First Order Conditions for Semidefinite Representations of Convex Sets Defined by Rational or Singular Polynomials *

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Abstract A set is called *semidefinite representable* or *semidefinite programming (SDP) representable* if it equals the projection of a higher dimensional set which is defined by some *Linear Matrix Inequality (LMI)*. This paper discusses the semidefinite representability conditions for convex sets of the form $S_{\mathcal{D}}(f) = \{x \in \mathcal{D} : f(x) \geq 0\}$. Here, $\mathcal{D} = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$ is a convex domain defined by some "nice" concave polynomials $g_i(x)$ (they satisfy certain concavity certificates), and $f(x)$ is a polynomial or rational function. When $f(x)$ is concave over $\mathcal{D}$, we prove that $S_{\mathcal{D}}(f)$ has some explicit semidefinite representations under certain conditions called *preordering concavity* or *q-module concavity*, which are based on the Positivstellensatz certificates for the first order concavity criteria:

$$f(u) + \nabla f(u)^T (x - u) - f(x) \geq 0, \quad \forall\, x, u \in \mathcal{D}.$$

When $f(x)$ is a polynomial or rational function having singularities on the boundary of $S_{\mathcal{D}}(f)$, a *perspective transformation* is introduced to find some explicit semidefinite representations for $S_{\mathcal{D}}(f)$ under certain conditions. In the particular case $n = 2$, if the Laurent expansion of $f(x)$ around one singular point has only two consecutive homogeneous parts, we show that $S_{\mathcal{D}}(f)$ always admits an explicitly constructible semidefinite representation.

Keywords convexity, linear matrix inequality, perspective transformation, polynomial, rational function, singularity, semidefinite programming, sum of squares

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1 Introduction

Semidefinite programming (SDP) [1,12,13,22] is an important convex optimization problem. It has wide applications in combinatorial optimization, control theory and nonconvex polynomial optimization as well as many other areas. There are efficient numerical algorithms and standard packages for solving semidefinite programming. Hence, a fundamental problem in optimization theory is what sets can be represented by semidefinite programming. This paper discusses this problem.

A set S is said to be *Linear Matrix Inequality (LMI) representable* if

$$S = \{x \in \mathbb{R}^n : A_0 + A_1 x_1 + \cdots + A_n x_n \succeq 0\}$$

for some symmetric matrices A_i. Here the notation $X \succeq 0 \, (\succ 0)$ means X is positive semidefinite (definite). The above is then called an LMI representation for S. If S is representable as the projection of

$$\hat{S} = \left\{ (x, u) \in \mathbb{R}^{(n+N)} : A_0 + \sum_{i=1}^{n} A_i x_i + \sum_{j=1}^{N} B_j u_j \succeq 0 \right\} \subset \mathbb{R}^{(n+N)},$$

that is, $S = \left\{ x \in \mathbb{R}^n : \exists u \in \mathbb{R}^n, \ (x, u) \in \hat{S} \right\}$, for some symmetric matrices A_i and B_j, then S is called *semidefinite representable* or *semidefinite programming (SDP) representable*. The lifted LMI above is then called a *semidefinite representation*, *SDP representation* or *lifted LMI representation* for S. Sometimes, we also say $S = \hat{S}$ if S equals the projection of the lift $\hat{S}$.

Nesterov and Nemirovski ([12]), Ben-Tal and Nemirovski ([1]), and Nemirovski ([13]) gave collections of examples of SDP representable sets. Thereby leading to the fundamental question which sets are SDP representable? Obviously, to be SDP representable, S must be convex and semialgebraic. Is this necessary condition also sufficient? What are the sufficient conditions for S to be SDP representable? Note that not every convex semialgebraic set is LMI representable (see Helton and Vinnikov [8]).

