June 8, 2007 **Math 140B – Sample Final**

Caution: This is only a sample final. The real final might be very different and could be harder!

Instructions. Answer all 8 questions. You may use without proof anything which was proved in class or in the text by Rudin. Either state theorems by name or by what the theorem says. However, you must reprove items which were given as exercises.

1. (20 pts.) Give an example of a sequence $\{f_n\}$ of continuous, real-valued functions on $[0,1]$ such that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$, but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\ dx \neq 0.$$

Explain your example briefly.

2. (20 pts.) If $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq (x-y)^2$$

for all $x, y \in \mathbb{R}$, show that f is constant.

3. (20 pts.) If $f_n : [a,b] \to \mathbb{R}$ is a sequence of continuous functions such that $f_n \to f$ converges uniformly to f, then the function $h(x) : [a,b] \to \mathbb{R}$ defined by

$$h(x) = \int_a^x f(t)dt$$

is differentiable on (a,b). (You should cite a couple of theorems to do this problem.)

4. (20 pts.) Suppose that $f : (-2,2) \to \mathbb{R}$ is differentiable with f' continuous and $f(1/n) = 0$ for all positive integers n. Prove that $f'(0) = 0$.

5. Suppose $f : [0,1] \to \mathbb{R}$ satisfies

$$\int_0^1 f(x)x^n\ dx = 0$$

for all positive integers n.

(a) (25 pts.)If f is continuous, show that $f = 0$.

(b) (20 pts.) Show by example that the conclusion of (a) need not follow if f is not continuous. Explain your example briefly.

6. (25 pts.) Show that the series $\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$ converges uniformly in the interval $[-2, 2]$.

7. (25 pts.) Prove (carefully) that

$$\lim_{n\to\infty} \int_0^{\pi} (\sin x)^n dx = 0.$$

Caution: You may not use the Dominated Convergence Theorem, even if you know what it is! 8. (25 pts.) If $\{f_n\}$ is a sequence of continuous, real-valued functions on $[0, 2]$ such that $f_n(0) = 0$ for all n and

$$|f_n(x) - f_n(y)| \leq |x - y| \quad \forall\, x, y \in [0, 2],$$

show that $\{f_n\}$ has a uniformly convergent subsequence.