# An Exact Jacobian SDP Relaxation for Polynomial Optimization

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### Abstract

Given polynomials $f(x), g_i(x), h_j(x)$, we study how to minimize $f(x)$ on the set

$$S = \{x \in \mathbb{R}^n : h_1(x) = \cdots = h_{m_1}(x) = 0, g_1(x) \geq 0, \ldots, g_{m_2}(x) \geq 0\}.$$

Let $f_{min}$ be the minimum of $f$ on $S$. Suppose $S$ is nonsingular and $f_{min}$ is achievable on $S$, which are true generically. This paper proposes a new type semidefinite programming (SDP) relaxation which is the first one for solving this problem exactly. First, we construct new polynomials $\varphi_1, \ldots, \varphi_r$, by using the Jacobian of $f, h_i, g_j$, such that the above problem is equivalent to

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & h_i(x) = 0, \; \varphi_j(x) = 0, \; 1 \leq i \leq m_1, 1 \leq j \leq r, \\ & g_1(x)^{\nu_1} \cdots g_{m_2}(x)^{\nu_{m_2}} \geq 0, \quad \forall \nu \in \{0,1\}^{m_2}. \end{aligned}$$

Second, we prove that for all $N$ big enough, the standard $N$-th order Lasserre's SDP relaxation is exact for solving this equivalent problem, that is, its optimal value is equal to $f_{min}$. Some variations and examples are also shown.

**Key words**  determinantal varieties, ideals, minors, polynomials, nonsingularity, semidefinite programming, sum of squares

**AMS subject classification**  14P99, 65K05, 90C22

## 1   Introduction

Consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & h_1(x) = \cdots = h_{m_1}(x) = 0 \\ & g_1(x) \geq 0, \ldots, g_{m_2}(x) \geq 0 \end{aligned} \tag{1.1}$$

where $f(x), g_i(x), h_j(x)$ are polynomial functions. When $m_1 = 0$ (resp. $m_2 = 0$), there are no equality (resp. inequality) constraints. Let $S$ be its feasible set and $f_{min}$ be its global minimum. We are interested in computing $f_{min}$. The problem is NP-hard [16].

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A standard approach for solving (1.1) is the hierarchy of semidefinite programming (SDP) relaxations proposed by Lasserre [16]. It is based on a sequence of sum of squares (SOS) type representations of polynomials that are nonnegative on $S$. The basic idea is, for a given integer $N > 0$ (called relaxation order), solve the SOS program

$$
\begin{aligned}
\max \quad & \gamma \\
\text{s.t.} \quad & f(x) - \gamma = \sum_{i=1}^{m_1} \phi_i h_i + \sum_{j=1}^{m_2} \sigma_j g_j, \\
& \deg(\phi_i h_i),\ \deg(\sigma_j g_j) \leq 2N \quad \forall\, i, j, \\
& \sigma_1, \ldots, \sigma_{m_2} \text{ are SOS.}
\end{aligned}
\tag{1.2}
$$

In the above, $g_0(x) \equiv 1$, the decision variables are the coefficients of polynomials $\phi_i$ and $\sigma_j$. Here a polynomial is SOS if it is a sum of squares of other polynomials. The SOS program (1.2) is equivalent to an SDP problem (see [16]). Let $p_N$ be the optimal value of (1.2). Clearly, $p_N \leq f_{min}$ for every $N$. Using Putinar's Positivstellensatz [21], Lasserre proved $p_N \to f_{min}$ as $N \to \infty$, under the archimedean condition. A stronger relaxation than (1.1) would be obtained by using cross products of $g_j$, which is

$$
\begin{aligned}
\max \quad & \gamma \\
\text{s.t.} \quad & f(x) - \gamma = \sum_{i=1,\ldots,m_1} \phi_i h_i + \sum_{\nu \in \{0,1\}^{m_2}} \sigma_\nu \cdot g_\nu, \\
& \deg(\phi_i h_i) \leq 2N,\ \deg(\sigma_\nu g_\nu) \leq 2N \quad \forall\, i, \nu, \\
& \sigma_\nu \text{ are all SOS.}
\end{aligned}
\tag{1.3}
$$

In the above, $g_\nu = g_1^{\nu_1} \cdots g_{m_2}^{\nu_{m_2}}$. Let $q_N$ be the optimal value of (1.3). When $S$ is compact, Lasserre showed $q_N \to f_{min}$ as $N$ goes to infinity, using Schmügen's Positivstellensatz [24]. An analysis for the convergence speed of $p_N, q_N$ to $f_{min}$ is given in [19, 25]. Typically, (1.2) and (1.3) are not exact for (1.1) with a finite $N$. Scheiderer [23] proved a very surprising result: whenever $S$ has dimension three or higher, there always exists $f$ such that $f(x) - f_{min}$ does not have a representation required in (1.3). Thus, we usually need to solve a big number of SDPs until convergence is met. This would be very inefficient in many applications. Furthermore, when $S$ is not compact, typically we do not have the convergence $p_N \to f_{min}$ or $q_N \to f_{min}$. This is another difficulty. Thus, people are interested in more efficient methods for solving (1.1).

Recently, the author, Demmel and Sturmfels [18] proposed a gradient type SOS relaxation. Consider the case of (1.1) without constraints, i.e., $m_1 = m_2 = 0$. If the minimum $f_{min}$ is achieved at a point $u$, then $\nabla f(u) = 0$, and the problem is equivalent to

$$
\min_{x \in \mathbb{R}^n} \quad f(x) \quad s.t. \quad \frac{\partial f}{\partial x_1} = \cdots = \frac{\partial f}{\partial x_n} = 0.
\tag{1.4}
$$

In [18], Lasserre's relaxation is applied to solve (1.4), and it was shown that a sequence of lower bounds converging to $f_{min}$ would be obtained. It has finite convergence if the gradient ideal, generated by the partial derivatives of $f(x)$, is radical. More recently, Demmel, the author and Powers [7] generalized the gradient SOS relaxation to solve (1.1) by using the Karush-Kuhn-Tucker (KKT) conditions of (1.1)

$$
\nabla f(x) = \sum_{i=1}^{m_1} \lambda_i \nabla h_i(x) + \sum_{j=1}^{m_2} \mu_j \nabla g_j(x), \quad \mu_j g_j(x) = 0,\ j = 1, \ldots, m_2.
$$

If a global minimizer of (1.1) is a KKT point, then (1.1) is equivalent to

$$
\begin{aligned}
\min_{x,\lambda,\mu} \quad & f(x) \\
\text{s.t.} \quad & h_1(x) = \cdots = h_{m_1}(x) = 0, \\
& \nabla f(x) = \sum_{i=1}^{m_1} \lambda_i \nabla h_i(x) + \sum_{j=1}^{m_2} \mu_j \nabla g_j(x), \\
& \mu_j g_j(x) = 0, \quad g_j(x) \geq 0, \ j = 1, \ldots, m_2.
\end{aligned}
\tag{1.5}
$$

Let $\{v_N\}$ be the sequence of lower bounds for (1.5) obtained by applying Lasserre's relaxation of type (1.3). It was shown in [7] that $v_N \to f_{min}$, no matter $S$ is compact or not. Furthermore, it holds that $v_N = f_{min}$ for a finite $N$ when the KKT ideal is radical, but it was unknown in [7] whether this property still holds without the KKT ideal being radical. A drawback for this approach is that the involved polynomials are in $(x, \lambda, \mu)$. There are totally $n + m_1 + m_2$ variables, which makes the resulting SDP very difficult to solve in practice.

**Contributions**  This paper proposes a new type SDP relaxation for solving (1.1) via using KKT conditions but the involved polynomials are only in $x$. Suppose $S$ satisfies a nonsingularity assumption (see Assumption 2.2 for its meaning) and $f_{min}$ is achievable on S, which are true generically. We construct new polynomials $\varphi_1(x), \ldots, \varphi_r(x)$, by using the minors of the Jacobian of $f, h_i, g_j$, such that (1.1) is equivalent to

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & h_i(x) = 0 \, (1 \leq i \leq m_1), \ \varphi_j(x) = 0 \, (1 \leq j \leq r), \\
& g_1(x)^{\nu_1} \cdots g_{m_2}(x)^{\nu_{m_2}} \geq 0, \quad \forall \nu \in \{0,1\}^{m_2}.
\end{aligned}
$$

Then we prove that for all $N$ big enough, the standard $N$-th order Lasserre's relaxation for the above returns the minimum $f_{min}$. That is, an exact SDP relaxation for (1.1) is obtained by using the Jacobian.

This paper is organized as follows. Section 2 gives the construction of this exact SDP relaxation by using Jacobian. Its exactness and genericity are proved in Section 3. Some efficient variations are proposed in Section 4. Some examples of how to apply it are shown in Section 5. Some conclusions and discussions are made in Section 6. Finally, we attach an appendix introducing some basics of algebraic geometry and real algebra that are used in the paper.

**Notations**  The symbol $\mathbb{N}$ (resp., $\mathbb{R}$, $\mathbb{C}$) denotes the set of nonnegative integers (resp., real numbers, complex numbers). For any $t \in \mathbb{R}$, $\lceil t \rceil$ denotes the smallest integer not smaller than $t$. For integer $n > 0$, $[n]$ denotes the set $\{1, \ldots, n\}$, and $[n]_k$ denotes the set of subsets of $[n]$ whose cardinality is $k$. For a subset $J$ of $[n]$, $|J|$ denotes its cardinality. For $x \in \mathbb{R}^n$, $x_i$ denotes the $i$-th component of $x$, that is, $x = (x_1, \ldots, x_n)$. For $\alpha \in \mathbb{N}^n$, denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$, $x^\alpha$ denotes $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The symbol $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ (resp. $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$) denotes the ring of polynomials in $(x_1, \ldots, x_n)$ with real (resp. complex) coefficients. A polynomial is called a form if it is homogeneous. The $\mathbb{R}[x]_{\leq d}$ denotes the subspace of polynomials in $\mathbb{R}[x]$ of degrees at most $d$. For a general set $T \subseteq \mathbb{R}^n$, $int(T)$ denotes its interior, and $\partial T$ denotes its boundary in standard Euclidean topology. For a symmetric matrix $X$, $X \succeq 0$ (resp., $X \succ 0$) means $X$ is positive semidefinite (resp. positive definite). For $u \in \mathbb{R}^N$, $\|u\|_2$ denotes the standard Euclidean norm.

# 2 Construction of the exact Jacobian SDP relaxation

Let $S$ be the feasible set of (1.1) and

$$m \quad = \quad \min\{m_1 + m_2, n - 1\}. \tag{2.1}$$

For convenience, we denote $h(x) = (h_1(x), \ldots, h_{m_1}(x))$ and $g(x) = (g_1(x), \ldots, g_{m_2}(x))$. For a subset $J = \{j_1, \ldots, j_k\} \subset [m_2]$, denote

$$g_J(x) = (g_{j_1}(x), \ldots, g_{j_k}(x)).$$

Let $x^*$ be a minimizer of (1.1). If $J$ is the active index set at $x^*$ such that $g_J(x^*) = 0$ and the KKT conditions hold at $x^*$, then there exist $\lambda_i$ and $\mu_j (j \in J)$ such that

$$h(x^*) = 0, \quad g_J(x^*) = 0, \quad \nabla f(x^*) = \sum_{i \in [m_1]} \lambda_i \nabla h_i(x^*) + \sum_{j \in J} \mu_j \nabla g_j(x^*).$$

The above implies the Jacobian matrix of $(f, h, g_J)$ is singular at $x^*$. For a subset $J \subset [m_2]$, denote the determinantal variety of $(f, h, g_J)$'s Jacobian being singular by

$$G_J = \left\{ x \in \mathbb{C}^n : \operatorname{rank} B^J(x) \leq m_1 + |J| \right\}, \quad B^J(x) = \begin{bmatrix} \nabla f(x) & \nabla h(x) & \nabla g_J(x) \end{bmatrix}. \tag{2.2}$$

Then, $x^* \in V(h, g_J) \cap G_J$ where $V(h, g_J) := \left\{ x \in \mathbb{C}^n : h(x) = 0, g_J(x) = 0 \right\}$.

