Semidefinite Relaxation Bounds for Indefinite Homogeneous Quadratic Optimization

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Abstract

This paper studies the relationship between the optimal value of a homogeneous quadratic optimization problem and that of its Semidefinite Programming (SDP) relaxation. We consider two quadratic optimization models: **(1)** $\min\{x^*Cx \mid x^*A_kx \geq 1,\ k=0,1,...,m,\ x \in \mathbb{F}^n\}$ and **(2)** $\max\{x^*Cx \mid x^*A_kx \leq 1,\ k=0,1,...,m,\ x \in \mathbb{F}^n\}$, where $\mathbb{F}$ is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, and A_k, C are symmetric matrices. For the minimization model **(1)**, we prove that, if the matrix C and all but *one* of A_k's are positive semidefinite, then the ratio between the optimal value of (1) and its SDP relaxation is upper bounded by $O(m^2)$ when $\mathbb{F}=\mathbb{R}$, and by $O(m)$ when $\mathbb{F}=\mathbb{C}$. Moreover, when two or more of A_k's in (1) are indefinite, this ratio can be arbitrarily large. For the maximization model **(2)**, we show that, if C and at most one of A_k's are indefinite while other A_k's are positive semidefinite, then the ratio between the optimal value of (2) and its SDP relaxation is bounded from below by $O(1/\log m)$ for both the real and complex case. This result improves the bound based on the so-called approximate S-Lemma of Ben-Tal *et al.* [3]. When two or more of A_k in (2) are indefinite, we derive a general bound in terms of the problem data and the SDP solution. For both optimization models, we present examples to show that the derived approximation bounds are essentially tight.

Keywords: Quadratic optimization, SDP relaxation, approximation ratio, probabilistic solution.

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1 Introduction

In this paper we study the relationship between the optimal values of a homogeneous quadratic optimization problem and its *Semidefinite Programming* (SDP) relaxation. Two specific optimization models are considered.

The minimization model. Let A_k $(k = 0, 1, ..., m)$ and C be $n \times n$ real symmetric or complex Hermitian matrices. Consider

$$\begin{array}{rl} \min & x^* C x \\ \text{s.t.} & x^* A_k x \geq 1, \; k = 0, 1, ..., m \\ & x \in \mathbb{F}^n, \end{array} \tag{1.1}$$

where $\mathbb{F}$ can be either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$, and the superscript * represents Hermitian transpose (or regular transpose in case of real matrices). The quadratic optimization problems of form (1.1) are NP-hard [14], even when all the data matrices, C and A_k, $k = 1, ..., m$, are positive semidefinite. Homogeneous quadratic optimization problems (1.1) arise naturally in telecommunications and robust control applications; see [22, 17, 14] and the references therein. In these applications, the optimization variables are naturally complex since they represent the in-phase and quadrature components of a complex signal. A popular approach to approximately solving the NP-hard quadratic problem (1.1) is to use the following SDP relaxations:

$$\begin{array}{rl} \min & \operatorname{Tr}(CX) \\ \text{s.t.} & \operatorname{Tr}(A_k X) \geq 1, \; k = 0, 1, ..., m \\ & X \in \mathbb{SF}^n_+, \end{array}$$

where $\operatorname{Tr}(\cdot)$ represents the trace of a matrix, $\mathbb{SF}^n_+$ denotes the convex cone of positive semidefinite matrices in the space of all (Hermitian) symmetric matrices $\mathbb{SF}^n$. The above SDP is convex and can be solved efficiently using interior point methods. After the SDP relaxation problems are solved, we can apply a *probabilistic method* to the corresponding optimal SDP solution to extract rank-one feasible solutions for (1.1). Theoretically, even though the probabilistic solutions obtained in this manner are not globally optimal for (1.1), they can be shown to be high quality approximate solutions; see, e.g. [3, 14]. Recently, Luo *et al.* [14] considered problem (1.1) and gave bounds for the SDP approximation ratio for (1.1) . When all the matrices A_k and C are positive semidefinite, Luo *et al.* [14] showed that the ratio between the original optimal value and the SDP relaxation optimal value is bounded above by $O(m^2)$ when $\mathbb{F} = \mathbb{R}$ and by $O(m)$ when $\mathbb{F} = \mathbb{C}$, where the factors in the big O notations are absolute constants and independent of data matrices A_k and C. All these bounds are shown to be tight in the worst case. However, the average performance can be much better than the stated worst-case bounds for randomly generated instances. The simulation studies in [14] showed that the average ratios are typically close to 1.

Our contributions. In Section 3, we analyze the approximation ratio of the SDP relaxation for homogeneous quadratic optimization problem (1.1) when some of the constraint matrices A_k are *indefinite*. We show that, for problem (1.1), the same upper bounds for the SDP approximation ratios as given in [14] ($O(m^2)$ when $\mathbb{F} = \mathbb{R}$ and $O(m)$ when $\mathbb{F} = \mathbb{C}$) still hold true even when *one* of the constraint matrices is indefinite. If there are more than one indefinite quadratic constraints, we show by an example that the SDP approximation ratio can be infinity. Therefore, our bounds are essentially the best possible.

The maximization model. We also consider the quadratic optimization problem of the form

$$\begin{array}{rl} \max & x^* C x \\ \text{s.t.} & x^* A_k x \le 1,\ k = 0, 1, ..., m \\ & x \in \mathbb{F}^n. \end{array} \tag{1.2}$$

This quadratic optimization problem is still NP-hard [3, 18], even when all the matrices C and A_k are positive semidefinite. Problem (1.2) arises naturally in telecommunications and robust control applications; see [22, 17, 3] and the references therein. The SDP relaxation for (1.2) can be written as follows:

$$\begin{array}{rl} \max & \operatorname{Tr}(CX) \\ \text{s.t.} & \operatorname{Tr}(A_k X) \le 1,\ k = 0, 1, ..., m \\ & X \in \mathbb{SF}^n_+. \end{array}$$

As in the minimization case, after the SDP relaxation problem is solved, some probabilistic method can be applied to extract a high quality rank-one feasible solution for (1.2). Various estimates exist for the quality of approximate solutions; see, e.g. [3, 18]. Specifically, Nemirovski *et al.* [18] proved that if all A_k's are positive semidefinite, then the ratio between the optimal value of the SDP relaxation problem and that of the original quadratic problem is bounded above by $O(\log m)$. More generally, Ben-Tal *et al.* [3] established a so-called *approximate S-Lemma* which shows that the approximation ratio for the SDP relaxation is at most $O(\log(n^2 m))$ when all but one of the matrices A_k, $k = 0, 1, ..., m$ are positive semidefinite.

Our contributions. In Section 4, we study the SDP approximation ratio for the homogeneous quadratic maximization problem (1.2) when some of the constraint matrices $\{A_k\}$ are *indefinite*. Our results are as follows. We strengthen the approximate S-Lemma of Ben-Tal *et al.* [3] by improving their upper bound on the SDP approximation ratio from $O(\log(n^2 m))$ to $O(\log m)$ when *one* quadratic inequality is indefinite. In the process of establishing this new bound, we give a universal lower bound of 0.03 on the probability that a homogeneous quadratic form of n binary i.i.d. Bernoulli random variables lies below its mean. The previous best known lower bound for this probability was $1/(8n^2)$ due to Ben-Tal *et al.* [3]. In this reference, the authors also conjectured that the actual lower bound should be 0.25. We also present a new and unifying upper bound on the ratio of the optimal value of SDP relaxation over that of the original quadratic maximization problem (1.2) *without* any positive definiteness assumptions. This new general bound involves the problem data and the SDP optimal solution, which is computable in polynomial time. We also present an example showing that this bound is essentially best possible.

Related literature. In addition to the work of Ben-Tal *et al.* [3], Luo *et al.* [14], and Nemirovski *et al.* [18], there is a sizeable literature on the quality bound of SDP relaxation for solving nonconvex quadratic optimization problems. For instance, for problem (1.2), when $m = n$, $A_i = e_i e_i^T$ (there is no A_0) and C is positive semidefinite with nonpositive non-diagonal entries and row sums 0 (which corresponds to the Maximum Cut problem), Goemans and Williamson [8] showed that the ratio of the optimal value of SDP relaxation over that of the original quadratic maximization problem (1.2) is bounded below by 0.87856.... Furthermore, if C is only positive semidefinite, Nesterov [19] showed the same ratio is bounded below by 0.6366.... For closely related results, see Ye [24] and Bertsimas and Ye [4]. Recently, So *et al.* [23] developed SDP relaxation methods for finding approximate low rank solutions for linear matrix inequalities. Their results unify and extend the approximation bounds of Nemirovski *et al.* [18] and Luo *et al.* [14] for the case when all the data matrices are positive semidefinite. Beck and Teboulle [2] discussed the nonconvex problem of minimizing the ratio of two nonconvex quadratic functions over a possibly degenerate ellipsoid, and showed that the SDP relaxation can return exact solutions under a certain condition. There is also some work on solving quadratic optimization problems using other methods, e.g., Hiriart-Urruty and Jean-Baptiste [10], Jeyakumar, Rubinov and Wu [12], and Madsen, Nielsen and Pinar [15, 16].

Outline of this paper. Section 2 is devoted to analyzing the probability of a general random variable to be above (or below) its mean value. Section 3 concentrates on the SDP approximation bound for the quadratic minimization problem (1.1), while Section 4 studies the SDP approximation bound for quadratic maximization problem (1.2). Some concluding remarks are given in the last section (Section 5).

