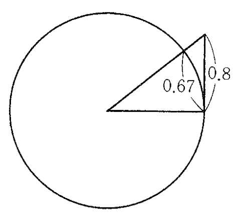

FIGURE 1.18

It has been shown that the right-hand side of (8) will converge even if $x = 1$, and we find the sum, in this case, to be $\frac{\pi}{4}$. If $x = 1$, this power series becomes the famous **Leibniz Series** for $\frac{\pi}{4}$, namely

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

The power series expansion for the function $y = \tan^{-1} x$ was known to mathematicians in India as early as the 15th century. In Europe, this series remained unknown till the mid-17th century, when it was discovered by Nicolaus Mercator (1620–1687) and John Wallis (1616–1703). In contrast to the function $y = \tan^{-1} x$, it is not easy to represent the function $y = \sin^{-1} x$

by a power series. The reason that it is not easy is that the integrand

(9) $$\frac{1}{\sqrt{1-t^2}}$$

was not well understood. So let us now study this expression. The first thing we note is that it cannot be changed into any other form (e.g., infinite series).

Recall that when we discussed the function $y = \tan^{-1} x$, we found that the sum of the infinite geometric series played a decisive role in deriving expression (8). Therefore, for the function $y = \sin^{-1} x$ there must also be some important clue hidden in expression (9), which will yield the infinite power series. This is indeed the case, as Newton was first to discover.

When Newton was in his early twenties (in 1664 or 1665), he studied the **General Binomial Theorem** and obtained the following formula:

If α is a positive or negative rational number, then

$$\begin{aligned} (1+x)^\alpha = 1 &+ \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 \\ &+ \cdots + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}x^k + \cdots \quad \text{for } |x| < 1. \end{aligned}$$

The notation $k!$ stands for k factorial.[2]

CHALLENGE 2. Show that

$$\begin{aligned} (1+x)^2 &= 1 + 2x + x^2, \\ (1+x)^3 &= 1 + 3x + 3x^2 + x^3. \end{aligned}$$

CHALLENGE 3. Using the binomial theorem, find the power series expansion for $(1+x)^6$.

[2] The value obtained by multiplying all the natural numbers from 1 to n is denoted by $n!$. Some examples are shown below:

$$\begin{aligned} 1! &= 1, \\ 2! &= 2 \cdot 1 = 2, \\ 3! &= 3 \cdot 2 \cdot 1 = 6, \\ 4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24, \\ 5! &= 120, \\ 6! &= 720, \\ 7! &= 5040, \\ 8! &= 40320, \\ 9! &= 362880, \\ 10! &= 3628800. \end{aligned}$$

CHALLENGE 4. Show that in order that the power series expansion for $(1+x)^\alpha$ be a polynomial (a finite series), α must be either a natural number or zero.

CHALLENGE 5. If $\alpha = -1$, show that $(1+x)^\alpha$ is equal to the sum of a geometric series.

CHALLENGE 6. If $\alpha = \frac{1}{2}$, approximate $\sqrt{1+x}$ by finding the first five terms of the infinite power series expansion.[3]

According to the general binomial theorem, if $\alpha = -\frac{1}{2}$ and $x = -t^2$, then

$$\frac{1}{\sqrt{1-t^2}} = (1-t^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}t^6 + \cdots.$$

Thus, it is clear that if we remove the integral notation from expression (6), then the reiationship between x and y is as follows (with y represented by the infinite series):

$$\begin{aligned} y = \sin^{-1} x &= \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt \\ &= \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{1\cdot 3}{2\cdot 4}t^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6} + \cdots\right) dt \\ &= x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots. \end{aligned}$$

The above fact is restated as

$$\text{(10) if } |x| < 1, \text{ then } y = \sin^{-1} x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots.$$

The right-hand side of this equation is called the infinite power series expansion of $y = \sin^{-1} x$. Pay attention to the coefficients of the terms of the series. For example, the coefficient of x^9, following x^7, is given by

$$\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{x^9}{9}.$$

As you can see, even numbers and odd numbers appear in an orderly sequence.

We tend to think that a circle has nothing to do with the world of numbers. Returning to our earlier analogy, this thought is reinforced when we look at the full moon. Certainty, we do not think of numbers when we gaze upon the moon's mysterious beauty. But, in fact, as in expression (10), even numbers and odd numbers appear in the relation between the arc length and the half-chord length of a circle. This fact is intriguing. No one knows

[3]The definition of $a^{\frac{n}{m}}$ and a^{-n} can be given as follows: if both m and n are natural numbers, then $a^{\frac{m}{n}} = \sqrt[m]{a^n}$ for $a > 0$. If n is a natural number, then $a^{-n} = \frac{1}{a^n}$ for $a > 0$.

why such a harmonious relation exists between a circle and the world of numbers. Mathematicians know only that equation (10) is valid, but they cannot explain the reason behind it.