This motivates us to use $g_J(x) = 0$ and $G_J$ to get tighter SDP relaxations for (1.1). To do so, a practical issue is how to get a "nice" description for $G_J$? An obvious one is that all its maximal minors vanish. But there are totally $\binom{n}{m_1+k+1}$ such minors (if $m_1 + k + 1 \leq n$), which is huge for big $n, m_1, k$. Can we define $G_J$ by a set of the smallest number of equations? Furthermore, the active index set $J$ is usually unknown in advance. Can we get an SDP relaxation that is independent of $J$?

## 2.1 Minimum defining equations for determinantal varieties

Let $k \leq n$ and $X = (X_{ij})$ be a $n \times k$ matrix of indeterminants $X_{ij}$. Define the determinantal variety

$$D_{t-1}^{n,k} = \left\{ X \in \mathbb{C}^{n \times k} : \operatorname{rank} X < t \right\}.$$

For any index set $I = \{i_1, \ldots, i_k\} \subset [n]$, denote by $\det_I(X)$ the $(i_1, \ldots, i_k) \times (1, \ldots, k)$-minor of matrix $X$, i.e., the determinant of the submatrix of $X$ whose row indices are $i_1, \ldots, i_k$ and column indices are $1, \ldots, k$. Clearly, it holds that

$$D_{k-1}^{n,k} = \left\{ X \in \mathbb{C}^{n \times k} : \det_I(X) = 0 \quad \forall I \in [n]_k \right\}.$$

The above has $\binom{n}{k}$ defining equations of degree $k$. An interesting fact is that we do not need $\binom{n}{k}$ equations to define $D_{k-1}^{n,k}$. Actually, this number would be significantly smaller. There is very nice work on this issue. Bruns and Vetter [3] showed that $nk - t^2 + 1$ equations are enough to define $D_{t-1}^{n,k}$. Later, Bruns and Schwänzl [2] showed that $nk - t^2 + 1$ is the smallest number of equations for defining $D_{t-1}^{n,k}$. Typically, $nk - t^2 + 1 \ll \binom{n}{k}$ for big $n$ and $k$. A general

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method for constructing $nk - t^2 + 1$ defining polynomial equations for $D_{t-1}^{n,k}$ was described in Chapt. 5 of [3]. Here we briefly show how it works for $D_{k-1}^{n,k}$.

Let $\Gamma(X)$ denote the set of all $k$-minors of $X$ (assume their row indices are strictly increasing). For convenience, for $i_1 < \cdots < i_k$, denote by $[i_1, \ldots, i_k]$ the $(i_1, \ldots, i_k) \times (1, \ldots, k)$-minor of $X$. Define a partial ordering on $\Gamma(X)$ as follows:

$$[i_1, \ldots, i_k] < [j_1, \ldots, j_k] \iff i_1 \leq j_1, \ldots, i_k \leq j_k, \ i_1 + \cdots + i_k < j_1 + \cdots + j_k.$$

If $I = \{i_1, \ldots, i_k\}$, we also write $I = [i_1, \ldots, i_k]$ as a minor in $\Gamma(X)$ for convenience. For any $I \in \Gamma(X)$, define its rank as

$$rk(I) = \max\left\{ \ell : I = I^{(\ell)} > \cdots > I^{(1)}, \quad \text{every} \quad I^{(i)} \in \Gamma(X) \right\}.$$

The maximum minor in $\Gamma(X)$ is $[n - k + 1, \ldots, n]$ and has rank $nk - k^2 + 1$. For every $1 \leq \ell \leq nk - k^2 + 1$, define

$$\eta_\ell(X) \quad = \sum_{I \in [n]_k, rk(I) = \ell} \det_I(X). \tag{2.3}$$

**Lemma 2.1** (Lemma (5.9), Bruns and Vetter [3]). *It holds that*

$$D_{k-1}^{n,k} = \left\{ X \in \mathbb{C}^{n \times k} : \eta_\ell(X) = 0, \ \ell = 1, \ldots, nk - k^2 + 1 \right\}.$$

When $k = 2$, $D_1^{n,2}$ would be defined by $2n - 3$ polynomials. The biggest minor is $[n-1, n]$ and has rank $2n - 3$. For each $\ell = 1, 2, \ldots, 2n - 3$, we clearly have

$$\eta_\ell(X) = \sum_{1 \leq i_1 < i_2 \leq n : i_1 + i_2 = \ell + 2} [i_1, i_2].$$

Every 2-minor of $X$ is a summand of some $\eta_\ell(X)$.

When $k = 3$, $D_2^{n,3}$ can be defined by $3n - 8$ polynomials of the form $\eta_\ell(X)$. For instance, when $n = 6$, the partial ordering on $\Gamma(X)$ is shown in the following diagram:
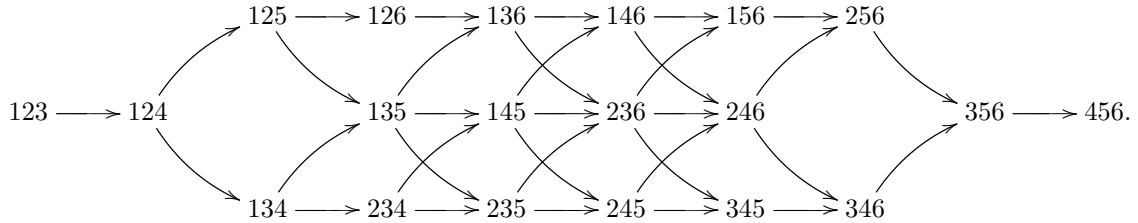


In the above, an arrow points to a bigger minor. Clearly, we have the expressions

$$\eta_1(X) = [1, 2, 3], \quad \eta_2(X) = [1, 2, 4], \quad \eta_3(X) = [1, 2, 5] + [1, 3, 4],$$

$$\eta_4(X) = [1, 2, 6] + [1, 3, 5] + [2, 3, 4], \quad \eta_5(X) = [1, 3, 6] + [1, 4, 5] + [2, 3, 5],$$

$$\eta_6(X) = [1, 4, 6] + [2, 3, 6] + [2, 4, 5], \quad \eta_7(X) = [1, 5, 6] + [2, 4, 6] + [3, 4, 5],$$

$$\eta_8(X) = [2, 5, 6] + [3, 4, 6], \quad \eta_9(X) = [3, 5, 6], \quad \eta_{10}(X) = [4, 5, 6].$$

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Every above $\eta_i(X)$ has degree 3. Note that the summands $[i_1, i_2, i_3]$ from the same $\eta_i(X)$ have a constant summation $i_1 + i_2 + i_3$. Thus, for each $\ell = 1, \ldots, 3n - 8$, we have

$$\eta_\ell(X) = \sum_{1 \leq i_1 < i_2 < i_3 \leq n: \, i_1 + i_2 + i_3 = \ell + 5} [i_1, i_2, i_3].$$

When $k > 3$ is general, $D_{k-1}^{n,k}$ can be defined by $nk - k^2 + 1$ polynomials of the form $\eta_\ell(X)$. For each $\ell = 1, 2, \ldots, nk - k^2 + 1$, we similarly have the expression

$$\eta_\ell(X) = \sum_{1 \leq i_1 < \cdots < i_k \leq n: \, i_1 + \cdots + i_k = \ell + \binom{k+1}{2} - 1} [i_1, \ldots, i_k].$$

## 2.2   The exact Jacobian SDP relaxation

For every $J = \{j_1, \ldots, j_k\} \subset [m_2]$ with $k \leq m - m_1$, by applying formula (2.3), let

$$\eta_1^J, \ldots, \eta_{len(J)}^J \quad \left( len(J) := n(m_1 + k + 1) - (m_1 + k + 1)^2 + 1 \right)$$

be the set of defining polynomials for the determinantal variety $G_J$ defined in (2.2) of the Jacobian of $(f, h, g_J)$ being singular. For each $i = 1, \ldots, len(J)$, define

$$\varphi_i^J(x) = \eta_i^J \left( B^J(x) \right) \cdot \prod_{j \in J^c} g_j(x), \quad \text{where} \quad J^c = [m_2] \backslash J. \tag{2.4}$$

Using the product $\prod_{j \in J^c} g_j(x)$ in the above is motivated by a characterization of critical points in [15]. For simplicity, list all possible $\varphi_i^J$ in (2.4) sequentially as

$$\varphi_1, \varphi_2, \ldots, \varphi_r, \quad \text{where} \quad r = \sum_{J \subset [m_2], |J| \leq m - m_1} len(J). \tag{2.5}$$

Now define the variety

$$W = \left\{ x \in \mathbb{C}^n : h_1(x) = \cdots = h_{m_1}(x) = \varphi_1(x) = \cdots = \varphi_r(x) = 0 \right\}. \tag{2.6}$$

If the minimum $f_{min}$ of (1.1) is achieved at a KKT point, then (1.1) is equivalent to

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & h_i(x) = 0 \, (1 \leq i \leq m_1), \, \varphi_j(x) = 0 \, (1 \leq j \leq r), \\ & g_\nu(x) \geq 0, \, \forall \nu \in \{0, 1\}^{m_2}. \end{aligned} \tag{2.7}$$

In the above, each $g_\nu = g_1^{\nu_1} \cdots g_{m_2}^{\nu_{m_2}}$. Let $f^*$ be the minimum of (2.7). If (1.1) has a minimizer that is a KKT point, then $f_{min} = f^*$. If (1.1) does not have a minimizer (i.e., $f_{min}$ is not achievable in (1.1)), or if (1.1) has one or several minimizers but none of them is a KKT point, then we might have $f_{min} < f^*$ (e.g., for $f(x) = x_1^2 + (1 - x_1 x_2)^2$ and $S = \mathbb{R}^2$, $f_{min} = 0 < f^* = 1$ [18]). But in any case the minimum $f^*$ is always finite as will be shown by Theorem 2.3, while $f_{min}$ might not.