2 Estimating the Asymmetry of a Random Variable

To facilitate the technical analysis in subsequent sections, we establish in this section a bound on the probability for a general random variable to be above (or symmetrically, below) its mean value, using only the high order moment information of the random variable. This problem is of importance on its own in statistics and probability theory. The following lemma is a generalization of Theorem 2.1 in [13].

Lemma 2.1. *Suppose that a random variable* Φ *satisfies* $\mathsf{E}\Phi = 0$, $\mathrm{Var}(\Phi) = 1$ *and* $\mathsf{E}|\Phi|^t \le \tau$ *for some* $t > 2$ *and* $\tau > 0$. *Then* $\mathrm{Prob}\,\{\Phi \ge 0\} > 0.25\tau^{-\frac{2}{t-2}}$ *and* $\mathrm{Prob}\,\{\Phi \le 0\} > 0.25\tau^{-\frac{2}{t-2}}$.

Proof. Let $p_1 = \mathrm{Prob}\,\{\Phi \ge 0\}$ and $p_2 = \mathrm{Prob}\,\{\Phi \le 0\}$. Also let $Y_1 = \max(\Phi, 0)$ and $Y_2 = -\min(\Phi, 0)$. Since $\mathsf{E}\Phi = 0$, we know $\mathsf{E}Y_1 - \mathsf{E}Y_2 = 0$. Let $s := \mathsf{E}Y_1 = \mathsf{E}Y_2$. By Hölder's inequality it follows that $(\mathsf{E}Y_1^t)^{1/(t-1)}(\mathsf{E}Y_1)^{(t-2)/(t-1)} \ge \mathsf{E}Y_1^2$ and $(\mathsf{E}Y_2^t)^{1/(t-1)}(\mathsf{E}Y_2)^{(t-2)/(t-1)} \ge \mathsf{E}Y_2^2$. Since $\mathsf{E}Y_1^t + \mathsf{E}Y_2^t = \mathsf{E}|\Phi|^t$, we have

$$\tau \ge \mathsf{E}|\Phi|^t = \mathsf{E}Y_1^t + \mathsf{E}Y_2^t \ge \frac{(\mathsf{E}Y_1^2)^{t-1} + (\mathsf{E}Y_2^2)^{t-1}}{s^{t-2}}.$$

Let $u = \mathsf{E}Y_1^2 \in [0,1]$. Since $\mathsf{E}Y_1^2 + \mathsf{E}Y_2^2 = \mathsf{E}\Phi^2 = \mathrm{Var}(\Phi) = 1$, it follows that $s^{t-2} \geq \frac{u^{t-1}+(1-u)^{t-1}}{\tau}$. On the other hand, by the Cauchy-Schwartz inequality, we have

$$s^2 = (\mathsf{E}Y_1)^2 = (\mathsf{E}(\mathrm{sign}(Y_1)Y_1))^2 \leq \mathsf{E}(\mathrm{sign}(Y_1)^2)\mathsf{E}Y_1^2 \leq p_1 u$$

which implies that

$$\begin{aligned} p_1 &\geq u^{-1}\left[\frac{u^{t-1}+(1-u)^{t-1}}{\tau}\right]^{\frac{2}{t-2}} \\ &= \frac{\left(u^{t-1}+(1-u)^{t-1}\right)^{\frac{2}{t-2}}}{u}\,\tau^{-\frac{2}{t-2}} \\ &\geq \left(u^{t-1}+(1-u)^{t-1}\right)^{\frac{2}{t-2}}\tau^{-\frac{2}{t-2}} \\ &\geq \left(2\left(\frac{1}{2}\right)^{t-1}\right)^{\frac{2}{t-2}}\tau^{-\frac{2}{t-2}} \\ &= 0.25\tau^{-\frac{2}{t-2}}, \end{aligned}$$

where the third inequality follows from the convexity of the function u^{t-1} when $t > 2$. Obviously, the equality can not hold throughout. Therefore, $p_1 > 0.25\tau^{-\frac{2}{t-2}}$. By symmetry, we also have $p_2 > 0.25\tau^{-\frac{2}{t-2}}$. $\square$

In case $t = 4$, Lemma 2.1 asserts that $\mathrm{Prob}\{\Phi \geq 0\} \geq \frac{1}{4\tau}$ and $\mathrm{Prob}\{\Phi \leq 0\} \geq \frac{1}{4\tau}$. However, in this particular case, this specific bound can in fact be further sharpened.

Lemma 2.2. *Suppose that a random variable Φ satisfies $\mathsf{E}\Phi = 0$, $\mathrm{Var}(\Phi) = 1$ and $\mathsf{E}\Phi^4 \leq \tau$. Then* $\mathrm{Prob}\{\Phi \geq 0\} \geq \frac{2\sqrt{3}-3}{\tau} > \frac{9}{20\tau}$ *and* $\mathrm{Prob}\{\Phi \leq 0\} \geq \frac{2\sqrt{3}-3}{\tau} > \frac{9}{20\tau}$.

Proof. It follows from the proof in the Lemma 2.1 that

$$p_1 \geq \frac{u^3 + (1-u)^3}{\tau u} = \left(\frac{1}{u} + 3u - 3\right)\frac{1}{\tau} \geq \frac{2\sqrt{3}-3}{\tau} > \frac{9}{20\tau}.$$

By symmetry, $p_2 > \frac{9}{20\tau}$ holds as well. $\square$

3 SDP Relaxation Bounds for the Quadratic Minimization Model

Consider the homogeneous quadratic optimization

$$\begin{array}{rll} v_{qp}^{\min} := \min & x^* C x & \\ \text{s.t.} & x^* A_k x \geq 1,\; k = 0,1,...,m & (3.1) \\ & x \in \mathbb{F}^n, & \end{array}$$

where $C, A_0, A_1, ..., A_m \in \mathbb{SF}^n$ are symmetric matrices. This problem is NP-hard [14]. A natural semidefinite programming (SDP) relaxation to the above quadratic optimization problem is

$$\begin{array}{rl} v_{sdp}^{\min} := \min & \mathrm{Tr}\,(CZ) \\ \text{s.t.} & \mathrm{Tr}\,(A_k Z) \geq 1,\ k = 0, 1, ..., m \\ & Z \in \mathbb{SF}_+^n. \end{array} \tag{3.2}$$

Obviously, the SDP relaxation provides a lower bound, i.e., $v_{sdp}^{\min} \leq v_{qp}^{\min}$. In the case $C = I_n$, and $A_0, A_1, ..., A_m$ are all positive semidefinite, Luo *et al.* [14] proved that $v_{qp}^{\min}/v_{sdp}^{\min} \leq \frac{27(m+1)^2}{\pi}$ for $\mathbb{F} = \mathbb{R}$, and $v_{qp}^{\min}/v_{sdp}^{\min} \leq 8(m+1)$ for $\mathbb{F} = \mathbb{C}$. Moreover, when there are two or more of $A_0, A_1, ..., A_m$ are indefinite, there is in general no data-independent upper bound on $v_{qp}^{\min}/v_{sdp}^{\min}$, as shown by the following example [14]:

$$\begin{array}{rl} \min & x_1^2 + x_2^2 \\ \text{s.t.} & x_1^2 \geq 1 \\ & x_1^2 + M x_1 x_2 \geq 1 \\ & x_1^2 - M x_1 x_2 \geq 1 \end{array}$$

where $M > 0$ is a constant. In the above example, $v_{sdp}^{\min} = 1$, and the last two constraints imply $x_1^2 \geq M|x_1||x_2| + 1$ which, together with the first constraint $x_2^2 \geq 1$, yield $x_1^2 \geq M|x_1| + 1$ or, equivalently, $|x_1| \geq (M + \sqrt{M^2+4})/2$. Therefore, $v_{qp}^{\min} \geq 1 + \frac{1}{4}(M + \sqrt{M^2+4})^2$. That is, $v_{qp}^{\min}/v_{sdp}^{\min} \geq 1 + \frac{1}{4}(M + \sqrt{M^2+4})^2$, which can be arbitrarily large, depending on the problem data $M > 0$.

In this section, we consider the homogeneous quadratic optimization (3.1) under the assumption that $C, A_1, A_2, ..., A_m \in \mathbb{SF}_+^n$ are positive semidefinite while $A_0 \in \mathbb{SF}^n$ can be indefinite. Throughout this section, we assume that (3.1) is feasible, and that there is $\mu_k \geq 0$, $k = 0, 1, ..., m$, such that $\sum_{k=0}^m \mu_k A_k \prec C$. This assumption guarantees that the SDP relaxation is primal feasible while its dual problem satisfies the Slater condition. Hence the strong duality holds and the primal problem (3.2) has an optimal solution that attains its infimum.

Our analysis shall treat the cases $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ separately, leading to strikingly different bounds.

3.1 The real case

Let us start with a useful lemma regarding a lower bound on worst asymmetric mass distributions for a χ^2-distribution around its mean vector. In fact this result is interesting on its own right.