Prior work When S is a convex set of the form $\{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$ defined by polynomials $g_i(x)$, there is recent work on the SDP representability of S. Parrilo [15] gave a construction of lifted LMIs using moments and sum of squares techniques, and proved the construction gives an SDP representation in the two dimensional case when the boundary of S is a single rational planar curve of genus zero. Lasserre [11] showed the construction can give arbitrarily accurate approximations to compact S, and the construction gives a lifted LMI for S under some algebraic properties called *S-BDR* or *PP-BDR*, i.e., requiring *almost all* positive affine polynomials on S have certain SOS representations with uniformly bounded degrees. Helton and Nie [6] proved that the convex sets of the form $\{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$ are SDP representable if every $g_i(x)$ is sos-concave ($-\nabla^2 g_i(x) = G_i(x)^T G_i(x)$ for some possibly nonsquare matrix polynomial $G_i(x)$), or every $g_i(x)$ is strictly quasi-concave on S, or a mixture of the both. Later, based on the work [6], Helton and Nie [7] proved a very general result that a compact convex semialgebraic set S is always SDP representable if the boundary of S is nonsingular and has positive curvature. This sufficient condition is not far away from being necessary: the boundary of a convex set has nonnegative curvature when it is nonsingular. So the only unaddressed cases for SDP representability are that the boundary of a convex set has zero curvature somewhere or has some singularities.

Contributions The results in [6,7,11] are more on the theoretical existence of SDP representations. The constructions given there might be too complicated to be useful for computational purposes. These results sometimes need to check conditions of Hessians of defining polynomials, which sometimes are difficult or inconvenient to verify in practice. However, in many applications, people often want explicit and simple semidefinite representations. Thus some "simple" SDP representations and conditions justifying them are favorable in practical applications. All these practical issues motivate this paper. Our contributions come in the following three aspects.

First, there are some convex sets defined by polynomials that are not concave in the whole space $\mathbb{R}^n$ but concave over a convex domain $\mathcal{D} \subset \mathbb{R}^n$. For instance, for convex set $\{x \in \mathbb{R}^2 : x_2 - x_1^3 \geq 0, x_1 \geq 0\}$, the defining polynomial $x_2 - x_1^3$ is not concave when $x_1 < 0$, but is concave over the domain $\mathbb{R}_+ \times \mathbb{R}$. Moreover, this set allows an SDP representation, e.g.,

$$\left\{ (x_1, x_2) : \exists\, u, \begin{bmatrix} x_1 & u \\ u & x_2 \end{bmatrix} \succeq 0, \begin{bmatrix} 1 & x_1 \\ x_1 & u \end{bmatrix} \succeq 0 \right\}.$$

For convex sets given in the form $S_{\mathcal{D}}(f) = \{x \in \mathcal{D} : f(x) \geq 0\}$, where $f(x)$ is a polynomial concave over a convex domain $\mathcal{D}$, we prove some sufficient conditions for semidefinite representability of $S_{\mathcal{D}}(f)$ and give explicit SDP representations. This will be discussed in Section 2.

Second, there are some convex sets defined by rational functions (also called rational polynomials) which are concave over a convex domain $\mathcal{D}$ of $\mathbb{R}^n$. If we redefine them by using polynomials, the concavity of rational functions might not be preserved. For instance, the unbounded convex set

$$\left\{ x \in \mathbb{R}_+^2 : 1 - \frac{1}{x_1 x_2} \geq 0 \right\}$$

is defined by a rational function concave over $\mathbb{R}_+^2$ ($\mathbb{R}_+$ is the set of nonnegative real numbers). This set can be equivalently defined by polynomials

$$\left\{ x \in \mathbb{R}^2 : x_1 x_2 - 1 \geq 0, x_1 \geq 0, x_2 \geq 0 \right\}.$$

But $x_1 x_2 - 1$ is not concave anywhere. The prior results in [6,7] do not imply the SDP representability of this set. However, this set is SDP representable, e.g., it equals

$$\left\{ x \in \mathbb{R}^2 : \begin{bmatrix} x_1 & 1 \\ 1 & x_2 \end{bmatrix} \succeq 0 \right\}.$$

For convex sets given in the form $S_{\mathcal{D}}(f) = \{x \in \mathcal{D} : f(x) \geq 0\}$, where $f(x)$ is a rational function concave over a convex domain $\mathcal{D}$, we prove some sufficient conditions for semidefinite representability of $S_{\mathcal{D}}(f)$ and give explicit SDP representations. This will be discussed in Section 3.