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To construct an SDP relaxation for (2.7), we need to define localizing moment matrices. Let $q(x)$ be a polynomial with $\deg(q) \leq 2N$. Define symmetric matrices $A_\alpha^{(N)}$ such that

$$q(x)[x]_d[x]_d^T = \sum_{\alpha \in \mathbb{N}^n : |\alpha| \leq 2N} A_\alpha^{(N)} x^\alpha, \quad \text{where} \quad d = N - \lceil \deg(q)/2 \rceil.$$

Then the $N$-th order localizing moment matrix of $q$ is defined as

$$L_q^{(N)}(y) = \sum_{\alpha \in \mathbb{N}^n : |\alpha| \leq 2N} A_\alpha^{(N)} y_\alpha.$$

Here $y$ is a moment vector indexed by $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2N$. Moreover, denote

$$L_f(y) = \sum_{\alpha \in \mathbb{N}^n : |\alpha| \leq \deg(f)} f_\alpha y_\alpha \quad \text{for} \quad f(x) = \sum_{\alpha \in \mathbb{N}^n : |\alpha| \leq \deg(f)} f_\alpha x^\alpha.$$

The $N$-th order Lasserre's relaxation for (2.7) is the SDP

$$
\begin{aligned}
f_N^{(1)} := \min \quad & L_f(y) \\
\text{s.t.} \quad & L_{h_i}^{(N)}(y) = 0 \,(1 \leq i \leq m_1),\ L_{\varphi_j}^{(N)}(y) = 0 \,(1 \leq j \leq r), \\
& L_{g_\nu}^{(N)}(y) \succeq 0,\ \forall \nu \in \{0,1\}^{m_2},\ y_0 = 1.
\end{aligned}
\tag{2.8}
$$

Compared to Schmüdgen type Lasserre's relaxation (1.3), the number of new constraints in (2.8) is $r = O\left(2^{m_2} \cdot n \cdot (m_1 + m_2)\right)$ by (2.5). That is, $r$ is of linear order in $n m_1$ for fixed $m_2$, but is exponential in $m_2$. So, when $m_2$ is small or moderately large, (2.8) is practical; but for big $m_2$, (2.8) becomes more difficult to solve numerically. Now we present the dual of (2.8). Define the truncated preordering $P^{(N)}$ generated by $g_j$ as

$$P^{(N)} = \left\{ \sum_{\nu \in \{0,1\}^{m_2}} \sigma_\nu g_\nu \,\middle|\, \deg(\sigma_\nu g_\nu) \leq 2N \right\}, \tag{2.9}$$

and the truncated ideal $I^{(N)}$ generated by $h_i$ and $\varphi_j$ as

$$I^{(N)} = \left\{ \sum_{i=1}^{m_1} p_i h_i + \sum_{j=1}^{r} q_j \varphi_j \,\middle|\, \begin{matrix} \deg(p_i h_i) \leq 2N & \forall i \\ deg(q_j \varphi_j) \leq 2N & \forall j \end{matrix} \right\}. \tag{2.10}$$

Then, as shown in Lasserre [16], the dual of (2.8) is the following SOS relaxation for (2.7):

$$
\begin{aligned}
f_N^{(2)} := \max \quad & \gamma \\
\text{s.t.} \quad & f(x) - \gamma \in I^{(N)} + P^{(N)}.
\end{aligned}
\tag{2.11}
$$

Note the relaxation (2.11) is stronger than (1.3). Then, by weak duality, we have

$$f_N^{(2)} \leq f_N^{(1)} \leq f^*. \tag{2.12}$$

We are going to show that when $N$ is big enough, (2.8) is an exact SDP relaxation for (2.7), i.e., $f_N^{(2)} = f_N^{(1)} = f^*$. For this purpose, we need the following assumption.

**Assumption 2.2.** *(i) $m_1 \leq n$. (ii) For any $u \in S$, at most $n - m_1$ of $g_1(u), \ldots, g_{m_2}(u)$ vanish. (iii) For every $J = \{j_1, \ldots, j_k\} \subset [m_2]$ with $k \leq n - m_1$, the variety $V(h, g_J) = \{x \in \mathbb{C}^n : h(x) = 0, \; g_J(x) = 0\}$ is nonsingular (its Jacobian has full rank on $V(h, g_J)$).*

**Theorem 2.3.** *Suppose Assumption 2.2 holds. Let $f^*$ be the minimum of (2.7). Then $f^* > -\infty$ and there exists $N^* \in \mathbb{N}$ such that $f_N^{(1)} = f_N^{(2)} = f^*$ for all $N \geq N^*$. Furthermore, if the minimum $f_{min}$ of (1.1) is achievable, then $f_N^{(1)} = f_N^{(2)} = f_{min}$ for all $N \geq N^*$.*

The proof of Theorem 2.3 is based on a new kind of Positivstellensatz using Jacobians of the objective and constraining polynomials: there exists an integer $N^* > 0$, which only depends on polynomials $f, h_i, g_j$, such that for all $\epsilon > 0$

$$f(x) - f^* + \epsilon \quad \in \quad I^{(N^*)} + P^{(N^*)}.$$

Note that the order $N^*$ in the above is independent of $\epsilon$. This new Positivstellensatz is given by Theorem 3.4 in the next section. Theorem 2.3 as well as this Positivstellensatz will be proved in Section 3. We would like to remark that in Theorem 2.3, for $N > N^*$, the optimal value of (2.11) might not be achievable (e.g., see Example 5.1), while the minimum of (2.8) is always achievable if (1.1) has a minimizer (e.g., $[x^*]_{2N}$ is one for any optimizer $x^*$ of (1.1)).

When the feasible set $S$ of (1.1) is compact, the minimum $f_{min}$ is always achievable. Thus, Theorem 2.3 implies the following.

**Corollary 2.4.** *Suppose Assumption 2.2 holds. If $S$ is compact, then $f_N^{(1)} = f_N^{(2)} = f_{min}$ for all $N$ big enough.*

A practical issue in applications is how to identify whether (2.8) is exact for a given $N$. This would be possible by applying the flat-extension condition (FEC) [6]. Let $y^*$ be a minimizer of (2.8). We say $y^*$ satisfies FEC if

$$\operatorname{rank} L_{g_0}^{(N)}(y^*) \quad = \quad \operatorname{rank} L_{g_0}^{(N-d_S)}(y^*), \quad \text{where}$$

$$d_S \quad = \quad \max_{i \in [m_1], j \in [r], \nu \in \{0,1\}^{m_2}} \left\{ \lceil \deg(h_i)/2 \rceil, \lceil \deg(\varphi_j)/2 \rceil, \lceil \deg(g_\nu)/2 \rceil \right\}.$$

Note that $g_0 \equiv 1$ and $L_{g_0}^{(N)}(y^*)$ reduces to an $N$-th order moment matrix. When FEC holds, (2.8) is exact for (1.1), and a finite set of global minimizers would be extracted from $y^*$. We refer to [13] for a numerical method on how to do this. A very nice software for solving SDP relaxations from polynomial optimization is *GloptiPoly 3* [14] which also provides routines for finding minimizers if FEC holds.

Now we discuss how general the conditions of Theorem 2.3 are. Define

$$B_d(S) = \left\{ f \in \mathbb{R}[x]_{\leq d} : \inf_{u \in S} f(u) > -\infty \right\}.$$

Clearly, $B_d(S)$ is convex and has nonempty interior. Define the projectivization of $S$ as

$$S^{prj} = \left\{ \tilde{x} \in \mathbb{R}^{n+1} : \tilde{h}_1(\tilde{x}) = \cdots = \tilde{h}_{m_1}(\tilde{x}) = 0, \quad \tilde{g}_1(\tilde{x}) \geq 0, \ldots, \tilde{g}_{m_2}(\tilde{x}) \geq 0 \right\}. \tag{2.13}$$

Here $\tilde{p}$ denotes the homogenization of $p$, and $\tilde{x} = (x_0, x_1, \ldots, x_n)$, i.e., $\tilde{p}(\tilde{x}) = x_0^{\deg(p)} p(x/x_0)$. We say $S$ is closed at $\infty$ if

$$S^{prj} \cap \{x_0 \geq 0\} = \text{closure}\left(S^{prj} \cap \{x_0 > 0\}\right).$$

Under some generic conditions, Assumption 2.2 holds and the minimum $f_{min}$ of (1.1) is achievable. These conditions are expressed as non-vanishing of the so-called *resultants Res* or *discriminants* $\Delta$, which are polynomial in the coefficients of $f, h_i, g_j$. We refer to Appendix for a short introduction about *Res* and $\Delta$.

**Theorem 2.5.** *Let $f, h_i, g_j$ be the polynomials in (1.1), and $S$ be the feasible set.*

(a) *If $m_1 > n$ and $Res(h_{i_1}, \ldots, h_{i_{n+1}}) \neq 0$ for some $\{i_1, \ldots, i_{n+1}\}$, then $S = \emptyset$.*

(b) *If $m_1 \leq n$ and for every $\{j_1, \ldots, j_{n-m_1+1}\} \subset [m_2]$*

$$Res(h_1, \ldots, h_{m_1}, g_{j_1}, \ldots, g_{j_{n-m_1+1}}) \neq 0,$$

*then item (ii) of Assumption 2.2 holds.*

(c) *If $m_1 \leq n$ and for every $\{j_1, \ldots, j_k\} \subset [m_2]$ with $k \leq n - m_1$*

$$\Delta(h_1, \ldots, h_{m_1}, g_{j_1}, \ldots, g_{j_k}) \neq 0,$$

*then item (iii) of Assumption 2.2 holds.*

(d) *Suppose $S$ is closed at $\infty$ and $f \in B_d(S)$. If the resultant of any $n$ of $h_i^{hom}$, $g_j^{hom}$ is nonzero (only when $m_1 + m_2 \geq n$), and for every $\{j_1, \ldots, j_k\}$ with $k \leq n - m_1 - 1$*

$$\Delta(f^{hom}, h_1^{hom}, \ldots, h_{m_1}^{hom}, g_{j_1}^{hom}, \ldots, g_{j_k}^{hom}) \neq 0,$$

*then there exists $v \in S$ such that $f_{min} = f(v)$. Here $p^{hom}$ denotes $p$'s homogeneous part of the highest degree.*

(e) *If $f \in B_d(\mathbb{R}^n)$ and $\Delta(f^{hom}) \neq 0$, then the minimum of $f(x)$ in $\mathbb{R}^n$ is achievable.*

Theorem 2.5 will be proved in Section 3. Now we consider the special case of (1.1) having no constraints. If $f_{min} > -\infty$ is achievable, then (1.1) is equivalent to (1.4). The item (e) of Theorem 2.5 tells us that this is generically true. The gradient SOS relaxation for solving (1.4) proposed in [18] is a special case of (2.11). The following is an immediate consequence of Theorem 2.3 and item (e) of Theorem 2.5.

**Corollary 2.6.** *If $S = \mathbb{R}^n$, $f(x)$ has minimum $f_{min} > -\infty$, and $\Delta(f^{hom}) \neq 0$, then the optimal values of (2.8) and (2.11) are equal to $f_{min}$ if $N$ is big enough.*

Corollary 2.6 is stronger than Theorem 10 of [18], where the exactness of gradient SOS relaxation for a finite order $N$ is only shown when the gradient ideal is radical.

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# 3 Proof of exactness and genericity

This section proves Theorems 2.3 and 2.5. First, we give some lemmas that are crucially used in the proofs.

**Lemma 3.1.** *Let $K$ be the variety defined by the KKT conditions*

$$K = \left\{ (x, \lambda, \mu) \in \mathbb{C}^{n+m_1+m_2} \,\middle|\, \begin{array}{c} \nabla f(x) = \sum_{i=1}^{m_1} \lambda_i \nabla h_i(x) + \sum_{j=1}^{m_2} \mu_j \nabla g_j(x) \\ h_i(x) = \mu_j g_j(x) = 0, \, \forall\, (i, j) \in [m_1] \times [m_2] \end{array} \right\}. \tag{3.1}$$

*If Assumption 2.2 holds, then $W = K_x$ where*

$$K_x = \{ x \in \mathbb{C}^n : (x, \lambda, \mu) \in K \text{ for some } \lambda, \mu \}.$$

*Proof.* First, we prove $W \subset K_x$. Choose an arbitrary $u \in W$. Let $J = \{ j \in [m_2] : g_j(u) = 0 \}$ and $k = |J|$. By Assumption 2.2, $m_1 + k \le n$. Recall from (2.2) that

$$B^J(x) = \begin{bmatrix} \nabla f(x) & \nabla h(x) & \nabla g_J(x) \end{bmatrix}.$$

**Case $m_1 + k = n$** By Assumption 2.2, the matrix $H(u) = \begin{bmatrix} \nabla h(u) & \nabla g_J(u) \end{bmatrix}$ is nonsingular. Note that $H(u)$ is now a square matrix. So, $H(u)$ is invertible, and there exist $\lambda_i$ and $\mu_j (j \in J)$ such that

$$\nabla f(u) = \sum_{i=1}^{m_1} \lambda_i \nabla h_i(u) + \sum_{j \in J} \mu_j \nabla g_j(u). \tag{3.2}$$

Define $\mu_j = 0$ for $j \notin J$, then we have $u \in K_x$.