Lemma 3.1. *Let τ_i be any real numbers, $i = 1, ..., n$, and let $\eta \sim N(0, I_n)$ be an n-dimensional normal distribution with zero mean and covariance matrix I_n. Then we have*

$$\mathrm{Prob}\left\{\sum_{i=1}^n \tau_i(\eta_i^2 - 1) \geq 0\right\} > \frac{3}{100}, \quad \mathrm{Prob}\left\{\sum_{i=1}^n \tau_i(\eta_i^2 - 1) \leq 0\right\} > \frac{3}{100}.$$

Proof. Note that $\mathsf{E}(\eta_i^2-1)^2 = \mathsf{E}(\eta_i^4 - 2\eta_i^2+1) = 3-2+1 = 2$. Let $\Psi = \sum_{i=1}^n \tau_i(\eta_i^2-1)$, and $\Phi = \frac{\Psi}{\sqrt{2\sum_{i=1}^n \tau_i^2}}$. Then $\mathsf{E}\Phi = 0$ and $\mathrm{Var}(\Phi) = 1$. Since $\mathsf{E}(\eta_i^2-1)^2 = 2$, and $\mathsf{E}(\eta_i^2-1)^4 = 60$, direct calculation shows

$$\mathsf{E}\Psi^4 = 48\sum_{i=1}^n \tau_i^4 + 12\left(\sum_{i=1}^n \tau_i^2\right)^2 \le 60\left(\sum_{i=1}^n \tau_i^2\right)^2.$$

Therefore, we have

$$\mathsf{E}\Phi^4 = \frac{\mathsf{E}\Psi^4}{4(\sum_{i=1}^n \tau_i^2)^2} \le 15.$$

It follows from Lemma 2.2 that $\mathrm{Prob}\,\{\Phi \ge 0\} > \frac{3}{100}$. Similarly, we have $\mathrm{Prob}\,\{\Phi \le 0\} > \frac{3}{100}$ by symmetry. □

Using Hölder's inequality, we also have $\mathsf{E}|\Psi|^3 \le 60^{\frac{3}{4}}(\sum_{i=1}^n \tau_i^2)^{\frac{3}{2}}$ and $\mathsf{E}|\Phi|^3 \le 15^{\frac{3}{4}}$ which can be used to lower $\mathrm{Prob}\,\{\Phi \ge 0\}$ (c.f. Theorem 2.1 in [13]). However, in this particular case, the bound so obtained is slightly worse than the one that we derived in Lemma 3.1.

Lemma 3.2. *Let A, Z be two real symmetric matrices with $Z \succeq 0$ and $\mathrm{Tr}\,(AZ) \ge 0$. Let $\xi \in N(0,Z)$ be a normal random vector with zero mean and covariance matrix Z. Then for any $0 \le \gamma \le 1$ we have*

$$\mathrm{Prob}\,\{\xi^T A\xi < \gamma \mathsf{E}(\xi^T A\xi)\} < 1 - \frac{3}{100}.$$

Proof. Let $r = \mathrm{rank}(AZ)$, and $Q \in \mathbb{R}^{n\times n}$ be an orthogonal matrix such that

$$Q^T(Z^{\frac{1}{2}}AZ^{\frac{1}{2}})Q = \mathrm{diag}(\lambda_1,\cdots,\lambda_r,0,\cdots,0).$$

Since $\mathrm{Tr}\,(AZ) \ge 0$ we have $\sum_{i=1}^r \lambda_i \ge 0$. Let $\bar\xi \in N(0,I_n)$ and $\xi := Z^{\frac{1}{2}}Q\bar\xi$. Then ξ follows a Gaussian distribution $N(0,Z)$. Moreover, we have $\xi^T A\xi = \sum_{i=1}^r \lambda_i\bar\xi_i^2$, where $\bar\xi_i$, $i = 1,...,r$, are independent and follow the normal distribution $N(0,1)$. Therefore, we have $\mathsf{E}(\xi^T A\xi) = \sum_{i=1}^r \lambda_i$ and

$$\begin{aligned}
\mathrm{Prob}\,\{\xi^T A\xi < \gamma\mathsf{E}(\xi^T A\xi)\} &= \mathrm{Prob}\left\{\sum_{i=1}^r \lambda_i\bar\xi_i^2 < \gamma\sum_{i=1}^r \lambda_i\right\} \\
&= \mathrm{Prob}\left\{\sum_{i=1}^r \lambda_i(\bar\xi_i^2-1) < (\gamma-1)\sum_{i=1}^r \lambda_i\right\} \\
&\le \mathrm{Prob}\left\{\sum_{i=1}^r \lambda_i(\bar\xi_i^2-1) < 0\right\} < 1 - \frac{3}{100},
\end{aligned}$$

where the first inequality follows from $\gamma \in [0,1]$ and $\sum_{i=1}^r \lambda_i \ge 0$, and the last step is due to Lemma 3.1. □

Now we are ready to establish the following quality bound for the SDP relaxation. The argument follows closely those of [14].

Theorem 3.3. *Consider the real quadratic program* (3.1) *and its SDP relaxation* (3.2)*, where* $\mathbb{F} = \mathbb{R}$*. Then, there holds*

$$\frac{v_{qp}^{\min}}{v_{sdp}^{\min}} \leq \frac{10^6 m^2}{\pi}.$$

Proof. Let $\hat{Z}$ be an optimal solution of the SDP relaxation (3.2) with rank r satisfying $\frac{(r+1)r}{2} \leq m$. The existence of such matrix solution is well known; cf. Pataki [20]. Moreover, this low rank matrix can be constructed in polynomial-time; cf. [11]. Clearly, $r < \sqrt{2m}$. Since $\hat{Z}$ is feasible, $\mathrm{Tr}\,(A_0\hat{Z}) \geq 1$. Define random vector $\xi = \mathcal{N}(0, \hat{Z})$. For any $0 < \gamma \leq 1$ and $\mu > 0$ we have

$$\begin{aligned}
& \mathrm{Prob}\left\{\min_{0\leq k\leq m} \xi^T A_k \xi \geq \gamma,\, \xi^T C\xi \leq \mu\,\mathrm{Tr}\,(C\hat{Z})\right\} \\
= \;& \mathrm{Prob}\left\{\xi^T A_k \xi \geq \gamma \text{ for all } k = 0,1,...,m, \text{ and } \xi^T C\xi \leq \mu\,\mathrm{Tr}\,(C\hat{Z})\right\} \\
\geq \;& \mathrm{Prob}\left\{\xi^T A_k \xi \geq \gamma\,\mathrm{Tr}\,(A_k\hat{Z}) \text{ for all } k = 0,1,...,m, \text{ and } \xi^T C\xi \leq \mu\,\mathrm{Tr}\,(C\hat{Z})\right\} \\
= \;& \mathrm{Prob}\left\{\xi^T A_k \xi \geq \gamma\,\mathsf{E}(\xi A_k \xi) \text{ for all } k = 0,1,...,m, \text{ and } \xi^T C\xi \leq \mu\,\mathsf{E}(\xi^T C\xi)\right\} \\
\geq \;& 1 - \sum_{k=0}^{m} \mathrm{Prob}\left\{\xi^T A_k \xi < \gamma\,\mathsf{E}(\xi A_k \xi)\right\} - \mathrm{Prob}\left\{\xi^T C\xi > \mu\,\mathsf{E}(\xi^T C\xi)\right\}.
\end{aligned}$$

Since $A_k \succeq 0$ for $k = 1,...,m$, it follows from Lemma 3.1 of [14] that

$$\mathrm{Prob}\left\{\xi^T A_k \xi < \gamma\mathsf{E}(\xi^T A_k \xi)\right\} \leq \max\left\{\sqrt{\gamma}, \frac{2(r-1)\gamma}{\pi - 2}\right\}.$$

Although A_0 is indefinite, we can use Lemma 3.2 to obtain

$$\mathrm{Prob}\left\{\xi^T A_0 \xi < \gamma\mathsf{E}(\xi^T A_0 \xi)\right\} < 1 - \frac{3}{100}.$$

Also, since $C \succeq 0$, we can apply Markov inequality to obtain

$$\mathrm{Prob}\left\{\xi^T C\xi > \mu\,\mathsf{E}(\xi^T C\xi)\right\} \leq \frac{1}{\mu}.$$

Combining the above estimates yields

$$\mathrm{Prob}\left\{\min_{0\leq k\leq m} \xi^T A_k \xi \geq \gamma,\, \xi^T C\xi \leq \mu\,\mathrm{Tr}\,(C\hat{Z})\right\} > \frac{3}{100} - m\max\left\{\sqrt{\gamma}, \frac{2(r-1)\gamma}{\pi-2}\right\} - \frac{1}{\mu}.$$

Let $\hat{\mu} = 100$ and $\hat{\gamma} = \frac{\pi}{10^4 m^2}$. Since $r < \sqrt{2m}$, we have $\sqrt{\hat{\gamma}} \geq \frac{2(r-1)\hat{\gamma}}{\pi-2}$. Then we have

$$\frac{3}{100} - m\max\left\{\sqrt{\hat{\gamma}}, \frac{2(r-1)\hat{\gamma}}{\pi-2}\right\} - \frac{1}{\hat{\mu}} = \frac{3}{100} - m\frac{\sqrt{\pi}}{100m} - \frac{1}{100} > \frac{1}{500}.$$

Therefore, there exists a vector $\xi \in \mathbb{R}^n$ such that

$$\xi^T A_k \xi \geq \hat{\gamma}, \quad k = 0,1,...,m, \quad \text{and} \quad \xi^T C\xi \leq \hat{\mu}\mathrm{Tr}\,(C\hat{Z}).$$