1.5. Reversing Infinite Power Series

So far, by working through the process of deriving infinite series from a circle, we have seen the power series expansions of the functions $y = \tan^{-1} x$ and $y = \sin^{-1} x$. However, geometry not withstanding, the most important trigonometric function for the development of mathematics in the various fields of science is $y = \sin x$. Can the function $y = \sin x$ be represented by a power series expansion as

$$\text{(11)} \qquad y = \sin x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

for suitable coefficients $a_0, a_1, a_2, \ldots$?

If it is possible to represent this function by the power series expansion, then what are the coefficients $a_0, a_1, a_2, \ldots$?

♣ If you studied calculus, then you know that $(\sin x)' = \cos x$. Hence, we know that $\sin x = \int_0^t \cos t\,dt$. In this integral representation of $\sin x$, $\cos t$ follows the integral notation. But in the integral representation of $\sin^{-1} x$, $\frac{1}{\sqrt{1-t^2}}$ follows the integral notation. Using the **Binomial Theorem** and the **Termwise Integration Theorem**, $\sin^{-1} x$ is represented by the power series expansion. We cannot apply the same method to $\cos t$ (which follows the integral notation) to find the power series expansion of $\sin x$.

Recognizing this, Isaac Newton instead employed a reverse method to obtain the power series. To use Newton's method, we must first assume that the function $y = \sin x$ is represented by the power series in expression (11) for suitable $a_0, a_1, a_2, \ldots$. In this function, x represents the arc length and y represents the half-chord length. The relation between the two functions sine and arcsine is given by the following diagram:

$$\text{the arc length} \xrightarrow{\sin} \text{the half-chord length}$$

$$(x) \xleftarrow[\sin^{-1}]{\qquad\qquad} (y)$$

This relation is simply stated as

$$y = \sin x \Leftrightarrow x = \sin^{-1} y.^4$$

Newton attempted to use the following method to find the coefficients a_0, a_1, a_2, ...

[4]The notation (⇔) is read "if and only if" and means equivalence.

First, he represented $x = \sin^{-1} y$ by the power series as

$$x = y + \frac{1}{2}\frac{y^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{y^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{y^7}{7} + \cdots .$$

Substituting the sum of the power series in (11) for y we obtain

$$\begin{aligned} x &= (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots) \\ &+ \frac{1}{2\cdot 3}(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)^3 \\ &+ \frac{1\cdot 3}{2\cdot 4\cdot 5}(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)^5 \\ &+ \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots)^7 + \cdots . \end{aligned} \tag{12}$$

Newton thought that using the **method of undetermined coefficients**[5] in the above expression, he might be able to determine the coefficients a_0, a_1, a_2, ... Newton called this approach the **reverse method** of solving the power series.

Some readers may be unfamiliar with the method of undetermined coefficients. Therefore, before we discuss the result of Newton's findings, let us use an example to explain briefly how the method of undetermined coefficients works. Consider the function $y = \tan^{-1} x$, which appeared in Challenge 6. Using the binomial theorem, we have

$$y = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots . \tag{13}$$

Let us now find the coefficients a_0, a_1, a_2, ... by using the method of undetermined coefficients.

First we have

$$y = \sqrt{1+x} \Leftrightarrow x = y^2 - 1.$$

Therefore, if $y = \sqrt{1+x}$ is represented by the power series as

$$y = \sqrt{1+x} = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots , \tag{14}$$

[5] The method of undetermined coefficients is a tool used to determine the coefficient of any power in a given polynomial by comparing it to the coefficient of the term of the same power in an identical polynomial. For example, if a polynomial $x^3 + 1$ is divided by the polynomial $x^2 - 2x$, then the quotient and the remainder are obtained by setting $x^3 + 1$ to be equal to the identical polynomial $(ax + b) \times (x^2 - 2x) + cx + d$, that is,

$$x^3 + 1 = (ax + b) \times (x^2 - 2x) + cx + d$$

or

$$x^3 + 1 = ax^3 + (b - 2a)x^2 + (c - 2b)x + d.$$

By comparing the coefficient of each term on the left-hand side of this equation with the coefficient at the term of the same power on the right-hand side of the equation, we have

$$a = 1, \quad b - 2a = 0, \quad c - 2b = 0, \quad d = 1.$$

From the above system of equations, we see that the quotient and remainder are given by polynomials $x + 2$ and $4x + 1$, respectively.