**Case $m_1 + k < n$** By the construction of polynomials $\varphi_i(x)$ in (2.5), some of them are

$$\varphi_i^J(x) := \eta_i(B^J(x)) \cdot \prod_{j \in J^c} g_j(x), \quad i = 1, \ldots, n(m_1 + k + 1) - (m_1 + k + 1)^2 + 1.$$

So the equations $\varphi_i(u) = 0$ imply every above $\varphi_i^J(u) = 0$ (see its definition in (2.4)). Hence $B^J(u)$ is singular. By Assumption 2.2, the matrix $H(u)$ is nonsingular. So there exist $\lambda_i$ and $\mu_j (j \in J)$ satisfying (3.2). Define $\mu_j = 0$ for $j \notin J$, then we also have $u \in K_x$.

Second, we prove $K_x \subset W$. Choose an arbitrary $u \in K_x$ with $(u, \lambda, \mu) \in K$. Let $I = \{ j \in [m_2] : g_j(u) = 0 \}$. If $I = \emptyset$, then $\mu = 0$, and $\begin{bmatrix} \nabla f(u) & \nabla h(u) \end{bmatrix}$ and $B^J(u)$ are both singular, which implies all $\varphi_i(u) = 0$ and $u \in W$. If $I \ne \emptyset$, write $I = \{ i_1, \ldots, i_t \}$. Let $J = \{ j_1, \ldots, j_k \} \subset [m_2]$ be an arbitrary index set with $m_1 + k \le m$.

**Case $I \nsubseteq J$** At least one $j \in J^c$ belongs to $I$. By choice of $I$, we know from (2.4)

$$\varphi_i^J(u) = \eta_i(B^J(u)) \cdot \prod_{j \in J^c} g_j(u) = 0.$$

**Case $I \subseteq J$** Then $\mu_j = 0$ for all $j \in J^c$. By definition of $K$, the matrix $B^J(u)$ must be singular. All polynomials $\varphi_i^J(x)$ vanish at $u$ by their construction.

Combining the above two cases, we know all $\varphi_i^J(x)$ vanish at $u$, that is, $\varphi_1(u) = \cdots = \varphi_r(u) = 0$. So $u \in W$. $\qquad \square$

**Lemma 3.2.** *Suppose Assumption 2.2 holds. Let $W$ be defined in (2.6), and $T = \{x \in \mathbb{R}^n : g_j(x) \geq 0, j = 1, \ldots, m_2\}$. Then there exist disjoint subvarieties $W_0, W_1, \ldots, W_r$ of $W$ and distinct $v_1, \ldots, v_r \in \mathbb{R}$ such that*

$$W = W_0 \cup W_1 \cup \cdots \cup W_r, \quad W_0 \cap T = \emptyset, \quad W_i \cap T \neq \emptyset, i = 1, \ldots, r,$$

*and $f(x)$ is constantly equal to $v_i$ on $W_i$ for $i = 1, \ldots, r$.*

*Proof.* Let $K = K_1 \cup \cdots \cup K_r$ be a decomposition of irreducible varieties. Then $f(x)$ is equaling a constant $v_i$ on each $K_i$, as shown by Lemma 3.3 in [7]. By grouping all $K_i$ on which $v_i$ are same into a single variety, we can assume all $v_i$ are distinct. Let $\widehat{W_i}$ be the projection of $K_i$ into $x$-space, then by Lemma 3.1 we get

$$W = \widehat{W_1} \cup \cdots \cup \widehat{W_r}.$$

Let $W_i = Zar(\widehat{W_i})$. Applying Zariski closure in the above gives

$$W = Zar(W) = W_1 \cup \cdots \cup W_r.$$

Note that $f(x)$ still achieves a constant value on each $W_i$. Group all $W_j$ for which $W_j \cap T = \emptyset$ into a single variety $W_0$ (if every $W_j \cap T \neq \emptyset$ we set $W_0 = \emptyset$). For convenience, we still write the resulting decomposition as $W = W_0 \cup W_1 \cup \cdots \cup W_r$. Clearly, $W_0 \cap T = \emptyset$, and for $i > 0$ the values $v_i$ are real and distinct (because $\emptyset \neq W_i \cap T \subset \mathbb{R}^n$ and $f(x)$ has real coefficients). Since $f(x)$ achieves distinct values on different $W_i$, we know $W_i$ must be disjoint from each other. Therefore, we get a desired decomposition for $W$. $\qquad\square$

**Lemma 3.3.** *Let $I_0, I_1, \ldots, I_k$ be ideals of $\mathbb{R}[x]$ such that $V(I_i) \cap V(I_j) = \emptyset$ for distinct $i, j$, and $I = I_0 \cap I_1 \cap \cdots \cap I_k$. Then there exist $a_0, a_1, \ldots, a_k \in \mathbb{R}[x]$ satisfying*

$$a_0^2 + \cdots + a_k^2 - 1 \in I, \quad a_i \in \bigcap_{i \neq j \in \{0, \ldots, k\}} I_j.$$

*Proof.* We prove by induction. When $k = 1$, by Theorem A.2, there exist $p \in I_0, q \in I_1$ such that $p + q = 1$. Then $a_0 = p, a_1 = q$ satisfy the lemma.

Suppose the lemma is true for $k = t$. We prove it is also true for $k = t + 1$. Let $J = I_0 \cap \cdots \cap I_t$. By induction, there exist $b_0, \ldots, b_t \in \mathbb{R}[x]$ such that

$$b_0^2 + \cdots + b_t^2 - 1 \in J, \quad b_i \in \bigcap_{i \neq j \in \{0, \ldots, t\}} I_j, \quad i = 0, \ldots, t.$$

Since $V(I_{t+1})$ is disjoint from $V(J) = V(I_0) \cup \cdots \cup V(I_t)$, by Theorem A.2, there exist $p \in I_{t+1}$ and $q \in J$ such that $p + q = 1$. Let $a_i = b_i p$ for $i = 0, \ldots, t$ and $a_{t+1} = q$. Then

$$a_i \in \bigcap_{i \neq j \in \{0, \ldots, t+1\}} I_j, \quad i = 0, \ldots, t + 1.$$

Since $(p + q)^2 = 1, I = I_{t+1} \cap J$, we have $pq \in I$, $(b_0^2 + \cdots + b_t^2 - 1)p^2 \in I$, and

$$a_0^2 + a_1^2 + \cdots + a_{t+1}^2 - 1 = (b_0^2 + \cdots + b_t^2 - 1)p^2 - 2pq \in I,$$

which completes the proof. $\qquad\square$

**Theorem 3.4.** *Suppose Assumption 2.2 holds and let $f^*$ be the minimum of (2.7). Then $f^* > -\infty$ and there exists $N^* \in \mathbb{N}$ such that for all $\epsilon > 0$*

$$f(x) - f^* + \epsilon \in I^{(N^*)} + P^{(N^*)}. \tag{3.3}$$

*Proof.* Note that the feasible set of (2.7) is contained in the variety $W$ defined by (2.6). Decompose $W$ as in Lemma 3.2. Thus, $f(x)$ achieves finitely many values on $W$ and

$$f^* = \min\{v_1, \ldots, v_r\} > -\infty.$$

So, we can generally assume $f^* = 0$. Reorder $W_i$ such that $v_1 > v_2 > \cdots > v_r = 0$. The ideal

$$I_W \quad = \quad \langle h_1, \ldots, h_{m_1}, \varphi_1, \ldots, \varphi_r \rangle \tag{3.4}$$

has a primary decomposition (see Sturmfels [28, Chapter 5])

$$I_W = E_0 \cap E_1 \cap \cdots \cap E_r$$

such that each ideal $E_i \subset \mathbb{R}[x]$ has variety $W_i = V(E_i)$.

When $i = 0$, we have $V_\mathbb{R}(E_0) \cap T = \emptyset$ ($T$ is defined in Lemma 3.2). By Theorem A.3, there exist SOS polynomials $\tau_\nu$ satisfying

$$-1 \equiv \sum_{\nu \in \{0,1\}^{m_2}} \tau_\nu \cdot g_\nu(x) \quad mod \quad E_0.$$

Thus, from $f = \frac{1}{4}(f+1)^2 - \frac{1}{4}(f-1)^2$, we have

$$f \equiv \frac{1}{4}\left\{ (f+1)^2 + (f-1)^2 \sum_{\nu \in \{0,1\}^{m_2}} \tau_\nu \cdot g_\nu \right\} \quad mod \quad E_0$$

$$\equiv \sum_{\nu \in \{0,1\}^{m_2}} \widehat{\tau}_\nu \cdot g_\nu \quad mod \quad E_0$$

for certain SOS polynomials $\widehat{\tau}_\nu$. Let

$$\sigma_0 = \epsilon + \sum_{\nu \in \{0,1\}^{m_2}} \widehat{\tau}_\nu \cdot g_\nu.$$

Clearly, if $N_0 > 0$ is big enough, then $\sigma_0 \in P^{(N_0)}$ for all $\epsilon > 0$. Let $q_0 = f + \epsilon - \sigma_0 \in E_0$, which is independent of $\epsilon$.

For each $i = 1, \ldots, r-1$, $v_i > 0$ and $v_i^{-1} f(x) - 1$ vanishes on $W_i$. By Theorem A.1, there exists $k_i > 0$ such that $(v_i^{-1} f(x) - 1)^{k_i} \in E_i$. Thus, it holds that

$$s_i(x) := \sqrt{v_i}\left(1 + \left(v_i^{-1} f(x) - 1\right)\right)^{1/2} \equiv \sqrt{v_i} \sum_{j=0}^{k_i - 1} \binom{1/2}{j}(v_i^{-1} f(x) - 1)^j \quad mod \quad E_i.$$

Let $\sigma_i = s_i(x)^2 + \epsilon$, and $q_i = f + \epsilon - \sigma_i \in E_i$, which is also independent of $\epsilon > 0$.

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When $i = r$, $v_r = 0$ and $f(x)$ vanishes on $W_r$. By Theorem A.1, there exists $k_r > 0$ such that $f(x)^{k_r} \in E_r$. Thus we obtain that

$$s_r(x) := \sqrt{\epsilon} \left(1 + \epsilon^{-1}f(x)\right)^{1/2} \equiv \sqrt{\epsilon} \sum_{j=0}^{k_r-1} \binom{1/2}{j} \epsilon^{-j} f(x)^j \mod E_r.$$

Let $\sigma_r = s_r(x)^2$, and $q_r = f + \epsilon - \sigma_r \in E_r$. Clearly, we have

$$q_r(x) = \sum_{j=0}^{k_r-2} c_j(\epsilon) f(x)^{k_r+j}$$

for some real scalars $c_j(\epsilon)$. Note each $f(x)^{k_r+j} \in E_r$.