Now let $x = \frac{1}{\sqrt{\hat{\gamma}}}\xi$. Then, $x^T A_k x \geq 1$, $k = 0, 1, ..., m$, and

$$v_{qp}^{\min} \leq x^T C x = \frac{1}{\hat{\gamma}} \xi^T C \xi \leq \frac{\hat{\mu}}{\hat{\gamma}} \mathrm{Tr}\,(C\hat{Z}) = \frac{10^6 m^2}{\pi}\, v_{sdp}^{\min},$$

which establishes the desired bound $\square$

3.2 The complex case

Recall that the density function of a complex-valued normal distribution[1] $\eta \sim N_c(0,1)$ is

$$\frac{1}{\pi} e^{-|u|^2}, \ \forall u \in \mathbb{C}.$$

In polar coordinates, the density function becomes

$$\frac{\rho}{\pi} e^{-\rho^2}, \ \forall \, \rho \in [0, +\infty), \ \theta \in [0, 2\pi).$$

The argument θ is uniformly distributed in $[0, 2\pi)$, and the modulus ρ has the distribution

$$f(\rho) = \begin{cases} 2\rho e^{-\rho^2}, & \text{if } \rho \geq 0; \\ 0, & \text{if } \rho < 0. \end{cases}$$

Thus squared modulus $|\eta|^2$ has the exponential distribution

$$\mathrm{Prob}\,\{|\eta|^2 \leq \alpha\} \leq 1 - e^{-\alpha}.$$

Lemma 3.4. *For any real numbers τ_i, and i.i.d. exponential random variables η_i with unit variance, $i = 1, ..., n$, there holds*

$$\mathrm{Prob}\left\{ \sum_{i=1}^n \tau_i (\eta_i - 1) \geq 0 \right\} > \frac{1}{20}, \qquad \mathrm{Prob}\left\{ \sum_{i=1}^n \tau_i (\eta_i - 1) \leq 0 \right\} > \frac{1}{20}.$$

Proof. Note that $\mathsf{E}(\eta_i - 1)^2 = 1$. Let $\Psi = \sum_{i=1}^n \tau_i(\eta_i - 1)$ and $\Phi = \frac{\Psi}{\sqrt{\sum_{i=1}^n \tau_i^2}}$. Clearly, $\mathsf{E}\Phi = 0$ and $\mathrm{Var}(\Phi) = 1$. Since $\mathsf{E}(\eta_i - 1)^4 = 9$, direct calculation shows

$$\mathsf{E}\Psi^4 = 6 \sum_{i=1}^n \tau_i^4 + 3 \left(\sum_{i=1}^n \tau_i^2 \right)^2 \leq 9 \left(\sum_{i=1}^n \tau_i^2 \right)^2.$$

This further implies

$$\mathsf{E}\Phi^4 = \frac{\mathsf{E}\Psi^4}{(\sum_{i=1}^n \tau_i^2)^2} \leq 9.$$

Using Lemma 2.2 we have $\mathrm{Prob}\,\{\Phi \geq 0\} > \frac{1}{20}$. Similarly, $\mathrm{Prob}\,\{\Phi \leq 0\} > \frac{1}{20}$. $\square$

[1] For a discussion on the complex normal distribution and the related references, see Zhang and Huang [26].

Interestingly, it is possible to find a closed formula (see e.g. [7] and [1]) for the above probability. In particular, if all the τ_i's are distinctive, then

$$\text{Prob}\left\{\sum_{i=1}^{n}\tau_i(\eta_i-1)\geq 0\right\}=\sum_{i=1}^{n}\frac{e^{-\frac{1}{\tau_i}}}{\prod_{j\neq i}\left(1-\frac{\tau_j}{\tau_i}\right)}.$$

Therefore, we have

$$\frac{1}{20}<\sum_{i=1}^{n}\frac{e^{-\frac{1}{\tau_i}}}{\prod_{j\neq i}\left(1-\frac{\tau_j}{\tau_i}\right)}<\frac{19}{20}$$

for any distinctive real values τ_i, $i=1,...,n$.

Lemma 3.5. *Let A, Z be two Hermitian matrices satisfying $Z\succeq 0$ and $\operatorname{Tr}(AZ)\geq 0$. Let $\xi\sim N_c(0,Z)$ be a complex normal random vector. Then, for any $0\leq\gamma\leq 1$, we have*

$$\text{Prob}\left\{\xi^*A\xi<\gamma\mathsf{E}(\xi^*A\xi)\right\}<1-\frac{1}{20}.$$

Proof. Let $Q\in\mathbb{C}^{n\times n}$ be an unitary matrix such that

$$Q^*(Z^{\frac{1}{2}}AZ^{\frac{1}{2}})Q=\text{diag}(\lambda_1,\cdots,\lambda_r,0,\cdots,0)$$

where $r=\text{rank}(AZ)$. Since $\operatorname{Tr}(AZ)\geq 0$, it follows that $\sum_{i=1}^{r}\lambda_i\geq 0$. Let $\hat{\xi}\in\mathbb{C}^n$ be a random Gaussian vector drawn from the complex normal distribution $N_c(0,I_n)$. Then the random vector $\xi=Z^{\frac{1}{2}}Q\hat{\xi}$ follows the Gaussian distribution $N_c(0,Z)$. As a result, there holds

$$\begin{aligned}\text{Prob}\left\{\xi^*A\xi<\gamma\mathsf{E}(\xi^*A\xi)\right) &= \text{Prob}\left\{\sum_{i=1}^{r}\lambda_i|\hat{\xi}_i|^2<\gamma\sum_{i=1}^{n}\lambda_i\right\}\\ &= \text{Prob}\left\{\sum_{i=1}^{n}\lambda_i(|\hat{\xi}_i|^2-1)<(\gamma-1)\sum_{i=1}^{n}\lambda_i\right\}\\ &\leq \text{Prob}\left\{\sum_{i=1}^{n}\lambda_i(|\hat{\xi}_i|^2-1)<0\right\},\end{aligned}$$

where the last step follows from $\gamma\in[0,1]$ and $\sum_{i=1}^{r}\lambda_i\geq 0$. Since $|\xi_i|^2$ is exponentially distributed, by Lemma 3.4, we have

$$\text{Prob}\left\{\sum_{i=1}^{n}\lambda_i(|\hat{\xi}_i|^2-1)\geq 0\right\}>\frac{1}{20}$$

which proves the lemma. ∎

Theorem 3.6. *Consider (3.1) and (3.2), where $\mathbb{F}=\mathbb{C}$. Then*

$$\frac{v_{qp}^{\min}}{v_{sdp}^{\min}}\leq 2400m.$$

Proof. It is known that in this case, if $v_{sdp}^{\min}$ is finite and $m \le 3$, then $v_{qp}^{\min}/v_{sdp}^{\min} = 1$ (cf. e.g. [11] and [25]). Below we shall only consider the case where $m \ge 4$. Let $\hat{Z}$ be a low rank optimal solution of the SDP relaxation (3.2), such that $r = \mathrm{rank}(\hat{Z}) \le \sqrt{m}$ (see [11], §5). The feasibility of $\hat{Z}$ implies that $\mathrm{Tr}\,(A_0\hat{Z}) \ge 1$. Similar to Theorem 3.3, we can use the union bound to obtain the following inequality

$$\begin{aligned} & \mathrm{Prob}\left\{\min_{0\le k\le m} \xi^* A_k \xi \ge \gamma,\ \xi^* C\xi \le \mu\,\mathrm{Tr}\,(C\hat{Z})\right\} \\ \ge\ & 1 - \sum_{k=0}^{m} \mathrm{Prob}\,\{\xi^* A_k\xi < \gamma\,\mathsf{E}(\xi^* A_k \xi)\} - \mathrm{Prob}\,\{\xi^* C\xi > \mu\,\mathsf{E}(\xi^* C\xi)\}. \end{aligned}$$

Since $A_k \succeq 0$, $k = 1, ..., m$, it follows from Lemma 3.4 in [14] that

$$\mathrm{Prob}\,\{\xi^* A_k\xi < \gamma\mathsf{E}(\xi^* A_k\xi)\} \le \max\left\{\frac{4}{3}\gamma, 16(r-1)^2\gamma^2\right\}.$$

Although A_0 is indefinite, Lemma 3.5, asserts that

$$\mathrm{Prob}\,\{\xi^* A_0\xi < \gamma\mathsf{E}(\xi^* A_0\xi)\} < 1 - \frac{1}{20}.$$

Therefore, combining these estimates and using Markov inequality, we have

$$\mathrm{Prob}\left\{\min_{0\le k\le m} \xi^* A_k\xi \ge \gamma, \xi^* C\xi \le \mu,\ \mathrm{Tr}\,(C\hat{Z})\right\} > \frac{1}{20} - m\max\left\{\frac{4}{3}\gamma, 16(r-1)^2\gamma^2\right\} - \frac{1}{\mu}.$$

Now choose $\hat{\mu} = 60$ and $\hat{\gamma} = \frac{1}{40m}$. In this case, $\frac{4}{3}\hat{\gamma} \ge \hat{1}6(r-1)^2\hat{\gamma}^2$. We also have a strict lower bound of the above probability

$$\mathrm{Prob}\left\{\min_{0\le k\le m} \xi^* A_k\xi \ge \hat{\gamma},\ \xi^* C\xi \le \hat{\mu}\,\mathrm{Tr}\,(C\hat{Z})\right\} > 0.$$

This implies that there exists $\xi \in \mathbb{C}^n$ such that

$$\xi^* A_k\xi \ge \hat{\gamma},\ k = 0, 1, ..., m; \quad \xi^* C\xi \le \hat{\mu}\mathrm{Tr}\,(C\hat{Z}).$$

Now let $x := \frac{1}{\sqrt{\hat{\gamma}}}\xi$. Then $x^* A_k x \ge 1$, $k = 0, 1, ..., m$, and so

$$v_{qp}^{\min} \le x^* Cx \le \frac{\xi^* C\xi}{\hat{\gamma}} \le \frac{\hat{\mu}\mathrm{Tr}\,(C\hat{Z})}{\hat{\gamma}} = 2400m \cdot v_{sdp}^{\min}.$$

The theorem is proven. □

Notice that there are examples (see [14]) which show that the worst-case ratios of $v_{qp}^{\min}/v_{sdp}^{\min}$ are indeed $O(m^2)$ and $O(m)$ in the real and complex case respectively, even in the absence of indefinite constraint $x^* A_0 x \ge 1$. Thus, the bounds of Theorems 3.3 and 3.6 are essentially tight.

