Math 140, Winter **Practice problems** **February, 2010**
Instructor: Professor Ni

1. Prove that if x is a real number and $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

2. The Cantor set are constructed as following. Let $K_0 = [0,1]$, remove $(\frac{1}{3}, \frac{2}{3})$ let $K_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Then remove the middle thirds of these intervals and let $K_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. Continue this process and construct K_n. Then $P = \cap_{n=1}^{\infty} K_n$.

   Prove that 1) P is nonempty and compact.

   2) P is a perfect set.

3. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

4. Define
$$e = \sum_{n=1}^{\infty} \frac{1}{n!}.$$
   Prove that 1) $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

   2) e is irrational.

5. Find the convergence radius of (i) $\sum_{n=1}^{\infty} n! z^n$, (ii) $\sum_{n=1}^{\infty} q^{n^2} z^n, |q| < 1$, (iii) $\sum_{n=1}^{\infty} z^{n!}$.

6. Suppose that $a_n \geq 0$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} a_n$ diverges. Prove that

   1) $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ and $\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$diverges.

   2)$\sum_{n=1}^{\infty} \frac{a_n}{1+n^2 a_n}$ converges.

   3) If s_n is the partial sum, prove that $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ diverges and $\sum_{n=1}^{\infty} \frac{a_n}{s_n^2}$ converges.

7. If k is a fixed positive integer and if $\{a_n\}$ satisfies that
$$\frac{1}{n^k} \leq a_n \leq n^k$$
   for all $n \geq 1$. Prove that $\lim_{n\to\infty} a_n^{1/n} = 1$.

8. Prove that if $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then
$$\sum_{n=1}^{\infty} \frac{a_n^{2/3}}{n^{2/3}}$$
   converges.

9. Assume that $\{s_n\}$ is a complex sequence, define
$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1}.$$
   Prove that if $\lim_{n\to\infty} s_n = s$ then $\lim_{n\to\infty} \sigma_n = s$.

10. Fix $a > 0$ and $x_1 > \sqrt{a}$. Define
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$
    Prove that $\{x_n\}$ converges and compute $\lim_{n\to\infty} x_n$.

11. Assume that $\{s_n\}$ is a complex sequence, define

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1}.$$

Prove that if $\lim_{n\to\infty} na_n = 0$ and $\{\sigma_n\}$ converges, then $\{s_n\}$ converges.

12. Prove that

$$\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^2}$$

converges.

13. Prove that

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}$$

converges.

14. Is there a nonempty perfect set of $\mathbf{R}^1$ which contains no rational numbers? If yes, give your example and prove that it has the required properties. If no, prove your claim.

15. Can a countably infinite set have an uncountable collection of nonempty subsets such that the intersection of any two of them is finite? If yes, give your example and prove that it has the required properties. If no, prove your claim.