Third, there are some convex sets that are defined by polynomials or rational functions which are singular on the boundary. For instance, the set

$$\left\{ x \in \mathbb{R}^2 : x_1^2 - x_1^3 - x_2^2 \geq 0, x_1 \geq 0 \right\}$$

is convex, and the origin is on the boundary. The polynomial $x_1^2 - x_1^3 - x_2^2$ is singular at the origin, i.e., its gradient vanishes at the origin. The earlier results in [6,7] do not imply the SDP representability of this set. However, this set is the same as

$$\left\{ (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R} : x_1 - x_1^2 - \frac{x_2^2}{x_1} \geq 0 \right\},$$

a convex set defined by a concave rational function over the domain $\mathbb{R}_+ \times \mathbb{R}$. By Schur's complement, we know it can be represented as

$$\left\{ (x_1, x_2) : \begin{bmatrix} x_1 & x_2 & x_1 \\ x_2 & x_1 & 0 \\ x_1 & 0 & 1 \end{bmatrix} \succeq 0 \right\}.$$

It is an LMI representation without projections. The technique of Schur's complement works only for very special concave rational functions, and is usually difficult to be applied for general cases. For singular convex sets of the form $S_{\mathcal{D}}(f) = \{x \in \mathcal{D} : f(x) \geq 0\}$, where $f(x)$ is a polynomial or rational function with singularities on the boundary, we give some sufficient conditions for semidefinite representability of $S_{\mathcal{D}}(f)$ and give explicit SDP representations. In the particular case $n = 2$, we show that $S_{\mathcal{D}}(f)$ always admits an explicitly constructible SDP representation when the Laurent expansion of $f(x)$ around one singular point has only two consecutive homogeneous parts. This will be discussed in Section 4.

In this paper, we always assume $\mathcal{D} = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$ is a convex domain defined by some nice concave polynomials $g_i(x)$. Here "nice" means that they satisfy certain concavity certificates. For instance, a very useful case is $\mathcal{D}$ is a polyhedron. We do not require $\mathcal{D}$ or $S_{\mathcal{D}}(f)$ to be compact, as required by [6,7, 11]. When $f(x)$ is concave over $\mathcal{D}$, the sufficient conditions for SDP representability of $S_{\mathcal{D}}(f)$ proven in this paper are based on some certificates for the first order concavity criteria:

$$f(u) + \nabla f(u)^T (x - u) - f(x) \geq 0, \quad \forall x, u \in \mathcal{D}.$$

Some Positivstellensatz certificates like Putinar's Positivstellensatz [17] or Schmüdgen's Positivstellensatz [20] for the above can be applied to justify some explicitly constructible SDP representations for $S_{\mathcal{D}}(f)$.

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{N}$) denotes the set of real numbers (resp. nonnegative integers). For $\alpha \in \mathbb{N}^n$ and $x \in \mathbb{R}^n$, denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. $B(u, r)$ denotes the ball $\{x \in \mathbb{R}^n : \|x - u\|_2 \leq r\}$. A vector $x \geq 0$ means all its entries are nonnegative. A polynomial $p(x)$ is said to be a sum of squares or sos if there exist finitely many polynomials $q_i(x)$ such that $p(x) = \sum q_i(x)^2$. A matrix polynomial is a matrix whose every entry is a polynomial. A matrix polynomial $H(x)$ is called a sum of squares or sos if there is a matrix polynomial $G(x)$ such that $H(x) = G(x)^T G(x)$.

2 Convex sets defined by polynomials concave over domains

In this section, consider the convex set $S_{\mathcal{D}}(f) = \{x \in \mathcal{D} : f(x) \geq 0\}$ defined by a polynomial $f(x)$. Here $\mathcal{D} = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$ is a convex domain. When $f(x)$ is concave on $\mathcal{D}$, it must hold that

$$-R_f(x, u) := f(u) + \nabla f(u)^T (x - u) - f(x) \geq 0 \quad \forall x, u \in \mathcal{D}.$$

The difference $R_f(x, u)$ is the first order Lagrange remainder.