What happens if there are more than one indefinite quadratic constraint? The following example shows that in this case the SDP relaxation does not admit any *finite* quality bound.

Example 3.7.

$$\begin{array}{ll} \min & x_4^2 \\ \text{s.t.} & x_1x_2 + x_3^2 + x_4^2 \geq 1 \\ & -x_1x_2 + x_3^2 + x_4^2 \geq 1 \\ & \frac{1}{2}x_1^2 - x_3^2 \geq 1 \\ & \frac{1}{2}x_2^2 - x_3^2 \geq 1 \\ & x_1, x_2, x_3, x_4 \in \mathbb{R}. \end{array}$$

The first two constraints are equivalent to $|x_1x_2| \leq x_3^2 + x_4^2 - 1$. At the same time, the last two constraints imply $|x_1x_2| \geq 2(x_3^2 + 1)$. Combining these two inequalities yields

$$x_3^2 + x_4^2 - 1 \geq 2(x_3^2 + 1),$$

which further implies $x_4^2 \geq 3$. Therefore, we must have $v_{qp}^{\min} \geq 3$ in this case. However,

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is feasible for the corresponding SDP relaxation problem and attains an objective value of 0. Thus, it must be optimal and thus $v_{sdp}^{\min} = 0$. Hence, $v_{qp}^{\min}/v_{sdp}^{\min} = \infty$ in this case.

4 SDP Relaxation Bounds for the Quadratic Maximization Model

In this section, we discuss the approximation bound of SDP relaxation for the the nonconvex homogeneous quadratic maximization problem (1.2). Subsection 4.1 considers the case that one constraint is indefinite, and Subsection 4.2 considers the case that two or more constraints are indefinite.

4.1 One Indefinite Constraint

In this subsection, consider the quadratic maximization problem

$$\begin{array}{rll} v_{qp}^{\max} := & \max & x^*Cx \\ & \text{s.t.} & x^*A_kx \leq 1,\ k = 0, 1, ..., m \\ & & x \in \mathbb{F}^n, \end{array} \tag{4.1}$$

where $A_k \in \mathbb{SF}_+^n$, $k = 1, ..., m$, are positive semidefinite, while $C, A_0 \in \mathbb{SF}^n$ may be indefinite. For convenience, from now on we shall focus on the case $\mathbb{F} = \mathbb{R}^n$. Unlike the case of minimization form, this choice does not significantly affect the quality of SDP approximation ratios, since in the complex case the bounds are of the same order of magnitude. We assume that there is $\mu_k \geq 0$, $k = 0, 1, ..., m$, such that

$$\sum_{k=0}^{m} \mu_k A_k \succ 0.$$

Under this condition, the SDP relaxation satisfies the dual Slater condition. Thus the primal-dual optimal solutions exist and the primal-dual optimal objective values are attainable. Let the SDP relaxation optimal value be

$$\begin{array}{rl} v_{sdp}^{\max} := \max & \operatorname{Tr}(CX) \\ \text{s.t.} & \operatorname{Tr}(A_k X) \leq 1,\ k = 0, 1, ..., m \\ & X \succeq 0. \end{array} \tag{4.2}$$

Obviously $v_{qp}^{\max} \leq v_{sdp}^{\max}$.

Lemma 4.1. *Let* w_{ij} $(1 \leq i < j \leq n)$ *be any real numbers, and* ξ_i $(1 \leq i \leq n)$ *be random variables such that* $\text{Prob}\,\{\xi_i = -1\} = \text{Prob}\,\{\xi_i = 1\} = 0.5$. *Then there holds*

$$\text{Prob}\left\{\sum_{1 \leq i < j \leq n} w_{ij}\xi_i\xi_j \leq 0\right\} > \frac{3}{100}.$$

Proof. Let $\Psi = \sum_{1 \leq i < j \leq n} w_{ij}\xi_i\xi_j$. Then $\mathsf{E}\Psi = 0$, $\mathsf{E}(\Psi^2) = \sum_{1 \leq i < j \leq n} w_{ij}^2$ and

$$\mathsf{E}(\Psi^4) = \sum_{1 \leq i < j \leq n} w_{ij}^4 + 6 \sum_{1 \leq i < j < k \leq n} (w_{ij}^2 w_{ik}^2 + w_{ij}^2 w_{jk}^2 + w_{ik}^2 w_{jk}^2) + W$$

where

$$\begin{aligned} W &= 24 \sum_{1 \leq i < j < k < \ell \leq n} (w_{ij}w_{ik}w_{j\ell}w_{k\ell} + w_{ij}w_{i\ell}w_{jk}w_{k\ell} + w_{ik}w_{i\ell}w_{jk}w_{j\ell}) \\ &\quad + 6 \sum_{1 \leq i < j < k < \ell \leq n} \left(w_{ij}^2 w_{k\ell}^2 + w_{ik}^2 w_{j\ell}^2 + w_{i\ell}^2 w_{jk}^2\right) \\ &\leq 30 \sum_{1 \leq i < j < k < \ell \leq n} \left(w_{ij}^2 w_{k\ell}^2 + w_{ik}^2 w_{j\ell}^2 + w_{i\ell}^2 w_{jk}^2\right). \end{aligned}$$

Therefore we have $\mathsf{E}(\Psi^4) \leq 15(\sum_{1 \leq i < j \leq n} w_{ij}^2)^2$. Now let $\Phi = \frac{\Psi}{\sqrt{\sum_{1 \leq i < j \leq n} w_{ij}^2}}$. Then $\mathsf{E}(\Phi) = 0$, $\text{Var}(\Phi) = 1$ and $\mathsf{E}(\Phi^4) \leq 15$. By Lemma 2.2, we have

$$\text{Prob}\,\{\Phi \leq 0\} > \frac{3}{100}.$$

The desired result follows. □

Lemma 4.1 settles in the affirmative an open question of Ben-Tal *et al.* [3, Conjecture A.5] who conjectured that

$$\text{Prob}\left\{\sum_{1 \leq i < j \leq n} w_{ij}\xi_i\xi_j \leq 0\right\} \geq \frac{1}{4}, \qquad \forall\ w_{ij},$$

except that we have a smaller constant of 3/100. The above inequality was needed to establish the so called *approximate S-Lemma* — an extension of the well-known S-Lemma, which is important in the context of robust optimization and is closely related to our analysis in this section. In their work [18], Ben-Tal *et al.* derived a weaker lower bound of $1/8n^2$, which goes to zero as $n \to \infty$.

We can now use Lemma 4.1 to analyze the performance of SDP relaxation for (4.2). Let $\hat{X} = UU^T$ be one optimal solution of (4.2), where $U \in \mathbb{R}^{n\times r}$ and $r = \text{rank}(\hat{X})$. Suppose $Q \in \mathbb{R}^{n\times r}$ is the orthogonal matrix such that $\hat{C} := Q^T U^T C U Q$ is diagonal. Let ξ_k, $k = 1, ..., r$, be i.i.d. random variables taking values -1 or 1 with equal probabilities, and let

$$x(\xi) := \frac{1}{\sqrt{\max\limits_{0\le k\le m} \xi^T \hat{A}_k \xi}}\, UQ\xi,$$

where $\hat{A}_k = Q^T U^T A_k U Q$. Note that the above random vector $x(\xi)$ is always well-defined, since the assumption $\sum_{k=0}^m \mu_k A_k \succ 0$ implies

$$\max_{0\le k\le m} \xi^T \hat{A}_k \xi > 0 \text{ for any } \xi \neq 0.$$

Let $\mu = \min\{m, \max_i \text{rank}(A_i \hat{X})\}$. We have the following estimate of the SDP approximation ratio.

