# Math 140A–Sample Final

**Caution**: This is not the real final. the real final will contain problems not on this sample final! It is possible that none of these problems will be on the real final! This sample final is intended to give you an idea of what I think a real final might look like. You can expect that the real final to be at this level of difficulty.

**Instructions**. Answer all questions. You may use without proof anything which was proved in the text by Rudin (unless the problem explicitly states otherwise). Cite a theorem either by name, if it has one, or by briefly stating what it says. However, you must reprove items which were given as exercises. Unless otherwise stated, $X$ is a metric space and $E$ is a subset of $X$. $E'$ denotes the set of limit points of $E$ in $X$, $E^o$ denotes the interior of $E$, and $\overline{E}$ denotes the closure of $E$. The notation $\mathbb{R}$ is used for the real numbers, with the usual metric, and $\mathbb{Q}$ for the rational numbers.

1. (10 pts.) Find the radius of convergence of the series $\sum \frac{2^n}{n^2} z^n$.

2. (10 pts.) Prove that every point of $\mathbb{Q}$ is a limit point of $\mathbb{Q}$.

3. (25 pts.) Suppose that $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences of real numbers with $\lim a_n = a$, $\lim b_n = 2$, and $\lim c_n = 1/2$. Prove the following, directly from the definitions, without citing any theorems such as the limit of the products is the product of the limits.

   (a) $\lim a_n c_n = a/2$

   (b) $\lim 1/b_n = 1/2$

   (c) $\lim a_n/b_n = a/2$.

4. (15 pts.)

   (a) Give an example of an open covering of the interval $(1, 2]$ that has no finite subcovering.

   (b) Prove (carefully!) that your example in (a) works.

5. (20 pts.) If $s_1 = \sqrt{2}$ and $s_{n+1}$ is inductively defined by $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$, prove that $\lim s_n$ exists.

6. (15 pts.) If $f : X \to \mathbb{R}$ is continuous and $E \subset X$, prove that $f(\overline{E}) \subset \overline{f(E)}$.

7.(25 pts.) Let $E \subset \mathbb{R}$ and suppose $f : E \to \mathbb{R}$ is uniformly continuous. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $E$ for which $\lim x_n = \lim y_n$.

   (a) Prove that $\lim f(x_n)$ exists. (Caution: You may NOT assume that $\lim x_n \in E$.)

   (b) Prove that $\lim f(x_n) = \lim f(y_n)$

   (c) Show by example that (a) may fail if $f$ is assumed only to be continuous, but not uniformly continuous.

8. (20 pts.) Suppose $a_n > 0$ and $\sum a_n$ diverges. Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

9. (10 pts. each part) True or false. For each part, determine if it is always true or sometimes false. If true give a reason or short proof. If false give a counterexample. No credit if reason is missing or incorrect. It's OK to be brief here, but "True. This is a theorem in Rudin." is never a correct answer.

(a) If $E \subset X$ has a finite open covering, then $E$ is compact.

(b) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\log 2n}$ converges.

(c) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f(\mathbb{R})$ is closed.

(d) If $f : [a, b] \to \mathbb{R}$ is continuous and $E \subset [a, b]$ is closed, then $f(E)$ is closed.

(e) If $A_n \subset X$ is closed and bounded with $A_n \supset A_{n+1} \supset \ldots$ and $A_n \neq \emptyset$ for any $n$, then $\cap A_n \neq \emptyset$.

(f) There is an uncountable metric space $X$ such that if $K \subset X$ is compact and $E \subset K$, then $E$ is also compact, i.e. every subset of a compact set is compact.