2.1. q-module convexity and preordering convexity

Now we introduce some types of definitions about convexity/concavity. Define $g_0(x) = 1$. We say $f(x)$ is *q-module convex over* $\mathcal{D}$ if it holds the identity

$$R_f(x, u) = \sum_{i=0}^{m} g_i(x) \left(\sum_{j=0}^{m} g_j(u) \sigma_{ij}(x, u) \right)$$

for some sos polynomials $\sigma_{ij}(x, u)$. Then define $f(x)$ to be *q-module concave over* $\mathcal{D}$ if $-f(x)$ is q-module convex over $\mathcal{D}$. We say $f(x)$ is *preordering convex over* $\mathcal{D}$ if it holds the identity

$$R_f(x, u) = \sum_{\nu \in \{0,1\}^m} g_1^{\nu_1}(x) \cdots g_m^{\nu_m}(x) \left(\sum_{\mu \in \{0,1\}^m} g_1^{\mu_1}(u) \cdots g_m^{\mu_m}(u) \sigma_{\nu,\mu}(x, u) \right)$$

for some sos polynomials $\sigma_{\nu,\mu}(x, u)$. Similarly, $f(x)$ is called *preordering concave over* $\mathcal{D}$ if $-f(x)$ is preordering convex over $\mathcal{D}$. The q-module convexity is based on Putinar's Positivstellensatz [17], while the preordering convexity is based on Schmüdgen's Positivstellensatz [20]. Obviously, the q-module convexity implies preordering convexity, which then implies the convexity, but the converse might not be true.

We remark that the defining polynomials $g_i(x)$ are not unique for the domain $\mathcal{D}$. When we say $f(x)$ is q-module or preordering convex/concave over $\mathcal{D}$, we usually assume certain set of defining polynomials $g_i(x)$ which should be clear from the context.

In the special case $\mathcal{D} = \mathbb{R}^n$, the definitions of q-module convexity and preordering convexity coincide and they are called *first order sos convexity*. The *first order sos concavity* is defined in a similar way. Recall that a polynomial $f(x)$ is *sos-convex* if its Hessian $\nabla^2 f(x)$ is sos (see [6]). An interesting fact is if $f(x)$ is sos-convex then it must also be first order sos convex. This is due to the fact that

$$f(x) - f(u) - \nabla f(u)^T (x - u) = (x - u)^T \left(\int_0^1 \int_0^t \nabla^2 f(u + s(x - u)) \, ds \, dt \right) (x - u).$$

If $f(x)$ is sos-convex, then $\nabla^2 f(x) = F(x)^T F(x)$ for some matrix polynomial $F(x)$ and

$$f(x) - f(u) - \nabla f(u)^T (x - u)$$
$$= (x - u)^T \left(\int_0^1 \int_0^t F(u + s(x - u))^T F(u + s(x - u)) \, ds \, dt \right) (x - u)$$

must be an sos polynomial (see Lemma 7 of [6]). Interestingly, if $f(x)$ is first order sos convex, $f(x)$ must also be sos convex, that is, these two kinds of convexity are equivalent. This fact was pointed out to the author by Amir Ali Ahmadi and Pablo Parrilo. They kindly give the following proof. For all $x, y \in \mathbb{R}^n$ and $t \neq 0$ it holds

$$\frac{1}{t^2} \left(f(x + ty) - f(x) - t \nabla f(x)^T y \right) = y^T \nabla^2 f(x) y + \frac{1}{t^2} o(t^2 \|y\|_2^2)).$$

If $f(x)$ is first order sos convex, the left hand side in the above must be sos in (x, y). As $t \to 0$, it has a bounded degree and converges to $y^T \nabla^2 f(x) y$, since $\frac{1}{t^2} o(t^2 \|y\|_2^2)) \to 0$. So $y^T \nabla^2 f(x) y$ must also be sos in (x, y), which means $f(x)$ is sos convex.