Theorem 4.2. *There holds*

$$v_{qp}^{\max} \le v_{sdp}^{\max} \le 2\log(67\, m\mu)\, v_{qp}^{\max}.$$

Proof. Notice that $\hat{C} = Q^T U^T C U Q$ is diagonal and hence

$$x(\xi)^T C x(\xi) = \frac{1}{\max\limits_{0\le k\le m} \xi^T \hat{A}_k \xi} \xi^T Q^T U^T C U Q \xi = \frac{1}{\max\limits_{0\le k\le m} \xi^T \hat{A}_k \xi} \text{Tr}\,(CX).$$

Therefore for any $\alpha > 1$ we have

$$\begin{aligned}
& \text{Prob}\left\{ x(\xi)^T C x(\xi) \ge \frac{1}{\alpha}\text{Tr}\,(CX) \right\} \\
= \;& \text{Prob}\left\{ \max_{0\le k\le m} \xi^T \hat{A}_k \xi \le \alpha \right\} \\
= \;& 1 - \text{Prob}\left\{ \max_{0\le k\le m} \xi^T \hat{A}_k \xi > \alpha \right\} \\
\ge \;& 1 - \text{Prob}\left\{ \max_{1\le k\le m} \xi^T \hat{A}_k \xi > \alpha \right\} - \text{Prob}\left\{ \xi^T \hat{A}_0 \xi > \alpha \right\}.
\end{aligned}$$

Since $\text{Tr}\,(A_0) \le 1$ and so $\alpha - \text{Tr}\,(A_0) \ge 0$, it follows from Lemma 4.1 that

$$\text{Prob}\left\{ \xi^T \hat{A}_0 \xi > \alpha \right\} \le \text{Prob}\left\{ \sum_{1\le i<j\le m} (\hat{A}_0)_{ij} \xi_i \xi_j > 0 \right\} < 1 - \frac{3}{100}.$$

Since $\hat{A}_k \succeq 0$ for $k = 1, ..., m$, and $\text{Tr}\,(\hat{A}_k) \le 1$, it follows from (12) in [18] that

$$\text{Prob}\left\{ \max_{1\le k\le m} \xi^T \hat{A}_k \xi > \alpha \right\} < 2m\mu e^{-\frac{1}{2}\alpha}.$$

Hence we have

$$\text{Prob}\left\{x(\xi)^T Cx(\xi) \geq \frac{1}{\alpha}\,\text{Tr}\,(CX)\right\} > \frac{3}{100} - 2m\mu e^{-\frac{1}{2}\alpha}.$$

Letting $\hat{\alpha} = 2\log(67\,m\mu)$ ensures the above probability to be positive. Therefore, there exists a random vector ξ such that $\text{Tr}\,(CX) \leq \hat{\alpha}\,x(\xi)^T Cx(\xi)$, and the theorem is proven. $\square$

We point out that Theorem 4.2 is an improvement of the so-called approximate S-Lemma of Ben-Tal, Nemirovski, and Roos [3] (Lemma A.6). In particular, Ben-Tal *et al.* showed that $\alpha \leq 2\log(16n^2\,m\mu)$, in contrast to our bound $\alpha \leq 2\log(67\,m\mu)$.

Notice that in (4.1) there is only one indefinite inequality. Can we allow more than one indefinite constraints? The following example shows that the answer is "no" if we wish to have a data-independent worst-case approximation ratio. (Data-dependent approximation ratio bounds will be discussed in Section 4.2 where we do allow multiple indefinite constraints.)

Example 4.3. Consider

$$\begin{array}{rl} \max & x_1^2 + \frac{1}{M}x_2^2 \\ \text{s.t.} & Mx_1x_2 + x_2^2 \leq 1 \\ & -Mx_1x_2 + x_2^2 \leq 1 \\ & M(x_1^2 - x_2^2) \leq 1, \end{array}$$

where $M > 0$ is an arbitrarily large positive constant. Its SDP relaxation is

$$\begin{array}{rl} \max & X_{11} + \frac{1}{M}X_{22} \\ \text{s.t.} & MX_{12} + X_{22} \leq 1,\ -MX_{12} + X_{22} \leq 1,\ M(X_{11} - X_{22}) \leq 1 \\ & \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0. \end{array}$$

For this quadratic program, the first two constraints imply that $|x_1x_2| \leq \frac{1-x_2^2}{M} \leq \frac{1}{M}$ and so $x_1^2 \leq \frac{1}{M^2x_2^2}$. The third inequality assures that $x_1^2 \leq \frac{1}{M} + x_2^2$. Therefore, $x_1^2 \leq \min\left\{\frac{1}{M^2x_2^2}, \frac{1}{M} + x_2^2\right\} \leq \frac{\sqrt{5}+1}{2M} \approx \frac{1.618}{M}$. Moreover, $x_2^2 \leq 1$, and so $v_{qp}^{\max} \leq \frac{2.618}{M}$.

The SDP relaxation satisfies both primal and dual Slater conditions, so the primal-dual optimal solutions exist. A feasible solution for the SDP relaxation (primal problem) is the 2 by 2 identity matrix, with the objective value being $1+\frac{1}{M} > 1$. On the other hand, since $X_{22} \leq M|X_{12}| + X_{22} \leq 1$, and $X_{11} \leq X_{22} + \frac{1}{M}$, an upper bound for the SDP optimal value is $1+\frac{2}{M}$. Therefore, for this example, the ratio $\frac{v_{sdp}^{\max}}{v_{qp}^{\max}} \geq \frac{M}{2.618} \approx 0.382M$, which can be arbitrarily large, depending on the size of M.

If there are at most two homogeneous quadratic constraints, and moreover if the SDP relaxation has a primal-dual complementary optimal solution, then the SDP optimal value will be equal to the optimal value of the quadratic model; see e.g. Ye and Zhang [25] (Corollary 2.6). In other words, if there are no more than two inequality constraints, then under the primal-dual Slater condition, we will have $v_{sdp}^{\max}/v_{qp}^{\max} = 1$. In this sense, Example 4.3 is the smallest possible in size. By removing the requirement that the SDP relaxation has a finite optimal value, then it is possible to construct an example which involves only two inequality constraints.

Example 4.4. Consider

$$\begin{array}{rl} \max & x_1x_2 + x_1^2 \\ \text{s.t.} & x_1x_2 \le 1 \\ & x_1^2 - x_2^2 \le 1, \end{array}$$

with the SDP relaxation

$$\begin{array}{rl} \max & X_{12} + X_{11} \\ \text{s.t.} & X_{12} \le 1,\ X_{11} - X_{22} \le 1, \\ & \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \succeq 0. \end{array}$$

In terms of polar coordinates, $(x_1, x_2) \longrightarrow (r\cos\theta, r\sin\theta)$, the original quadratic problem can be turned into

$$\begin{array}{rl} \max & r^2(\sin 2\theta + \cos 2\theta + 1)/2 \\ \text{s.t.} & r^2 \sin 2\theta \le 2 \\ & r^2 \cos 2\theta \le 1. \end{array}$$

By a further change of variables $(r^2\cos 2\theta, r^2 \sin 2\theta) \longrightarrow (y_1, y_2)$, we can reformulate the original quadratic problem as

$$\begin{array}{rl} \max & \frac{1}{2}\left(y_1 + y_2 + \sqrt{y_1^2 + y_2^2}\right) \\ \text{s.t.} & y_1 \le 2 \\ & y_2 \le 1. \end{array}$$

This optimization problem has a unique optimal solution at $(y_1^*, y_2^*) = (2, 1)$ with the optimal value being $\frac{3+\sqrt{5}}{2} \approx 2.618$. The SDP relaxation problem is clearly unbounded, as any positive multiple of the identity matrix is feasible. Therefore, $v_{sdp}^{\max}/v_{qp}^{\max} = +\infty$. This example is possible because the dual of the SDP relaxation problem is infeasible.

4.2 Multiple Indefinite Constraints

Unlike the minimization form (1.1) for which the SDP approximation ratio can be infinite when there are more than one indefinite constraints (see Example 3.7), the maximization form (1.2) can still admit a finite SDP approximation ratio in this case. In particular, consider a general homogeneous quadratic maximization problem

$$\begin{array}{rl} \max & x^T C x \\ \text{s.t.} & x^T A_k x \le 1,\ k = 0, 1, ..., m \\ & x \in \mathbb{F}^n. \end{array} \tag{4.3}$$

Suppose that $\mathcal{I}, \mathcal{D}$ are two index sets, $\mathcal{I} \cup \mathcal{D} = \{0, 1, ..., m\}$ and $\mathcal{I} \cap \mathcal{D} = \emptyset$, such that $A_k \succeq 0$ for $k \in \mathcal{D}$ and A_k indefinite for $k \in \mathcal{I}$. The SDP relaxation for (4.3) is

$$\begin{array}{rl} \max & \operatorname{Tr}(CX) \\ \text{s.t.} & \operatorname{Tr}(A_k X) \le 1, \; k = 0, 1, ..., m \\ & X \succeq 0. \end{array} \tag{4.4}$$

We begin our analysis with a technical lemma which bounds the probability of an exponential tail. Similar bounds exist in the literature, e.g. [6]. However, the lemma below serves our needs exactly; for completeness we include a proof here.