then by substituting for y in the equation $x = y^2 - 1$ the sum of the power series in (14), we have

$$x = (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots)^2 - 1.$$

If we expand the right-hand side of the above expression, we obtain

$$\begin{aligned} x &= (a_0^2 - 1) + 2a_0a_1x + (a_1^2 + 2a_0a_2)x^2 + 2(a_1a_2 + a_0a_3)x^3 \\ &+ \{a_2^2 + 2(a_0a_4 + a_1a_3)\}x^4 + \cdots . \end{aligned}$$

Since this is an identity with respect to x, comparing the two sides of the equation gives us

$$\begin{aligned} &a_0 - 1 = 0, \\ &2a_0a_1 = 1, \\ &a_1^2 + 2a_0a_2 = 0, \\ &2(a_1a_2 + a_0a_3) = 0, \\ &a_2^2 + 2(a_0a_4 + a_1a_3) = 0, \\ &\cdots \end{aligned}$$

From the first formula we have $a_0 = 1$. By substituting $a_0 = 1$ in the second formula, we obtain $a_1 = 1/2$. From the third formula we get

$$a_2 = -\frac{a_1^2}{2a_0} = -\frac{1}{2}\left(\frac{1}{2}\right)^2 = -\frac{1}{8}.$$

Similarly we have

$$a_3 = -\frac{a_1a_2}{a_0} = -\frac{1}{2}\left(-\frac{1}{8}\right) = \frac{1}{16},$$

$$a_4 = -\frac{a_2^2 + 2a_1a_3}{2a_0} = -\frac{1}{2}\left\{\left(-\frac{1}{8}\right)^2 + 2\cdot\frac{1}{2}\cdot\frac{1}{16}\right\} = -\frac{5}{128}.$$

Using this technique of successive substitutions, we obtain

$$a_0 = 1, \quad a_1 = \frac{1}{2}, \quad a_2 = -\frac{1}{8}, \quad a_3 = \frac{1}{16} \quad \text{and} \quad a_4 = -\frac{5}{128},$$

which are identical to the coefficients in equation (13).

If we do not mind spending a lot of time, the method of undetermined coefficients helps us to find finite numbers of coefficients in expression (13) by determining the first coefficient, then the second, and so on. The right-hand side of (13) has infinite number of terms; therefore, unless we find some general rule, we cannot say that $y = \sqrt{1+x}$ is represented by the power series expansion.

One uses the method of undetermined coefficients to obtain the power series by the reverse method. The reverse method is a heuristic method whereby we first find a limited sequence of coefficients, and then make a conjecture about the nth term of the sequence. We then need to prove that our conjecture is valid.

In 1669, Newton solved equation (12) by the method of undetermined coefficients and obtained the first five terms of the power series expansion for $y = \sin x$ as follows:

$$y = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \cdots .$$

He then concluded that the coefficient of x^{2n+1} (up to the sign) is $\frac{1}{(2n+1)!}$.

This surprising assertion was valid! The function $y = \sin x$ is indeed represented by the following power series:[6]

$$\sin x = x - \frac{1}{3!} + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \cdots .$$

This representation is valid for any value of x.

This discovery opened doors to further study in many disparate areas. For example, if we substitute $\sin\frac{\pi}{6} = \frac{1}{2}$, $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\sin\frac{\pi}{2} = 1$, $\sin\pi = 0$ into the above expressions, we obtain from each of the substitutions a mysterious relation expression relating to the circumference ratio π.

Now, let us compare the power series expansions (8) and (10) of the functions $y = \tan^{-1} x$ and $y = \sin^{-1} x$ respectively. We realize that the power series representing $y = \tan^{-1} x$ is much simpler than the power series representing $y = \sin^{-1} x$. Therefore, it seems that we should be able to obtain the power series expansion of the function $y = \tan x$ by solving in reverse the power series expansion of $y = \tan^{-1} x$ just as we obtained the power series expansion of $y = \sin x$ by the method of undetermined coefficients. But in fact, it is not possible to solve the function $y = \tan^{-1} x$ in reverse. We know this because the power series expansion of $y = \tan^{-1} x$ is known today to look as follows:

$$\text{if } |x| < \frac{\pi}{2}, \text{ then } \tan x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_n}{(2n)!} x^{2n-1}.$$

[6] $y = \cos x$ is represented by the power series expansion as follows:

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + (-1)^n \frac{1}{(2n)!}x^{2n} + \cdots .$$

The numbers B_n, which appear in the numerator above, are called Bernoulli numbers.[7] The sequence of Bernoulli numbers is an irregular sequence of rational numbers. The first nine terms of the sequence are given as follows:

$$B_1 = \frac{1}{6},\quad B_2 = \frac{1}{30},\quad B_3 = \frac{1}{42},\quad B_4 = \frac{1}{30},\quad B_5 = \frac{5}{66},$$
$$B_6 = \frac{691}{2730},\quad B_7 = \frac{7}{6},\quad B_8 = \frac{3617}{510},\quad B_9 = \frac{43867}{798},\quad \ldots .$$