Example 2.1 The bivariate polynomial $f(x) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$ is convex over the nonnegative orthant $\mathbb{R}_+^2$. It is also q-module convex with respect to $\mathbb{R}_+^2$. This is due to the identity

$$R_f(x, u) = \left(\frac{1}{3} u_1 + \frac{1}{6} x_1 \right) \left(4(x_1 - u_1)^2 + 2(x_1 + x_2 - u_1 - u_2)^2 \right) +$$
$$\left(\frac{1}{3} u_2 + \frac{1}{6} x_2 \right) \left(4(x_2 - u_2)^2 + 2(x_1 + x_2 - u_1 - u_2)^2 \right).$$

2.2. SDP representations

Throughout this subsection, we assume the polynomials $f(x)$ and $g_i(x)$ are either all q-module concave or all preordering concave over $\mathcal{D}$. For any $h(x)$ from the set $\{f(x), g_1(x), \cdots, g_m(x)\}$, we thus have either

$$-R_h(x, u) = \sum_{i=0}^{m} g_i(x) \left(\sum_{j=0}^{m} g_j(u) \sigma_{ij}^h(x, u) \right)$$

for some sos polynomials $\sigma_{ij}^h(x, u)$, or

$$-R_h(x, u) = \sum_{\nu \in \{0,1\}^m} g_1^{\nu_1}(x) \cdots g_m^{\nu_m}(x) \left(\sum_{\mu \in \{0,1\}^m} g_1^{\mu_1}(u) \cdots g_m^{\mu_m}(u) \sigma_{\nu,\mu}^h(x, u) \right)$$

for some sos polynomials $\sigma_{\nu,\mu}^h(x, u)$. Let $d_i = \lceil \frac{1}{2} \deg(g_i) \rceil$, $d_\nu = \lceil \frac{1}{2} \deg_x(g_1^{\nu_1} \cdots g_m^{\nu_m}) \rceil$ and

$$d_{qmod}^{(P)} = \max_{h \in \{f, g_1, \cdots, g_m\}} \max_{0 \le i,j \le m} \lceil \tfrac{1}{2} \deg_x(g_i \sigma_{ij}^h) \rceil$$
$$d_{pre}^{(P)} = \max_{h \in \{f, g_1, \cdots, g_m\}} \max_{\nu \in \{0,1\}^m, \mu \in \{0,1\}^m} \lceil \tfrac{1}{2} \deg_x(g_1^{\nu_1} \cdots g_m^{\nu_m} \sigma_{\nu,\mu}^h) \rceil \tag{2.1}$$

where $\deg_x(\cdot)$ denotes the degree of a polynomial in x. Then define $d = d_{qmod}^{(P)}$ (resp. $d = d_{pre}^{(P)}$) when $f(x)$ and all $g_i(x)$ are q-module (resp. preordering) concave over $\mathcal{D}$.

Define matrices $G_\alpha^{(i)}$ and $G_\alpha^{(\nu)}$ such that

$$g_i(x) \mathfrak{m}_{d-d_i}(x) \mathfrak{m}_{d-d_i}(x)^T = \sum_{\alpha \in \mathbb{N}^n : |\alpha| \le 2d} G_\alpha^{(i)} x^\alpha$$
$$g_1^{\nu_1}(x) \cdots g_m^{\nu_m}(x) \mathfrak{m}_{d-d_\nu}(x) \mathfrak{m}_{d-d_\nu}(x)^T = \sum_{\alpha \in \mathbb{N}^n : |\alpha| \le 2d} G_\alpha^{(\nu)} x^\alpha. \tag{2.2}$$