Lemma 4.5. *Let $\{\lambda_i\}_{i=1}^n$ be any given real numbers and $\{\eta_i\}_{i=1}^n$ be i.i.d. random variables drawn from either the real or complex valued zero mean Gaussian distribution with unit variance. Let $\sigma = \sqrt{\sum_{i=1}^n \lambda_i^2}$ and $\delta = \max\{\max\{\lambda_i \mid 1 \le i \le n\}, 0\}$. Then, for any $\alpha > 0$ there holds*

$$\operatorname{Prob}\left\{\sum_{i=1}^n \lambda_i \eta_i^2 - \sum_{i=1}^n \lambda_i \ge \alpha\sigma\right\} \le \begin{cases} \exp\left(-\min\left\{\alpha, \frac{\sigma}{\delta}\right\}\frac{\alpha}{8}\right), & \text{if } \eta_i \sim N(0,1) \text{ is real Gaussian,} \\ \exp\left(-\min\left\{\alpha, \frac{\sigma}{\delta}\right\}\frac{\alpha}{4}\right), & \text{if } \eta_i \sim N_c(0,1) \text{ is complex Gaussian.} \end{cases}$$

Proof. We will only prove the real Gaussian case; the complex case is similar and therefore omitted. Let $\beta := \frac{1}{4}\min\{\frac{1}{\delta}, \frac{\alpha}{\sigma}\}$. Then, $2\beta\lambda_i \le 1/2$ for all $i = 1, ..., n$, and $\beta\sigma = \frac{1}{4}\min\{\frac{\sigma}{\delta}, \alpha\}$. Note that for any $t \le 1/2$ the following inequality holds:

$$\frac{1}{1-t} \le e^{t+t^2}. \tag{4.5}$$

Let $\zeta := e^{\beta\sum_{i=1}^n \lambda_i \eta_i^2}$. Since $\{\eta_i^2\}_{i=1}^n$ are standard i.i.d. χ^2 random variables, it follows that

$$\mathsf{E}(\zeta) = \prod_{i=1}^n \mathsf{E}\left(e^{\beta\lambda_i\eta_i^2}\right) = \prod_{i=1}^n \frac{1}{\sqrt{1-2\beta\lambda_i}} = \left(\prod_{i=1}^n \frac{1}{1-2\beta\lambda_i}\right)^{\frac{1}{2}} \le \left(\prod_{i=1}^n e^{2\beta\lambda_i + 4\beta^2\lambda_i^2}\right)^{\frac{1}{2}} = e^{2\beta^2\sigma^2 + \beta\sum_{i=1}^n \lambda_i}$$

where the inequality is due to (4.5). This together with the Markov inequality implies

$$\begin{aligned} \operatorname{Prob}\left\{\sum_{i=1}^n \lambda_i\eta_i^2 - \sum_{i=1}^n \lambda_i \ge \alpha\sigma\right\} &= \operatorname{Prob}\left\{\zeta \ge e^{\beta(\alpha\sigma + \sum_{i=1}^n \lambda_i)}\right\} \\ &\le \frac{\mathsf{E}(\zeta)}{e^{\beta(\alpha\sigma + \sum_{i=1}^n \lambda_i)}} \\ &\le e^{2\beta^2\sigma^2 - \beta\sigma\alpha} = e^{\beta\sigma(2\beta\sigma - \alpha)} \le e^{\beta\sigma(\frac{\alpha}{2} - \alpha)} \\ &= e^{-\min\left\{\alpha, \frac{\sigma}{\delta}\right\}\frac{\alpha}{8}}. \end{aligned}$$

The lemma is proven. ∎

We are now ready to pursue the performance analysis for the real case $\mathbb{F} = \mathbb{R}$. Assume that (4.4) has an optimal solution $\hat{X}$. Denote the set of (real) eigenvalues of $A_k\hat{X}$ as $\lambda_1^k, ..., \lambda_n^k$, $k = 0, 1, ..., m$.

Since $\mathrm{Tr}\,(A_k\hat{X}) \leq 1$, it follows that $\sum_{i=1}^n \lambda_i^k \leq 1$. Moreover, $\|A_k\hat{X}\|_F^2 \geq \sum_{i=1}^n (\lambda_i^k)^2$, $k = 0, 1, ..., m$, where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix.

Let ξ be a random vector drawn from the Gaussian distribution $N(0, \hat{X})$. For any $\alpha > 1$ and $0 \leq k \leq m$, we consider the probability of the event $\mathrm{Prob}\,\{\xi^T A_k \xi > \alpha\}$. By diagonalization, we have $\mathrm{Prob}\,\{\xi^T A_k \xi > \alpha\} = \mathrm{Prob}\,\{\sum_{i=1}^n \lambda_i^k \eta_i^2 > \alpha\}$, where $\eta = (\eta_1, ..., \eta_n)^T$ is a random vector following the normal distribution $N(0, I_n)$.

If we let $\sigma^k := \sqrt{\sum_{i=1}^n (\lambda_i^k)^2} \leq \|A_k\hat{X}\|_F$, and $\delta^k := \max\left\{0, \max\{\lambda_i^k \mid 1 \leq i \leq n\}\right\}$, then Lemma 4.5 leads to

$$\mathrm{Prob}\,\{\xi^T A_k \xi > \alpha\} \leq \exp\left(-\min\left\{\frac{\alpha - \sum_{i=1}^n \lambda_i^k}{\sigma^k}, \frac{\sigma^k}{\delta^k}\right\} \frac{\alpha - \sum_{i=1}^n \lambda_i^k}{8\sigma^k}\right), \quad \forall\, 0 \leq k \leq m. \tag{4.6}$$

Alternatively, we can bound the tail probability using Chebyshev's inequality. In particular, since $\mathrm{Var}(\sum_{i=1}^n \lambda_i^k \eta_i^2) = 2\sum_{i=1}^n (\lambda_i^k)^2 \leq 2\|A_k\hat{X}\|_F^2$, it follows from Chebyshev's inequality

$$\begin{aligned}
\mathrm{Prob}\left\{\sum_{i=1}^n \lambda_i^k \eta_i^2 > \alpha\right\} &= \mathrm{Prob}\left\{\sum_{i=1}^n \lambda_i^k \eta_i^2 - \sum_{i=1}^n \lambda_i^k > \alpha - \sum_{i=1}^n \lambda_i^k\right\} \\
&\leq \mathrm{Prob}\left\{\left|\sum_{i=1}^n \lambda_i^k \eta_i^2 - \sum_{i=1}^n \lambda_i^k\right| > \alpha - \sum_{i=1}^n \lambda_i^k\right\} \\
&\leq \frac{\mathrm{Var}(\sum_{i=1}^n \lambda_i^k \eta_i^2)}{\left(\alpha - \sum_{i=1}^n \lambda_i^k\right)^2} \leq \frac{2\|A_k\hat{X}\|_F^2}{(\alpha-1)^2}, \quad \forall\, 0 \leq k \leq m,
\end{aligned} \tag{4.7}$$

where we have used the fact $\alpha > 1 \geq \sum_{i=1}^n \lambda_i^k$. Applying Lemma 3.1 and using (4.7)–(4.6) gives

$$\begin{aligned}
&\mathrm{Prob}\left\{\xi^T A_k \xi \leq \alpha,\ k = 0, 1, ..., m;\ \xi^T C \xi \geq \mathrm{Tr}\,(C\hat{X})\right\} \\
\geq\ & 1 - \mathrm{Prob}\left\{\xi^T C \xi < \mathrm{Tr}\,(C\hat{X})\right\} - \sum_{k=0}^m \mathrm{Prob}\,\left\{\xi^T A_k \xi > \alpha\right\} \\
\geq\ & \frac{3}{100} - \sum_{k=0}^m \min\left\{\exp\left(-\min\left\{\frac{\alpha - \sum_{i=1}^n \lambda_i^k}{\sigma^k}, \frac{\sigma^k}{\delta^k}\right\} \frac{\alpha - \sum_{i=1}^n \lambda_i^k}{8\sigma^k}\right), \frac{2\|A_k\hat{X}\|_F^2}{(\alpha-1)^2}\right\}.
\end{aligned}$$

Notice that $\delta^k \leq \sigma^k$ and $\sum_{i=1}^n \lambda_i^k \leq 1$ for any k. Therefore, we have, for any $\alpha > 1$,

$$\begin{aligned}
&\mathrm{Prob}\left\{\xi^T A_k \xi \leq \alpha,\ k = 0, 1, ..., m;\ \xi^T C \xi \geq \mathrm{Tr}\,(C\hat{X})\right\} \\
\geq\ & \frac{3}{100} - \sum_{i\in\mathcal{D}} \exp\left(-\min\left\{\frac{\alpha-1}{\sigma^k}, 1\right\} \frac{\alpha-1}{8\sigma^k}\right) \\
& - \sum_{i\in\mathcal{I}} \min\left\{\exp\left(-\min\left\{\frac{\alpha-1}{\sigma^k}, 1\right\} \frac{\alpha-1}{8\sigma^k}\right), \frac{2\|A_k\hat{X}\|_F^2}{(\alpha-1)^2}\right\}.
\end{aligned}$$

Let us choose

$$\hat{\alpha} = 1 + \max\left\{20 + 8\log|\mathcal{D}|, \min\left\{(20 + 8\log|\mathcal{I}|)\max_{k\in\mathcal{I}}\|A_k\hat{X}\|_F, \sqrt{200\sum_{k\in\mathcal{I}}\|A_k\hat{X}\|_F^2}\right\}\right\}.$$

Since $\sigma^k \leq \sum_{i=1}^n \lambda_i^k \leq 1$ for $k \in \mathcal{D}$, it follows from the choice of $\hat{\alpha}$ that

$$\exp\left(-\min\left\{\frac{\hat{\alpha}-1}{\sigma^k}, 1\right\}\frac{\hat{\alpha}-1}{8\sigma^k}\right) = \exp\left(-\frac{\hat{\alpha}-1}{8\sigma^k}\right) \leq \exp\left(-\frac{\hat{\alpha}-1}{8}\right) \leq \frac{1}{100|\mathcal{D}|}, \quad \forall\, k \in \mathcal{D},$$

and

$$\sum_{i\in\mathcal{I}} \min\left\{\exp\left(-\min\left\{\frac{\hat{\alpha}-1}{\sigma^k}, 1\right\}\frac{\hat{\alpha}-1}{8\sigma^k}\right), \frac{2\|A_k\hat{X}\|_F^2}{(\hat{\alpha}-1)^2}\right\} \leq \frac{1}{100}.$$

This further implies that

$$\text{Prob}\left\{\xi^T A_k \xi \leq \hat{\alpha},\, k = 0, 1, ..., m;\, \xi^T C \xi \geq \text{Tr}\,(C\hat{X})\right\} \geq \frac{1}{100}.$$

Summarizing, we obtain the following worst-case performance ratio bounds on the SDP relaxation for a real-valued homogeneous (indefinite) quadratic maximization problem. [We also state the complex case without proof.]