Applying Lemma 3.3 to $E_0, E_1, \ldots, E_r$, we can find $a_0, \ldots, a_r \in \mathbb{R}[x]$ satisfying

$$a_0^2 + \cdots + a_r^2 - 1 \in I_W, \quad a_i \in \bigcap_{i \neq j \in \{0,1,\ldots,r\}} E_j.$$

Let $\sigma = \sigma_0 a_0^2 + \sigma_1 a_1^2 + \cdots + \sigma_r a_r^2$, then

$$f(x) + \epsilon - \sigma = \sum_{i=0}^{r} (f + \epsilon - \sigma_i)a_i^2 + (f + \epsilon)(1 - a_0^2 - \cdots - a_r^2).$$

Since $q_i = f + \epsilon - \sigma_i \in E_i$, it holds that

$$(f + \epsilon - \sigma_i)a_i^2 \in \bigcap_{j=0}^{r} E_j = I_W.$$

For each $0 \leq i < r$, $q_i$ is independent of $\epsilon$. There exists $N_1 > 0$ such that for all $\epsilon > 0$

$$(f + \epsilon - \sigma_i)a_i^2 \in I^{(N_1)}, \quad i = 0, 1, \ldots, r-1.$$

For $i = r$, $q_r = f + \epsilon - \sigma_r$ depends on $\epsilon$. By the choice of $q_r$, it holds that

$$(f + \epsilon - \sigma_r)a_r^2 = \sum_{j=0}^{k_r-2} c_j(\epsilon) f^{k_r+j} a_r^2.$$

Note each $f^{k_r+j}a_r^2 \in I_W$, since $f^{k_r+j} \in E_r$. So, there exists $N_2 > 0$ such that for all $\epsilon > 0$

$$(f + \epsilon - \sigma_r)a_r^2 \in I^{(N_2)}.$$

Since $1 - a_1^2 - \cdots - a_r^2 \in I_W$, there also exists $N_3 > 0$ such that for all $\epsilon > 0$

$$(f + \epsilon)(1 - a_1^2 - \cdots - a_r^2) \in I^{(N_3)}.$$

Combining the above, we know if $N^*$ is big enough, then $f(x) + \epsilon - \sigma \in I^{(N^*)}$ for all $\epsilon > 0$. From the constructions of $\sigma_i$ and $a_i$, we know their degrees are independent of $\epsilon$. So, $\sigma \in P^{(N^*)}$ for all $\epsilon > 0$ if $N^*$ is big enough, which completes the proof. $\square$

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Theorem 3.4 is a kind of Positivstellensatz of representing $f(x) - f^* + \epsilon$, which is positive on $S$ for all $\epsilon > 0$, by the preordering generated by $g_j$ modulo the ideal $I_W$ in (3.4) of variety $W$. Usually, we can not conclude $f(x) - f^* \in I^{(N^*)} + P^{(N^*)}$ by setting $\epsilon = 0$, because the coefficients of the representing polynomials of $f(x) - f^* + \epsilon$ in $I^{(N^*)} + P^{(N^*)}$ go to infinity as $\epsilon \to 0$ (see $s_r(x)$ in the proof). It is possible that $f(x) - f^* \notin I^{(N)} + P^{(N)}$ for every $N > 0$. Such a counterexample is Example 5.1. However, Theorem 3.4 shows that the degree bound $N^*$ required for representing $f(x) - f^* + \epsilon$ is independent of $\epsilon$. This is a crucial property justifying the exactness of the SDP relaxation (2.8). Now we present its proof below.

*Proof of Theorem 2.3*　By Theorem 3.4, we know $f^* > -\infty$ and there exists $N^* \in \mathbb{N}$ such that for all $\epsilon > 0$

$$f(x) - (f^* - \epsilon) \quad \in \quad I^{(N^*)} + P^{(N^*)}.$$

Since $f_{N^*}^{(1)}, f_{N^*}^{(2)}$ are the optimal values of (2.8) and (2.11) respectively, we know

$$f^* - \epsilon \leq f_{N^*}^{(2)} \leq f_{N^*}^{(1)} \leq f^*.$$

Because $\epsilon > 0$ is arbitrary, the above implies $f_{N^*}^{(1)} = f_{N^*}^{(2)} = f^*$. Since the sequence $\{f_N^{(2)}\}$ is monotonically increasing and every $f_N^{(2)} \leq f_N^{(1)} \leq f^*$ by (2.12), we get $f_N^{(1)} = f_N^{(2)} = f^*$ for all $N \geq N^*$. If the minimum $f_{min}$ of (1.1) is achievable, then there exists $x^* \in S$ such that $f_{min} = f(x^*)$. By Assumption 2.2, we must have $x^* \in W$. So $x^*$ is feasible for (2.7), and $f^* = f_{min}$. Thus, we also have $f_N^{(1)} = f_N^{(2)} = f_{min}$ for all $N \geq N^*$. $\qquad\square$

Last we prove Theorem 2.5 by using the properties of resultants and discriminants described in Appendix.

*Proof of Theorem 2.5*　(a) If $Res(h_{i_1}, \ldots, h_{i_{n+1}}) \neq 0$, then the polynomial system

$$h_{i_1}(x) = \cdots = h_{i_{n+1}}(x) = 0$$

does not have a complex solution. Hence, $V(h) = \emptyset$ and consequently $S = \emptyset$.

(b) For a contradiction, suppose $n - m_1 + 1$ of $g_j$ vanish at $u \in S$, say, $g_{j_1}, \ldots, g_{j_{n-m_1+1}}$. Then the polynomial system

$$h_1(x) = \cdots = h_{m_1}(x) = g_{j_1}(x) = \cdots = g_{j_{n-m_1+1}}(x) = 0$$

has a solution, which contradicts $Res(h_1, \ldots, h_{m_1}, g_{j_1}, \ldots, g_{j_{n-m_1+1}}) \neq 0$.

(c) For every $J = \{j_1, \ldots, j_k\} \subset [m_2]$ with $k \leq n - m_1$, if

$$\Delta(h_1, \ldots, h_{m_1}, g_{j_1}, \ldots, g_{j_k}) \neq 0,$$

then the polynomial system

$$h_1(x) = \cdots = h_{m_1}(x) = g_{j_1}(x) = \cdots = g_{j_k}(x) = 0$$

has no singular solution, i.e., the variety $V(h, g_J)$ is smooth.

(d) Let $f_0(x) = f(x) - f_{min}$. Then $f_0$ lies on the boundary of the set

$$P_d(S) = \Big\{ p \in B_d(S) : p(x) \geq 0 \, \forall \, x \in S \Big\}.$$

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Since $S$ is closed at $\infty$, by Prop. 6.1 of [20], $f_0 \in \partial P_d(S)$ implies

$$0 = \min_{\tilde{x} \in S^{prj}, \|\tilde{x}\|_2 = 1, x_0 \geq 0} \tilde{f}_0(\tilde{x}).$$

Let $\tilde{u} = (u_0, u_1, \ldots, u_n) \neq 0$ be a minimizer of the above, which must exist because the feasible set is compact. We claim that $u_0 \neq 0$. Otherwise, suppose $u_0 = 0$. Then $u = (u_1, \ldots, u_n) \neq 0$ is a minimizer of

$$\begin{aligned} 0 = \min \quad & f^{hom}(x) \\ \text{s.t.} \quad & h_1^{hom}(x) = \cdots = h_{m_1}^{hom}(x) = 0, \\ & g_1^{hom}(x) \geq 0, \ldots, g_{m_2}^{hom}(x) \geq 0. \end{aligned}$$

Let $j_1, \ldots, j_k \in [m_2]$ be the indices of active constraints. By Fritz-John optimality condition (see Sec. 3.3.5 in [1]), there exists $(\lambda_0, \lambda_1, \ldots, \lambda_{m_1}, \mu_1, \ldots, \mu_k) \neq 0$ satisfying

$$\lambda_0 \nabla f^{hom}(u) + \sum_{i=1}^{m_1} \lambda_i \nabla h_i^{hom}(u) + \cdots + \sum_{\ell=1}^{k} \mu_\ell \nabla g_{j_\ell}^{hom}(u) = 0,$$
$$f^{hom}(u) = h_1^{hom}(u) = \cdots = h_{m_1}^{hom}(u) = g_{j_1}^{hom}(u) = \cdots = g_{j_k}^{hom}(u) = 0.$$

Thus, the homogeneous polynomial system

$$f^{hom}(x) = h_1^{hom}(x) = \cdots = h_{m_1}^{hom}(x) = g_{j_1}^{hom}(x) = \cdots = g_{j_k}^{hom}(x) = 0$$

has a nonzero singular solution. Since the resultant of any $n$ of $h_i^{hom}, g_j^{hom}$ is nonzero, we must have $m_1 + k \leq n - 1$. So the discriminant

$$\Delta(f^{hom}, h_1^{hom}, \ldots, h_{m_1}^{hom}, g_{j_1}, \ldots, g_{j_k}^{hom})$$

is defined and must vanish, which is a contradiction. So $u_0 \neq 0$. Let $v = u/u_0$, then $\tilde{u} \in S^{prj}$ implies $v \in S$ and $f(v) - f_{min} = u_0^{-d} \tilde{f}_0(\tilde{u}) = 0$.

Clearly, (e) is true since it is a special case of (d). $\qquad \square$

## 4   Some variations

This section presents some variations of the exact SDP relaxation (2.8) and its dual (2.11).

### 4.1   A refined version based on all maximal minors

An SDP relaxation tighter than (2.8) would be obtained by using all the maximal minors to define the determinantal variety $G_J$ in (2.2), while the number of equations would be significantly larger. For every $J = \{j_1, \ldots, j_k\} \subset [m_2]$ with $m_1 + k \leq m$, let $\tau_1^J, \ldots, \tau_\ell^J$ be all the maximal minors of $B^J(x)$ defined in (2.2). Then define new polynomials

$$\psi_i^J := \tau_i^J \cdot \prod_{j \in J^c} g_j, \quad i = 1, \ldots, \ell. \tag{4.1}$$

List all such possible $\psi_i^J$ as

$$\psi_1, \psi_2, \ldots, \psi_t, \quad \text{where} \quad t = \sum_{J \subset [m_2], |J| \leq m - m_1} \binom{n}{|J| + m_1 + 1}.$$

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Like (2.7), we formulate (1.1) equivalently as

$$\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & h_i(x) = \psi_j(x) = 0,\ i \in [m_1], j \in [t], \\
& g_\nu(x) \geq 0,\ \forall \nu \in \{0,1\}^{m_2}.
\end{aligned} \tag{4.2}$$

The standard $N$-th order Lasserre's relaxation for the above is

$$\begin{aligned}
\min \quad & L_f(y) \\
\text{s.t.} \quad & L_{h_i}^{(N)}(y) = 0, L_{\psi_j}^{(N)}(y) = 0,\ i \in [m_1], j \in [t], \\
& L_{g_\nu}^{(N)}(y) \succeq 0,\ \forall \nu \in \{0,1\}^{m_2},\ y_0 = 1.
\end{aligned} \tag{4.3}$$

Note that every $\varphi_i^J$ in (2.4) is a sum of polynomials like $\psi_i^J$ in (4.1). So the equations $L_{\psi_j}^{(N)}(y) = 0$ in (4.3) implies $L_{\varphi_j}^{(N)}(y) = 0$ in (2.8). Hence, (4.3) is stronger than (2.8). Its dual is an SOS program like (2.11). Theorem 2.3 then implies the following.

