June 13, 2007

Math 140B – Solutions to Sample Final

1. (20 pts.) Give an example of a sequence $\{f_n\}$ of continuous, real-valued functions on $[0,1]$ such that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0,1]$, but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\ dx \neq 0.$$

Explain your example briefly.

Solution: Let $f_n(0) = 0$, $f_n(1/2n) = n$, $f_n(x) = 0, 1/n \leq x \leq 1$, with $f(x)$ linear from 0 to $1/2n$ and from $1/2n$ to $1/n$. Then by calculating the area of a triangle of height n and base $1/n$, $\int_0^1 f_n(x)dx = 1/2$, but $\lim_{n\to\infty} f_n(x) = 0$.

2. (20 pts.) If $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \leq (x-y)^2$$

for all $x, y \in \mathbb{R}$, show that f is constant.

Solution: $\frac{|f(x+h)-f(x)|}{|h|} \leq |h|$. Hence $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists and is 0. This means that f is differentiable and $f'(x) \equiv 0$. By a theorem in Rudin, f must be constant.

3. (20 pts.) If $f_n : [a,b] \to \mathbb{R}$ is a sequence of continuous functions such that $f_n \to f$ uniformly, then the function $h(x) : [a,b] \to \mathbb{R}$ defined by

$$h(x) = \int_a^x f(t)dt$$

is differentiable on (a,b). (You should cite a couple of theorems to do this problem.)

Solution: The uniform limit of a sequence of continuous functions is again continuous (theorem in Rudin). Hence f is continuous on $[a,b]$. By part of the Fundamental Theorem of Calculus, it follows that h is differentiable (and $h'(x) = f(x)$).

4. (20 pts.) Suppose that $f : (-2,2) \to \mathbb{R}$ is differentiable with f' continuous and $f(1/n) = 0$ for all positive integers n. Prove that $f'(0) = 0$.

Solution: Since f is continuous, $f(0) = \lim_{n\to\infty} f(1/n) = 0$. By the Mean Value Theorem, $\frac{f(1/n)-f(0)}{1/n} = f'(a_n)$ for some a_n, $0 < a_n < 1/n$. Hence $f'(a_n) = 0$ with $\lim a_n = 0$. Since f' is continuous, $f'(0) = \lim f'(a_n) = 0$.

5. Suppose $f : [0,1] \to \mathbb{R}$ satisfies

$$\int_0^1 f(x)x^n \, dx = 0$$

for all positive integers n.

(a) (25 pts.)If f is continuous, show that $f = 0$.

Solution: Let $\mathcal{P}$ be the set of all polynomials with real coefficients. By the linearity of the integral, it follows that $\int_0^1 f(x)P(x) \, dx = 0$ for all $P \in \mathcal{P}$. By the Weierstrass Approximation Theorem, there is a sequence P_n in $\mathcal{P}$ such that $P_n \to f$ uniformly. There is a theorem in Rudin that says that if $g_n \to g$ uniformly on $[a,b]$ with g_n and g Riemann integrable, then $\int_a^b g_n(x)dx \to \int_a^b g(x)dx$ as $n \to \infty$. Hence as $n \to \infty$

$$0 = \int_0^1 f(x)P_n(x) \, dx \to \int_0^1 f(x)f(x) \, dx = \int_0^1 f^2(x) \, dx.$$

Since f^2 is continuous, nonnegative and $\int_0^1 f^2(x) \, dx = 0$, I claim that $f \equiv 0$. If not, then there exists $x_0 \in [0,1]$ such that $f^2(x_0) = a > 0$. Since f^2 is continuous, there exists $\delta > 0$ such that $f^2(x) > a/2$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Then

$$\int_0^1 f^2(x)dx \geq \int_{x_0-\delta}^{x_0+\delta} f^2(x)dx \geq (a/2)(2\delta) = a\delta > 0,$$

contradicting the assumption. Therefore $f \equiv 0$.

(b) (20 pts.) Show by example that the conclusion of (a) need not follow if f is not continuous. Explain your example briefly.

Solution: Let $f(x) = 0, x \neq 1/2$, and $f(x) = 1, x = 1/2$. Then for any $\epsilon > 0$, any $n \in \mathbb{N}$,

$$\int_0^1 f(x)x^n dx = \int_{1/2-\epsilon/2}^{1/2+\epsilon/2} |f(x)x^n| dx \leq \int_{1/2-\epsilon/2}^{1/2+\epsilon/2} 1 dx = \epsilon.$$

Since ϵ is arbitrary, it follows that $\int_0^1 f(x)x^n dx = 0$ for all $n \in \mathbb{N}$.

6. (25 pts.) Show that the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$ converges uniformly in the interval $[-2,2]$.

Solution:

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2} = \sum_{n=1}^{\infty} (-1)^n \frac{x^2}{n^2} + \sum_{n=1}^{\infty} (-1)^n \frac{1}{n},$$

since $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ converges (alternating series with decreasing terms). It is clear that $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ converges uniformly, since it is independent of x. To show $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{x^2}{n^2}$ also converges uniformly, note that $|(-1)^n\frac{x^2}{n^2}| \le \frac{4}{n^2}$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2} < \infty$, the uniform convergence follows from the Weierstrass M-test. Since the sum of two uniformly convergent series is uniformly convergent (check it), it follows that the series is uniformly convergent.

7. (25 pts.) Prove (carefully) that

$$\lim_{n\to\infty}\int_0^{\pi}(\sin x)^n dx = 0.$$

Caution: You may not use the Dominated Convergence Theorem, even if you know what it is!

Solution: You can use the Mean Value Theorem to prove that $|\sin x| \le |x|$. (Check it!) Hence $|\int_0^{\pi}(\sin x)^n dx| \le \int_0^{\pi} x^n dx = \frac{\pi^{n+1}}{n+1} \to 0$ as $n \to \infty$.

Note: You can also do this without the inequality $|\sin x| \le |x|$.

8. (25 pts.) If $\{f_n\}$ is a sequence of continuous, real-valued functions on $[0,2]$ such that $f_n(0) = 0$ for all n and

$$|f_n(x) - f_n(y)| \le |x - y| \quad \forall\, x, y \in [0,2],$$

show that $\{f_n\}$ has a uniformly convergent subsequence.

Solution: I shall show that $\{f_n\}$ is equicontinuous and pointwise bounded. Then the conclusion will follow immediately from the Arzela-Ascoli Theorem.

Pointwise bounded:

$$|f_n(x)| = |f_n(x) - f_n(0)| \le |x - 0| \le 2 \;\; \forall n.$$

Equicontinuous: Let $\epsilon > 0$, and take $\delta < \epsilon$. Then

$$|f_n(x) - f_n(y)| \le |x - y| < \epsilon \;\text{ for }\; |x - y| < \delta.$$