Theorem 4.6. *For the quadratic optimization problem* (4.3) *with* $\mathbb{F} = \mathbb{R}$ *and its SDP relaxation* (4.4)*, suppose that an optimal solution, say* $\hat{X}$*, for* (4.4) *exists. Then,*

$$\frac{v_{sdp}^{\max}}{v_{qp}^{\max}} \leq 1 + \max\left\{20 + 8\log|\mathcal{D}|, \min\left\{(20 + 8\log|\mathcal{I}|)\max_{k\in\mathcal{I}}\|A_k\hat{X}\|_F, \sqrt{200\sum_{k\in\mathcal{I}}\|A_k\hat{X}\|_F^2}\right\}\right\}.$$

Similarly, for the complex case $\mathbb{F} = \mathbb{C}$*, we have*

$$\frac{v_{sdp}^{\max}}{v_{qp}^{\max}} \leq 1 + \max\left\{15 + 4\log|\mathcal{D}|, \min\left\{(15 + 4\log|\mathcal{I}|)\max_{k\in\mathcal{I}}\|A_k\hat{X}\|_F, \sqrt{40\sum_{k\in\mathcal{I}}\|A_k\hat{X}\|_F^2}\right\}\right\}.$$

Let us consider two special cases of Theorem 4.6. First, if $\mathcal{I} = \emptyset$, then Theorem 4.6 becomes $\frac{v_{sdp}^{\max}}{v_{qp}^{\max}} \leq 20 + 8\log m$ (in the real case), which recovers the approximation result of Nemirovski *et al.* [18]. The second case is $\mathcal{D} = \emptyset$, where Theorem 4.6 becomes

$$\frac{v_{sdp}^{\max}}{v_{qp}^{\max}} \leq 1 + \min\left\{(20 + 8\log(m+1))\max_{0\leq k\leq m}\|A_k\hat{X}\|_F, \sqrt{200\sum_{k=0}^m\|A_k\hat{X}\|_F^2}\right\}.$$

Below is an example showing that this bound is also tight (in the order of magnitude). Specifically, consider Example 4.3 again:

$$\begin{array}{rl} \max & x_1^2 + \frac{1}{M}x_2^2 \\ \text{s.t.} & Mx_1x_2 + x_2^2 \leq 1 \\ & -Mx_1x_2 + x_2^2 \leq 1 \\ & M(x_1^2 - x_2^2) \leq 1. \end{array}$$

In this case we know that the SDP relaxation has an optimal solution $\hat{X} = \begin{bmatrix} 1+\frac{1}{M} & 0 \\ 0 & 1 \end{bmatrix}$, while the approximation ratio is $v_{sdp}^{\max}/v_{qp}^{\max} = O(M)$. There are three constraints, all indefinite, $\mathcal{I} = \{1,2,3\}$, with

$$A_1 = \begin{bmatrix} 0 & \frac{M}{2} \\ \frac{M}{2} & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & -\frac{M}{2} \\ -\frac{M}{2} & 1 \end{bmatrix},\ A_3 = \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix},$$

and so one may compute that

$$A_1\hat{X} = \begin{bmatrix} 0 & \frac{M}{2} \\ \frac{M}{2}+\frac{1}{2} & 1 \end{bmatrix},\ A_2\hat{X} = \begin{bmatrix} 0 & -\frac{M}{2} \\ -\frac{M}{2}-\frac{1}{2} & 1 \end{bmatrix},\ A_3\hat{X} = \begin{bmatrix} M+1 & 0 \\ 0 & -M \end{bmatrix}.$$

Thus, $\|A_k\hat{X}\|_F^2 = O(M^2)$, for $k = 1,2,3$. Theorem 4.6 predicts that $v_{sdp}^{\max}/v_{qp}^{\max} \le O(M)$, and this upper bound is exactly attained in this example.

5 Discussions

This paper studies the quality bound of SDP relaxations for some classes of nonconvex quadratic optimization problems. Our analysis reveals interesting differences in the quality bounds for the optimization models expressed in either maximization or minimization form, and for optimization variables defined over either the real or complex field. It provides a complete picture on the performance of the SDP relaxation techniques for quadratic optimization problems involving indefinite constraints.

Theoretically, the minimization model (1.1) and maximization model (1.2) are intrinsically different, and they cannot be directly transformed from one to the other. In general, the feasible region of problem (1.1) can be nonconvex, unbounded or even disconnected, while its objective function is usually assumed to be convex. In contrast, the maximization model (1.2) typically has a convex and bounded feasible region, but the nonconvexity of the objective function makes it difficult. These essential differences have led to the qualitatively different behaviors in the respective SDP approximation ratios.

It is equally interesting to note that the choice of field in which the optimization variables reside can also impact the quality of SDP relaxation. In a natural way, every complex quadratic program can be turned into an equivalent real quadratic program by doubling the dimension. Such a transformation will not affect the resulting approximation ratio. Since the SDP approximation ratio is weaker in the real case, we cannot derive the desired approximation ratio for the complex case by this simple reduction. It is worth noting that the tighter SDP approximation ratio for the complex case has been observed in digital communication applications [22, 17, 14] where the signals are naturally complex due to their in-phase and quadrature components.

An interesting byproduct of our work is a universal lower bound of $\mathrm{Prob}\left(\sum_{i=1}^n \tau_i(\eta_i - 1) \ge 0\right)$ for the independently distributed exponential random variables η_i (Lemma 4.1). The lower bounds of this type are interesting on their own and are related to the well-known inequality of Grünbaum [9] in convex analysis. In particular, by a different analytic argument, it is possible to further improve the universal lower bound obtained in this paper to the following

$$\mathrm{Prob}\left(\sum_{i=1}^{n} \tau_i(\eta_i - 1) \geq 0\right) = \sum_{i=1}^{n} \frac{e^{-\frac{1}{\tau_i}}}{\prod_{j\neq i}\left(1 - \frac{\tau_j}{\tau_i}\right)} > \frac{1}{e} \tag{5.1}$$

where τ_i, $i = 1, ..., n$, are any real numbers. [The above equality can be derived by evaluating a multidimensional integral.] The inequality (5.1) admits a simple geometric interpretation. For the joint standard exponential distribution on $\mathbb{R}^n_+$, the center of gravity of $\mathbb{R}^n_+$ is $x^c := \mathsf{E}(\eta) = (1, 1, ..., 1)^T$, and the inequality (5.1) can be interpreted as follows:

$$\mathrm{Prob}(\mathbb{R}^n_+ \cap \mathcal{H}_+) \geq e^{-1}, \quad \text{for any hyperplane } \mathcal{H} \text{ passing through } x^c. \tag{5.2}$$

Here $\mathcal{H}_+$ denotes the positive side of the hyperplane $\mathcal{H}$. The inequality (5.2) is an extension of the Grünbaum inequality [9]:

$$\mathrm{Volume}\,(\mathcal{C} \cap \mathcal{H}_+) \geq \; e^{-1}\; \mathrm{Volume}\,(\mathcal{C})$$

for any bounded convex set $\mathcal{C}$ in $\mathbb{R}^n$ and any hyperplane $\mathcal{H}$ passing through the center of gravity of $\mathcal{C}$

$$x^c = \frac{1}{\mathrm{Volume}(\mathcal{C})} \int_{\mathcal{C}} dx.$$

In particular, if we assign the uniform distribution to $\mathcal{C}$, then the mean vector of this distribution is given by the center of gravity x^c and the probability in (5.2) can be expressed in terms of volume. In this way, Grünbaum's inequality can be equivalently written as (5.2). This shows that the inequality (5.2) generalizes Grünbaum's theorem [9] from the uniform distribution over a compact convex set to the exponential distribution over $\mathbb{R}^n_+$. Interestingly, it is possible to establish the inequality (5.2) for any log-concave distributions defined over any (possibly unbounded) convex set in $\mathbb{R}^n$. The proof of this inequality relies on a result of Bobkov [5, Lemma 3.3] and a result of Prekopa [21] on the projection of any log-concave distribution. We plan to report the details of this proof elsewhere in future.

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