Here $\mathfrak{m}_k(x)$ is the vector of all monomials with degrees $\le k$. Let y be a vector multi-indexed by integer vectors in $\mathbb{N}^n$. Then define

$$N_i(y) = \sum_{\alpha \in \mathbb{N}^n : |\alpha| \le 2d} G_\alpha^{(i)} y_\alpha, \quad i = 0, 1, \ldots, m,$$

$$N_\nu(y) = \sum_{\alpha \in \mathbb{N}^n : |\alpha| \le 2d} G_\alpha^{(\nu)} y_\alpha, \quad \nu \in \{0,1\}^m.$$

Suppose the polynomial $f(x)$ is given in the form

$$f(x) = \sum_{\alpha \in \mathbb{N}^n : |\alpha| \le 2d} f_\alpha x^\alpha,$$

then define vector f such that

$$f^T y = \sum_{\alpha \in \mathbb{N}^n : |\alpha| \leq 2d} f_\alpha \, y_\alpha.$$

Define two sets

$$\mathcal{L}_{qmod}^{\mathcal{D}}(f) = \left\{ y \in \mathbb{R}_{\alpha \in \mathbb{N}^n, |\alpha| \leq 2d} : y_0 = 1, \ f^T y \geq 0, \ N_i(y) \succeq 0 \ \forall \, 0 \leq i \leq m \right\} \qquad (2.3)$$

$$\mathcal{L}_{pre}^{\mathcal{D}}(f) = \left\{ y \in \mathbb{R}_{\alpha \in \mathbb{N}^n, |\alpha| \leq 2d} : y_0 = 1, \ f^T y \geq 0, \ N_\nu(y) \succeq 0 \ \forall \, \nu \in \{0,1\}^m \right\} \qquad (2.4)$$

via their LMI representations. Then $S_{\mathcal{D}}(f)$ is contained in the image of both $\mathcal{L}_{qmod}^{\mathcal{D}}(f)$ and $\mathcal{L}_{pre}^{\mathcal{D}}(f)$ of the projection map: $\rho(y) = (y_{10\ldots0}, \ y_{010\ldots0}, \ \ldots, \ y_{00\ldots01})$, because for any $x \in S_{\mathcal{D}}(f)$ we can choose $y_\alpha = x^\alpha$ such that $y \in \mathcal{L}_{qmod}^{\mathcal{D}}(f)$ and $y \in \mathcal{L}_{pre}^{\mathcal{D}}(f)$. We say $S_{\mathcal{D}}(f)$ equals $\mathcal{L}_{qmod}^{\mathcal{D}}(f)$ (resp. $\mathcal{L}_{pre}^{\mathcal{D}}(f)$) if $S_{\mathcal{D}}(f)$ equals the image of $\mathcal{L}_{qmod}^{\mathcal{D}}(f)$ (resp. $\mathcal{L}_{pre}^{\mathcal{D}}(f)$) under this projection. Similarly, we say $x \in \mathcal{L}_{qmod}^{\mathcal{D}}(f)$ (resp. $x \in \mathcal{L}_{pre}^{\mathcal{D}}(f)$) if there exists $y \in \mathcal{L}_{qmod}^{\mathcal{D}}(f)$ (resp. $y \in \mathcal{L}_{pre}^{\mathcal{D}}(f)$) such that $x = \rho(y)$.

Lemma 2.2 Assume $S_{\mathcal{D}}(f)$ has nonempty interior. Let $\{x \in \mathbb{R}^n : a^T x = b\}$ be a supporting hyperplane of $S_{\mathcal{D}}(f)$ such that $a^T x \geq b$ for all $x \in S_{\mathcal{D}}(f)$ and $a^T u = b$ for some point $u \in S_{\mathcal{D}}(f)$.

(i) If $f(x)$ and every $g_i(x)$ are q-module concave over $\mathcal{D}$, then it holds

$$a^T x - b - \lambda f(x) = \sum_{i=0}^m g_i(x) \sigma_i(x)$$

for some $\lambda \geq 0$ and sos polynomials $\sigma_i(x)$ with $\deg(g_i \sigma_i) \leq 2 d_{qmod}^{(P)}$.