**Corollary 4.1.** *Suppose Assumption 2.2 is true, and the minimum $f_{min}$ of (1.1) is achievable. If $N$ is big enough, then the optimal value of (4.3) is equal to $f_{min}$.*

## 4.2 A Putinar type variation without using cross products of $g_j$

If the minimum $f_{min}$ of (1.1) is achieved at a KKT point, then (1.1) is equivalent to

$$\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & h_i(x) = \varphi_j(x) = 0,\ i \in [m_1], j \in [r], \\
& g_1(x) \geq 0, \ldots, g_{m_2}(x) \geq 0.
\end{aligned} \tag{4.4}$$

The standard $N$-th order Lasserre's relaxation for (4.4) is

$$\begin{aligned}
\min \quad & L_f(y) \\
\text{s.t.} \quad & L_{h_i}^{(N)}(y) = 0, L_{\varphi_j}^{(N)}(y) = 0,\ i \in [m_1], j \in [r], \\
& L_{g_i}^{(N)}(y) \succeq 0,\ i = 0, 1, \ldots, m_2,\ y_0 = 1.
\end{aligned} \tag{4.5}$$

The difference between (4.5) and (2.8) is that the cross products of $g_j(x)$ are not used in (4.5), which makes the number of resulting LMIs much smaller. Similar to $P^{(N)}$, define the truncated quadratic module $M^{(N)}$ generated by $g_i$ as

$$M^{(N)} = \left\{ \sum_{i=0}^{m_2} \sigma_i g_i \,\middle|\, \deg(\sigma_i g_i) \leq 2N \right\}. \tag{4.6}$$

The dual of (4.5) would be shown to be the following SOS relaxation for (4.4):

$$\begin{aligned}
\max \quad & \gamma \\
\text{s.t.} \quad & f(x) - \gamma \in I^{(N)} + M^{(N)}.
\end{aligned} \tag{4.7}$$

Clearly, for the same $N$, (4.7) is stronger than the standard Lasserre's relaxation (1.2). To prove (4.5) and (4.7) are exact for some $N$, we need the archimedean condition (AC) for $S$, i.e., there exist $R > 0$, $\phi_1, \ldots, \phi_{m_1} \in \mathbb{R}[x]$ and SOS $s_0, \ldots, s_{m_2} \in \mathbb{R}[x]$ such that

$$R - \|x\|_2^2 = \sum_{i=1}^{m_1} \phi_i h_i + \sum_{j=0}^{m_2} s_j g_j.$$

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**Theorem 4.2.** *Suppose Assumption 2.2 and the archimedean condition hold. If $N$ is big enough, then the optimal values of (4.5) and (4.7) are equal to $f_{min}$.*

To prove Theorem 4.2, we need the following.

**Theorem 4.3.** *Suppose Assumption 2.2 and the archimedean condition hold. Let $f^*$ be the optimal value of (4.4). Then there exists an integer $N^* > 0$ such that for every $\epsilon > 0$*

$$f(x) - f^* + \epsilon \in I^{(N^*)} + M^{(N^*)}. \tag{4.8}$$

*Proof.* The proof is almost same as for Theorem 3.4. We follow the same approach used there. The only difference occurs for the case $i = 0$ and $V_{\mathbb{R}}(E_0) \cap T = \emptyset$. By Theorem A.3, there exist SOS polynomials $\eta_\nu$ satisfying

$$-2 \equiv \sum_{\nu \in \{0,1\}^{m_2}} \eta_\nu \cdot g_\nu \quad mod \quad E_0.$$

Clearly, each $\frac{1}{2^{m_2}} + \eta_\nu \cdot g_1^{\nu_1} \cdots g_{m_2}^{\nu_{m_2}}$ is positive on $S$. Since AC holds, by Putinar's Positivtellensatz (Theorem A.4), there exist SOS polynomials $\theta_{\nu,i}$ such that

$$\frac{1}{2^{m_2}} + \eta_\nu \cdot g_\nu = \sum_{i=0}^{m_2} \theta_{\nu,i} g_i \quad mod \quad \langle h_1, \ldots, h_{m_1} \rangle.$$

Hence, it holds that

$$-1 \equiv \sum_{\nu \in \{0,1\}^{m_2}} \left( \frac{1}{2^{m_2}} + \eta_\nu \cdot g_\nu \right) \quad mod \quad \langle h_1, \ldots, h_{m_1} \rangle + E_0$$

$$\equiv \sum_{i=0}^{m_2} \left( \sum_{\nu \in \{0,1\}^{m_2}} \theta_{\nu,i} \right) g_i \quad mod \quad E_0.$$

The second equivalence above is due to the relation

$$\langle h_1, \ldots, h_{m_1} \rangle \subset I_W \subset E_0.$$

Letting $\tau_i = \sum_{\nu \in \{0,1\}^{m_2}} \theta_{\nu,i}$, which is clearly SOS, we get

$$-1 \equiv \tau_0 + \tau_1 g_1 + \cdots + \tau_{m_2} g_{m_2} \quad mod \quad E_0.$$

The rest of the proof is almost same as for Theorem 3.4. $\qquad\square$

*Proof of Theorem 4.2* For convenience, still let $f_N^{(1)}, f_N^{(2)}$ be the optimal values of (4.5) and (4.7) respectively. From Theorem 4.3, there exists an integer $N^*$ such that for all $\epsilon > 0$

$$f(x) - (f^* - \epsilon) \quad \in \quad I^{(N^*)} + M^{(N^*)}.$$

Like in the proof of Theorem 2.3, we can similarly prove $f_N^{(1)} = f_N^{(2)} = f^*$ for all $N \geq N^*$. Since AC holds, the set $S$ must be compact. So the minimum $f_{min}$ of (1.1) must be achievable. By Assumption 2.2, we know $f^* = f_{min}$, and the proof is complete. $\qquad\square$

## 4.3 A simplified version for inactive constraints

Suppose in (1.1) we are only interested in a minimizer making all the inequality constraints inactive. Consider the problem

$$\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & h_1(x) = \cdots = h_{m_1}(x) = 0, \\
& g_1(x) > 0, \ldots, g_{m_2}(x) > 0.
\end{aligned} \tag{4.9}$$

Let $u$ be a minimizer of (4.9). If $V(h)$ is smooth at $u$, there exist $\lambda_i$ such that

$$\nabla f(u) = \lambda_1 \nabla h_1(u) + \cdots + \lambda_{m_1} \nabla h_{m_1}(u).$$

Thus, $u$ belongs to the determinantal variety

$$G_h = \left\{ x : \operatorname{rank} \begin{bmatrix} \nabla f(x) & \nabla h(x) \end{bmatrix} \leq m_1 \right\}.$$

If $m_1 < n$, let $\phi_1, \ldots, \phi_s$ be a minimum set of defining polynomials for $G_h$ by using formula (2.3). If $m_1 = n$, then $G_h = \mathbb{R}^n$ and we do not need these polynomials; set $s = 0$, and $[s]$ is empty. Then, (4.9) is equivalent to

$$\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & h_i(x) = \phi_j(x) = 0, \ i \in [m_1], j \in [s], \\
& g_1(x) > 0, \ldots, g_{m_2}(x) > 0.
\end{aligned} \tag{4.10}$$

The difference between (4.10) and (2.7) is that the number of new equations in (4.10) is $s = O\left(nm_1\right)$, which is much smaller than $r$ in (2.7). So, (4.10) is preferable to (2.7) when the inequality constraints are all inactive. The $N$-th order Lasserre's relaxation for (4.10) is

$$\begin{aligned}
\min \quad & L_f(y) \\
\text{s.t.} \quad & L_{h_i}^{(N)}(y) = 0, L_{\phi_j}^{(N)}(y) = 0, \ i \in [m_1], j \in [s], \\
& L_{g_j}^{(N)}(y) \succeq 0, \ j = 1, \ldots, m_2, \ y_0 = 1.
\end{aligned} \tag{4.11}$$

A tighter version than the above using cross products of $g_j$ is

$$\begin{aligned}
\min \quad & L_f(y) \\
\text{s.t.} \quad & L_{h_i}^{(N)}(y) = 0, L_{\phi_j}^{(N)}(y) = 0, \ i \in [m_1], j \in [s], \\
& L_{g_\nu}^{(N)}(y) \succeq 0, \ \forall \nu \in \{0, 1\}^{m_2}, \ y_0 = 1.
\end{aligned} \tag{4.12}$$

Define the truncated ideal $J^{(N)}$ generated by $h_i(x)$ and $\phi_j$ as

$$J^{(N)} = \left\{ \sum_{i=1}^{m_1} p_i h_i + \sum_{j=1}^{s} q_j \phi_j \ \middle| \ \begin{array}{ll} \deg(p_i h_i) \leq 2N & \forall i \\ deg(q_j \phi_j) \leq 2N & \forall j \end{array} \right\}.$$

The dual of (4.11) is the SOS relaxation

$$\begin{aligned}
\max \quad & \gamma \\
\text{s.t.} \quad & f(x) - \gamma \in J^{(N)} + M^{(N)}.
\end{aligned} \tag{4.13}$$

The dual of (4.12) is the SOS relaxation

$$\begin{aligned} \max \quad & \gamma \\ \text{s.t.} \quad & f(x) - \gamma \in J^{(N)} + P^{(N)}. \end{aligned} \tag{4.14}$$

The exactness of the above relaxations is summarized as follows.

**Theorem 4.4.** *Suppose the variety $V(h)$ is nonsingular and the minimum $f_{min}$ of (4.9) is achieved at some feasible $u$ with every $g_j(u) > 0$. If $N$ is big enough, then the optimal values of (4.12) and (4.14) are equal to $f_{min}$. If, in addition, the archimedean condition holds for $S$, the optimal values of (4.11) and (4.13) are also equal to $f_{min}$ for $N$ big enough.*

*Proof.* The proof is almost same as for Theorems 2.3 and 4.2. We can first prove a decomposition result like Lemma 3.2, and then prove there exists $N^* > 0$ such that for all $\epsilon > 0$ (like in Theorem 3.4)

$$f(x) - f_{min} + \epsilon \in J^{(N^*)} + P^{(N^*)}.$$

Furthermore, if AC holds, we can similarly prove there exists $N^* > 0$ such that for all $\epsilon > 0$ (like in Theorem. 4.3)

$$f(x) - f_{min} + \epsilon \in J^{(N^*)} + M^{(N^*)}.$$

The rest of the proof is almost same as for Theorems 2.3 and 4.2. Due to its repeating, we omit the details here for cleanness of the paper. $\square$

## 5 Examples

This section presents some examples on how to apply the SDP relaxation (2.8) and its dual (2.11) to solve polynomial optimization problems. The software *GloptiPoly 3* [14] is used to solve (2.8) and (2.11).

First, consider unconstrained polynomial optimization. Then the resulting SOS relaxation (2.11) is reduced to the gradient SOS relaxation in [18], which is a special case of (2.11).