B_{15} is given by the following (rational) monstrosity:

$$B_{15} = \frac{8615841276005}{14322}.$$

The coefficients of terms in the power series expansion of $\tan x$ do not follow any particular rule. Hence, even if we try to employ the reverse method and obtain the values of the first several coefficients, we will not know the coefficient of the general term of the series. Without being able to express the coefficient of the general term of the series, it is impossible to obtain the power series expansion of $\tan x$ from that of $\tan^{-1} x$.

1.6. History

At the beginning of the 17th century, John Wallis (1616–1703) and his contemporaries developed what is known as **infinite operation**. This operation later was expanded by Newton and Leibniz. In the 1680s, these two mathematical geniuses introduced a completely new mathematical system called calculus. Even at that early date, both Newton and Leibniz knew how to represent the functions $\tan^{-1} x$, $\sin^{-1} x$, and $\sin x$ by power series expansions. They also knew the power series representation of the function $y = e^x - 1$. They started from the power series expansion of $\log(1+x)$, which is given by

$$\log(1+x) = \int_0^x \frac{1}{1+t}\,dt = \int_0^x (1 - t + t^2 - t^3 + t^4 - \cdots)\,dt$$
$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots .$$

They then solved the above power series equation in reverse and obtained the following:

$$y = e^x - 1 = x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots .$$

[7] Bernoulli numbers are defined by the following series:

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n B_n x^{2n}}{(2n)!} \quad \text{for } |x| < 2\pi.$$

Since Newton was able to represent a number of functions by the power series, he likely became convinced that any function could, in fact, be represented by a power series. Quite probably, he arrived at this conclusion after observing both the above power series expansions, and also the power series obtained by the general binomial theorem.[8] However, his conjecture was incorrect, as current knowledge about functions reflects. But if we narrow our focus to those functions which appear in Newtonian mechanics, then Newton's conjecture is correct.

In 1715, Brook Taylor (1685–1731) showed that if a function $f(x)$ is represented by a power series expansion, then the power series is given by the higher order derivatives of f. This series is called the **Taylor series** for $f(x)$. In 1742, by taking a special case of the Taylor series expansion, Colin Maclaurin (1698–1746) discovered the following power series expansion for $f(x)$, called the **Maclaurin series**:

$$(*)\ f(x) = f(0) + \frac{f'(0)}{1!}\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots + \frac{f^{(n)}(0)}{n!}\,x^n + \cdots .$$

Most students encounter the above expansion in lecures on differential and integral calculus. However, the general term of this series can only be found once we obtain $f^{(n)}(0)$. Hence, the Maclaurin series for $f(x)$ will be found only when f has derivatives of all orders.

CHALLENGE 7. (For those of you who studied calculus.) Find the Maclaurin series for e^x, $\sin x$, and $\cos x$ using the power series expansion $(*)$.

As we have seen, once we know that a function can be represented by a power series, then the power series is the Taylor series. Representing a function by a power series via integration and differentiation of higher orders, we catch a glimpse of the world of infinite operations. Since the core of that world is the system of differential and integral calculus, we can fairly say that the 18th century was an era of analysis which blossomed from the study of calculus.

Isaac Newton (1642–1727) was born in the remote countryside of eastern England. He was, by all accounts, an ordinary child who spent a great deal of his time tinkering with mechanical devices. He entered the Trinity College of Cambridge University, but in 1665 a plague epidemics spread throughout England. As a result, the University was closed and Newton returned to his home in the English countryside. It was during this forced hiatus that Newton began his deep study of both physics and mathematics. Newton later stated that the time he spent in the countryside while the University was closed was the most creative and productive period of his life. In fact, he discovered universal gravitation, which is amongst his three greatest discoveries ever. During this time, he also established the study of differential and integral calculus and the fundamental idea of the spectral

[8]It is believed that Newton's study of the general binomial theorem was the starting point of his mathematical research.

nature of light. These amazing scientific achievements he incorporated into *Principia*, published in 1687 when Newton was 46 years old.

Newton's research laid the foundation of modern science; his influence is felt at every level of modern technology. Not long ago, Cambridge University created the Isaac Newton Institute for Mathematical Sciences to commemorate his numerous scientific achievements. Today, this institute is world renowned for cutting-edge research in the fields of mathematics and physics.[9]

[9]Toward the end of his life, Newton turned to matters of religion, and he also became the Master of the Royal Mint!