**Example 5.1.** Consider the optimization problem

$$\min_{x \in \mathbb{R}^3} \quad x_1^8 + x_2^8 + x_3^8 + x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2.$$

This example was studied in [18]. Its global minimum is zero. The SDP relaxation (2.8) and its dual (2.11) for this problem are equivalent to gradient SOS relaxations in [18] (there are no constraints). We apply (2.8) of order $N = 4$, and get a lower bound $-9.7 \cdot 10^{-9}$. The resulting SDP (2.8) has a single block of size $35 \times 35$. The minimizer $(0,0)$ is extracted. In [18], it was shown that $f(x)$ is not SOS modulo its gradient ideal $I_{grad}$. But for every $\epsilon > 0$, $f(x) + \epsilon \equiv s_\epsilon(x)$ modulo $I_{grad}$ for some SOS polynomial $s_\epsilon(x)$, whose degree is independent of $\epsilon$ (see equation (10) of [18]). But its coefficients go to infinity as $\epsilon \to 0$. This shows that the optimal value of (2.11) might not be achievable. On the other hand, if (1.1) has a minimizer (say, $x^*$) that is a KKT point (then $x^* \in W$), then its dual problem (2.8) always achieves its optimal value $f_{min}$ for $N$ big enough, e.g., $[x^*]_{2N}$ is a minimizer. Thus, for this example, the minimum of the dual (2.8) is achievable. $\square$

Second, consider polynomial optimization having only equality constraints:

$$\min_{x \in \mathbb{R}^n} \quad f(x) \quad \text{s.t.} \quad h_1(x) = \cdots = h_m(x) = 0. \tag{5.1}$$

When $V(h)$ is nonsingular, its equivalent version (2.7) reduces to

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & h_1(x) = \cdots = h_m(x) = 0, \\
& \sum_{\substack{I \in [n]_{m+1} \\ sum(I) = \ell}} \det_I F(x) = 0, \quad \ell = \binom{m+2}{2}, \ldots, (n - \tfrac{m}{2})(m+1).
\end{aligned} \tag{5.2}
$$

In the above $sum(I)$ denotes the summation of the indices in $I$, $F(x) = \begin{bmatrix} \nabla f(x) & \nabla h(x) \end{bmatrix}$, and $\det_I F(x)$ denotes the maximal minor of $F(x)$ whose row indices are in $I$. When $m \geq n$, there are no minor equations in (5.2).

**Example 5.2.** Consider the optimization problem

$$
\begin{aligned}
\min_{x \in \mathbb{R}^3} \quad & x_1^6 + x_2^6 + x_3^6 + 3x_1^2 x_2^2 x_3^2 - (x_1^2(x_2^4 + x_3^4) + x_2^2(x_3^4 + x_1^4) + x_3^2(x_1^4 + x_2^4)) \\
\text{s.t.} \quad & x_1 + x_2 + x_3 - 1 = 0.
\end{aligned}
$$

The objective is the Robinson polynomial, which is nonnegative everywhere but not SOS [22]. So the minimum $f_{min} = 0$. We apply SDP relaxation (2.8) of order $N = 4$, and get a lower bound $-4.4600 \cdot 10^{-9}$. The resulting SDP (2.8) has a single block of size $35 \times 35$. The minimizer $(1/3, 1/3, 1/3)$ is also extracted. Applying Lasserre's relaxation (1.2) of orders $N = 3, 4, 5, 6, 7$, we get lower bounds respectively

$$-0.0582, \quad -0.0479, \quad -0.0194, \quad -0.0053, \quad -4.8358 \cdot 10^{-5}.$$

We can see that (1.2) is weaker than (2.8). It is not clear whether the sequence of lower bounds returned by (1.2) converges to zero or not, because it is not guaranteed when the feasible set is noncompact, which is the case in this example. The objective $f(x)$ here is not SOS modulo the constraint. Otherwise, suppose there exist polynomials $\sigma(x)$ being SOS and $\phi(x)$ such that

$$f(x) = \sigma(x) + \phi(x)(x_1 + x_2 + x_3 - 1).$$

In the above, replacing every $x_i$ by $x_i/(x_1 + x_2 + x_3)$ gives

$$f(x) = (x_1 + x_2 + x_3)^6 \sigma(x/(x_1 + x_2 + x_3)).$$

So, there exist polynomials $p_1, \ldots, p_k, q_1, \ldots, q_\ell$ such that

$$f(x) = p_1^2 + \cdots + p_k^2 + \frac{q_1^2}{(x_1 + x_2 + x_3)^2} + \cdots + \frac{q_\ell^2}{(x_1 + x_2 + x_3)^{2\ell}}.$$

Since the objective $f(x)$ does not have any pole, every $q_i$ must vanish on the plane $x_1 + x_2 + x_3 = 0$. Thus $q_i = (x_1 + x_2 + x_3)^i w_i$ for some polynomials $w_i$. Hence, we get

$$f(x) = p_1^2 + \cdots + p_r^2 + w_1^2 + \cdots + w_\ell^2$$

is SOS, which is a contradiction. $\square$

Third, consider polynomial optimization having only a single inequality constraint.

$$\min_{x\in\mathbb{R}^n} \quad f(x) \quad \text{s.t.} \quad g(x) \geq 0. \tag{5.3}$$

Its equivalent problem (2.7) becomes

$$
\begin{aligned}
\min_{x\in\mathbb{R}^n} \quad & f(x) \\
\text{s.t.} \quad & g(x)\frac{\partial f(x)}{\partial x_i} = 0, \quad i = 1, \ldots, n, \quad g(x) \geq 0, \\
& \sum_{i+j=\ell}\left(\frac{\partial f(x)}{\partial x_i}\frac{\partial g(x)}{\partial x_j} - \frac{\partial f(x)}{\partial x_j}\frac{\partial g(x)}{\partial x_i}\right) = 0, \quad \ell = 3, \ldots, 2n-1.
\end{aligned}
\tag{5.4}
$$

There are totally $3(n-1)$ new equality constraints.

**Example 5.3.** Consider the optimization problem

$$
\begin{aligned}
\min_{x\in\mathbb{R}^3} \quad & x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2 \\
\text{s.t.} \quad & x_1^2 + x_2^2 + x_3^2 \leq 1.
\end{aligned}
$$

The objective is the Motzkin polynomial which is nonnegative everywhere but not SOS [22]. So its minimum $f_{min} = 0$. We apply SDP relaxation (2.8) of order $N = 4$, and get a lower bound $-1.6948 \cdot 10^{-8}$. The resulting SDP (2.8) has two blocks of sizes $35 \times 35$ and $20 \times 20$. The minimizer $(0, 0, 0)$ is also extracted. Now we apply Lasserre's relaxation (1.2). For orders $N = 4, 5, 6, 7, 8$, (1.2) returns the lower bounds respectively

$$-2.0331 \cdot 10^{-4}, -2.9222 \cdot 10^{-5}, -8.2600 \cdot 10^{-6}, -4.2565 \cdot 10^{-6}, -2.3465 \cdot 10^{-6}.$$

Clearly, (1.2) is weaker than (2.8). The sequence of lower bounds given by (1.2) converges to zero for this example, because the archimedean condition holds. The feasible set has nonempty interior. Hence, for every $N$, there is no duality gap between (1.2) and its dual, and (1.2) has an optimizer. The objective does not belong to the preordering generated by the ball condition. This fact was kindly pointed out to the author by Claus Scheiderer (implied by his proof of Prop. 6.1 in [23], since the objective is a nonnegative but non-SOS form vanishing at origin). Therefore, for every $N$, the optimal value of (1.2) as well as its dual is strictly smaller than the minimum $f_{min}$. □

**Example 5.4.** Consider Example 5.3 but the constraint is the outside of the unit ball:

$$
\begin{aligned}
\min_{x\in\mathbb{R}^3} \quad & x_1^4 x_2^2 + x_1^2 x_2^4 + x_3^6 - 3x_1^2 x_2^2 x_3^2 \\
\text{s.t.} \quad & x_1^2 + x_2^2 + x_3^2 \geq 1.
\end{aligned}
$$

Its minimum is still 0. We apply SDP relaxation (2.8) of order $N = 4$, and get a lower bound $1.7633 \cdot 10^{-9}$ (its sign is not correct due to numerical issues). The resulting SDP (2.8) has two blocks of sizes $35 \times 35$ and $20 \times 20$. Now we compare it with Lasserre's relaxation (1.2). When $N = 4$, (1.2) is not feasible. When $N = 5, 6, 7, 8$, (1.2) returns the following lower bounds respectively

$$-4.8567 \cdot 10^5, \quad -98.4862, \quad -0.7079, \quad -0.0277.$$

So we can see (1.2) is much weaker than (2.8). Again, it is not clear whether the sequence of lower bounds given by (1.2) converges to zero or not, because it is only guaranteed when the feasible set is compact, which is not the case in this example. □

Last, we show some general examples.

**Example 5.5.** Consider the following polynomial optimization

$$\min_{x\in\mathbb{R}^2} \quad x_1^2 + x_2^2$$
$$\text{s.t.} \quad x_2^2 - 1 \geq 0,$$
$$x_1^2 - Mx_1x_2 - 1 \geq 0,$$
$$x_1^2 + Mx_1x_2 - 1 \geq 0.$$

This problem was studied in [7, 12]. Its global minimum is $2 + \frac{1}{2}M(M + \sqrt{M^2 + 4})$. Let $M = 5$ here. Applying (2.8) of order $N = 4$, we get a lower bound $27.9629$ which equals the global minimum, and four global minimizers $(\pm 5.1926, \pm 1.0000)$. The resulting SDP (2.8) has eight blocks whose matrix lengths are $15, 10, 10, 10, 6, 6, 6, 3$ respectively. However, if we apply the Lasserre's relaxation either (1.2) or (1.3), the best lower bound we would obtain is 2, no matter how big the relaxation order $N$ is (see Example 4.5 of [7]). The reason for this is that its feasible set is noncompact, while Lasserre's relaxations (1.2) or (1.3) are only guaranteed to converge for compact sets. $\square$

**Example 5.6.** Consider the polynomial optimization

$$\min_{x\in\mathbb{R}^3} \quad x_1^4x_2^2 + x_2^4x_3^2 + x_3^4x_1^2 - 3x_1^2x_2^2x_3^2$$
$$\text{s.t.} \quad 1 - x_1^2 \geq 0, 1 - x_2^2 \geq 0, 1 - x_3^2 \geq 0.$$

The objective is a nonnegative form being non-SOS [22, Sec. 4c]. Its minimum $f_{min} = 0$. We apply SDP relaxation (2.8) of order $N = 6$, and get a lower bound $-9.0752\cdot 10^{-9}$. A minimizer $(0, 0, 0)$ is also extracted. The resulting SDP (2.8) has eight blocks whose matrix lengths are $84, 56, 56, 56, 35, 35, 35, 20$ respectively. Now we apply Lasserre's relaxation of type (1.3). Let $f_N^{smg}$ be the optimal value of (1.3) for an order $N$, and $f_N^{mom}$ be the optimal value of its dual optimization problem (an analogue of (2.8) by using moments and localizing matrices, see Lasserre [16]). Since the feasible set here has nonempty interior, the dual optimization problem of (1.3) has an interior point, so (1.3) always has an optimizer and there is no duality gap, i.e., $f_N^{smg} = f_N^{mom}$ for every order $N$ (cf. [16]). For $N = 6, 7, 8$, (1.3) returns the lower bounds $f_N^{smg}$ respectively

$$-3.5619 \cdot 10^{-5}, \quad -1.0406 \cdot 10^{-5}, \quad -7.6934 \cdot 10^{-6}.$$

The sequence of lower bounds given by (1.3) converges to zero, since the feasible set is compact. However, the objective does not belong to the preordering generated by the constraints, which is implied by the proof of Prop. 6.1 of [23] (the objective is a nonnegative but non-SOS form vanishing at origin). Because (1.3) has an optimizer for every order $N$, this implies that $f_N^{smg} = f_N^{mom} < f_{min} = 0$ for every $N$. Thus, the relaxation (1.3) and its dual could not be exact for any order $N$. $\square$

# 6 Some conclusions and discussions

This paper proposes an exact type SDP relaxation (2.8) and its dual (2.11) for polynomial optimization (1.1) by using the Jacobian of its defining polynomials. Under some generic

conditions, we showed that the minimum of (1.1) would be found by solving the SDP (2.8) for a finite relaxation order.

A drawback of the proposed relaxation (2.8) and its dual (2.11) is that there are totally $O(2^{m_2} \cdot n \cdot (m_1 + m_2))$ new constraints. This would make the computation very difficult to implement if either $m_1$ or $m_2$ is big. Thus, this method is more interesting theoretically. However, this paper discovers an important fact: it is possible to solve the polynomial optimization (1.1) exactly by a single SDP relaxation, which was not known in the prior existing literature. Currently, it is not clear for the author whether the number of newly introduced constraints would be significantly dropped while the exactness is still wanted. This is an interesting future research topic. On the other hand, the relaxation (2.8) is not too bad in applications. For problems that have only a few constraints, the method would also be efficiently implemented. For instance, in all the examples of Section 5, they are all solved successfully, and the advantages of the method over the prior existing ones are very clear. Thus, this method would also be computationally attractive in such applications.

The results of this paper improve the earlier work [7, 18], where the exactness of gradient or KKT type SOS relaxations for a finite relaxation order is only proved when the gradient or KKT ideal is radical. There are other conditions like boundary hessian condition (BHC) guaranteeing this property, like in [15, 17]. In [17], Marshall showed that the gradient SOS relaxation is also exact for a finite relaxation order by assuming BHC, in unconstrained optimization. In this paper, the exactness of (2.8) and (2.11) for a finite $N$ is proved without the conditions like radicalness or BHC. The only assumptions required are nonsingularity of $S$ and the minimum $f_{min}$ being achievable (the earlier related work also requires this), but they are generically true as shown by Theorem 2.5.

We would like to point out that the KKT type SOS relaxation proposed in [7] using Lagrange multipliers is also exact for a finite order, no matter the KKT ideal is radical or not. This would be proved in a similar way as we did in Section 3. First, we can get a similar decomposition for the KKT variety like Lemma 3.2. Second, we can prove a similar representation for $f(x) - f^* + \epsilon$ like in Theorem 3.4, with degree bounds independent of $\epsilon$. Based on these two steps, we can similarly prove its exactness for a finite relaxation order. Since the proof is almost a repeating of Section 3, we omit it for cleanness of the paper.

The proof of the exactness of (2.8) provides a representation of polynomials that are positive on $S$ through using the preordering of $S$ and the Jacobian of all the involved polynomials. A nice property of this representation is that the degrees of the representing polynomials are independent of the minimum value. This is presented by Theorem 3.4. A similar representation result using the quadratic module of $S$ is given by Theorem 4.3.

An issue that is not addressed by the paper is that the feasible set $S$ has singularities. If a global minimizer $x^*$ of (1.1) is singular on $S$, then the KKT condition might no longer hold, and $x^* \notin W$. In this case, the original optimization (1.1) is not equivalent to (2.7), and the SDP relaxation (2.8) might not give a correct lower bound for $f_{min}$. It is not clear how to handle singularities generally in an efficient way.

Another issue that is not addressed by the paper is the minimum $f_{min}$ of (1.1) is not achievable, which happens only if $S$ is noncompact. For instance, when $S = \mathbb{R}^2$, the polynomial $x_1^2 + (x_1 x_2 - 1)^2$ has minimum 0 but it is not achievable. If applying the relaxation (2.8) for this instance, we would not get a correct lower bound. Generally, this case will not happen, as shown by items (d), (e) of Theorem 2.5. In unconstrained optimization, when

$f_{min}$ is not achievable, excellent approaches are proposed in [9, 10, 26]. It is an interesting future work to generalize them to constrained optimization.

An important question is for what concrete relaxation order $N^*$ the SDP relaxation (2.8) is exact for solving (1.1). No good estimates for $N^*$ in Theorem 2.3 are available currently. Since the original problem (1.1) is NP-hard, any such estimates would be very bad if they exist. This is another interesting future work.

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# A   Some basics in algebraic geometry and real algebra

In this appendix, we give a short review on basic algebraic geometry and real algebra. More details would be found in the books [4, 11].

An ideal $I$ of $\mathbb{R}[x]$ is a subset such that $I \cdot \mathbb{R}[x] \subseteq I$ and $I + I \subseteq I$. Given polynomials $p_1, \ldots, p_m \in \mathbb{R}[x]$, $\langle p_1, \cdots, p_m \rangle$ denotes the smallest ideal containing every $p_i$, which is the set $p_1\mathbb{R}[x] + \cdots + p_m\mathbb{R}[x]$. The ideals in $\mathbb{C}[x]$ are defined similarly. An algebraic variety is a subset of $\mathbb{C}^n$ that are common complex zeros of polynomials in an ideal. Let $I$ be an ideal of $\mathbb{R}[x]$. Define

$$V(I) = \{x \in \mathbb{C}^n : p(x) = 0 \quad \forall\, p \in I\},$$
$$V_{\mathbb{R}}(I) = \{x \in \mathbb{R}^n : p(x) = 0 \quad \forall\, p \in I\}.$$

The $V(I)$ is called an algebraic variety or just a variety, and $V_{\mathbb{R}}(I)$ is called a real algebraic variety or just a real variety. Every subset $T \subset \mathbb{C}^n$ is contained in a variety in $\mathbb{C}^n$. The smallest one containing $T$ is called the *Zariski* closure of $T$, and is denoted by $Zar(T)$. In the Zariski topology on $\mathbb{C}^n$, the varieties are called closed sets, and the complements of varieties are called open sets. A variety $V$ is irreducible if there exist no proper subvarieties $V_1, V_2$ of $V$ such that $V = V_1 \cup V_2$. Every variety is a finite union of irreducible varieties.

**Theorem A.1** (Hilbert's Strong Nullstellensatz). *Let $I \subset \mathbb{R}[x]$ be an ideal. If $p \in \mathbb{R}[x]$ vanishes on $V(I)$, then $p^k \in I$ for some integer $k > 0$.*

If an ideal $I$ has empty variety $V(I)$, then $1 \in I$. This is precisely the Hilbert's weak Nullstellensatz.

**Theorem A.2** (Hilbert's Weak Nullstellensatz). *Let $I \subset \mathbb{R}[x]$ be an ideal. If $V(I) = \emptyset$, then $1 \in I$.*

Now we consider $I$ to be an ideal generated by polynomials having real coefficients. Let $T$ be a basic closed semialgebraic set. There is a certificate for $V_{\mathbb{R}}(I) \cap T = \emptyset$. This is the so-called Positivstellensatz.

**Theorem A.3** (Positivstellensatz, [27]). *Let $I \subset \mathbb{R}[x]$ be an ideal, and $T = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$ be defined by real polynomials $g_i$. If $V_{\mathbb{R}}(I) \cap T = \emptyset$, then there exist SOS polynomials $\sigma_\nu$ such that*

$$-1 \equiv \sum_{\nu \in \{0,1\}^r} \sigma_\nu \cdot g_1^{\nu_1} \cdots g_r^{\nu_r} \quad mod \quad I.$$

**Theorem A.4** (Putinar's Positivstellensatz, [21])**.** *Let $I$ be an ideal of $\mathbb{R}[x]$ and $T = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_r(x) \geq 0\}$ be defined by real polynomials $g_i$. Suppose there exist $R > 0$ and SOS polynomials $s_0(x), \ldots, s_m(x)$ such that (the archimedean condition holds)*

$$R - \|x\|_2^2 \equiv s_0(x) + s_1(x)g_1(x) + \cdots + s_m(x)g_m(x) \quad mod \quad I.$$

*If a polynomial $f(x)$ is positive on $V_{\mathbb{R}}(I) \cap T$, then there exist SOS polynomials $\sigma_i$ such that*

$$f(x) \equiv \sigma_0(x) + \sigma_1(x)g_1(x) + \cdots + \sigma_m(x)g_m(x) \quad mod \quad I.$$

In the following, we review some elementary background about resultants and discriminants. More details would be found in [5, 8, 28].

Let $f_1, \ldots, f_n$ be homogeneous polynomials in $x = (x_1, \ldots, x_n)$. The resultant $Res(f_1, \ldots, f_n)$ is a polynomial in the coefficients of $f_1, \ldots, f_n$ satisfying

$$Res(f_1, \ldots, f_n) = 0 \quad \Longleftrightarrow \quad \exists 0 \neq u \in \mathbb{C}^n, \ f_1(u) = \cdots = f_n(u) = 0.$$

The resultant $Res(f_1, \ldots, f_n)$ is homogeneous, irreducible and has integer coefficients. When $f(x)$ is a single homogeneous polynomial, its discriminant is defined to be

$$\Delta(f) = Res\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Thus, we have the relation

$$\Delta(f) = 0 \quad \Longleftrightarrow \quad \exists 0 \neq u \in \mathbb{C}^n, \ \nabla f(u) = 0.$$

The discriminants and resultants are also defined for inhomogeneous polynomials. Let $f_0, f_1, \ldots, f_n$ be general polynomials in $x = (x_1, \ldots, x_n)$. Their resultant $Res(f_0, f_1, \ldots, f_n)$ is then defined to be $Res(\tilde{f}_0(\tilde{x}), \tilde{f}_1(\tilde{x}), \ldots, \tilde{f}_n(\tilde{x}))$, where each $\tilde{f}_i(\tilde{x}) = x_0^{\deg(f_i)} f(x/x_0)$ is the homogenization of $f_i(x)$. Clearly, if the polynomial system

$$f_0(x) = f_1(x) = \cdots = f_n(x) = 0$$

has a solution in $\mathbb{C}^n$, then the homogeneous system

$$\tilde{f}_0(\tilde{x}) = \tilde{f}_1(\tilde{x}) = \cdots = \tilde{f}_n(\tilde{x}) = 0$$

has a nozero solution in $\mathbb{C}^{n+1}$, and hence $Res(f_0, f_1, \ldots, f_n) = 0$. The reverse is not always true, because the latter homogeneous system might have a solution at infinity $x_0 = 0$. If $f(x)$ is a single nonhomogeneous polynomial, its discriminant is defined similarly as $\Delta(\tilde{f})$.

The discriminants are also defined for several polynomials. More details are in [20, Sec. 3]. Let $f_1(\tilde{x}), \ldots, f_m(\tilde{x})$ be forms in $x = (x_1, \ldots, x_n)$ of degrees $d_1, \ldots, d_m$ respectively, and $m \leq n - 1$. Suppose at least one $d_i > 1$. The discriminant for $f_1, \ldots, f_m$, denoted by $\Delta(f_1, \ldots, f_m)$, is a polynomial in the coefficients of $f_i$ such that

$$\Delta(f_1, \ldots, f_m) = 0$$

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if and only if the polynomial system

$$f_1(x) = \cdots = f_m(x) = 0$$

has a solution $u \neq 0$ such that the matrix $\begin{bmatrix} \nabla f_1(u) & \cdots & \nabla f_m(u) \end{bmatrix}$ does not have full rank. When $m = 1$, $\Delta(f_1, \ldots, f_m)$ reduces to the standard discriminant of a single polynomial.

When $f_1, \ldots, f_m$ are nonhomogeneous polynomials in $x = (x_1, \ldots, x_n)$ and $m \leq n$, the discriminant $\Delta(f_1, \ldots, f_m)$ is then defined to be $\Delta(\tilde{f}_1(\tilde{x}), \ldots, \tilde{f}_m(\tilde{x}))$, where each $\tilde{f}_i(\tilde{x})$ is the homogenization of $f_i(x)$.

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