(ii) If $f(x)$ and every $g_i(x)$ are preordering concave over $\mathcal{D}$, then it holds

$$a^T x - b - \lambda f(x) = \sum_{\nu \in \{0,1\}^m} g_1^{\nu_1}(x) \cdots g_m^{\nu_m}(x) \sigma_\nu(x)$$

for some $\lambda \geq 0$ and sos polynomials $\sigma_\nu(x)$ with $\deg(g_1^{\nu_1} \cdots g_m^{\nu_m} \sigma_\nu) \leq 2 d_{pre}^{(P)}$.

Proof Since $S_{\mathcal{D}}(f)$ has nonempty interior and the polynomials $f(x), g_1(x), \cdots, g_m(x)$ are all concave, the first order optimality condition holds at u for the convex optimization problem (u is a minimizer)

$$\min_x \quad a^T x \quad \text{subject to} \quad f(x) \geq 0, \ g_i(x) \geq 0, \ i = 1, \ldots, m.$$

Hence there exist Lagrange multipliers $\lambda \geq 0, \lambda_1 \geq 0, \ldots, \lambda_m \geq 0$ such that

$$a = \lambda \nabla f(u) + \sum_{i=1}^m \lambda_i \nabla g_i(u), \quad \lambda f(u) = \lambda_1 g_1(u) = \cdots = \lambda_m g_m(u) = 0.$$

Thus the Lagrange function $a^T x - b - \lambda f(x) - \sum_{i=1}^m \lambda_i g_i(x)$ has representation

$$\lambda \Big(f(u) + \nabla f(u)^T (x - u) - f(x) \Big) + \sum_{i=1}^m \lambda_i \Big(g_i(u) + \nabla g_i(u)^T (x - u) - g_i(x) \Big).$$

Therefore, the claims (i) and (ii) can be implied immediately from the definition of q-module concavity or preordering concavity and plugging in the value of u. $\qquad \square$

Theorem 2.3 Assume $\mathcal{D}$ and $S_{\mathcal{D}}(f)$ are both convex and have nonempty interior.

(i) If $f(x)$ and every $g_i(x)$ are q-module concave over $\mathcal{D}$, then $S_{\mathcal{D}}(f) = \mathcal{L}_{qmod}^{\mathcal{D}}(f)$.
(ii) If $f(x)$ and every $g_i(x)$ are preordering concave over $\mathcal{D}$, then $S_{\mathcal{D}}(f) = \mathcal{L}_{pre}^{\mathcal{D}}(f)$.

Proof (i) Since $S_{\mathcal{D}}(f)$ is contained in the projection of $\mathcal{L}_{qmod}^{\mathcal{D}}(f)$, we only need to prove $S_{\mathcal{D}}(f) \supseteq \mathcal{L}_{qmod}^{\mathcal{D}}(f)$. For a contradiction, suppose there exists some $\hat{y} \in \mathcal{L}_{qmod}^{\mathcal{D}}(f)$ such that $\hat{x} = \rho(\hat{y}) \notin S_{\mathcal{D}}(f)$. By the convexity of $S_{\mathcal{D}}(f)$, it holds

$$S_{\mathcal{D}}(f) = \bigcap_{\substack{\{a^T x = b\} \text{ is a} \\ \text{supporting hyperplane}}} \left\{ x \in \mathbb{R}^n : a^T x \geq b \right\}.$$

If $\hat{x} \notin S_{\mathcal{D}}(f)$, then there exists one hyperplane $\{a^T x = b\}$ of $S_{\mathcal{D}}(f)$ such that $a^T \hat{x} < b$. By Lemma 2.2, we have representation

$$a^T x - b = \lambda f(x) + \sum_{i=0}^{m} g_i(x)\sigma_i(x) \tag{2.5}$$

for some $\lambda \geq 0$ and sos polynomials $\sigma_i(x)$ with $\deg(g_i\sigma_i) \leq 2d_{qmod}^{(P)} = 2d$. Write

$$\sigma_i(x) = \mathfrak{m}_{d-d_i}(x)^T W_i \mathfrak{m}_{d-d_i}(x), \ i = 0, 1, \ldots, m$$

for some symmetric matrices $W_i \succeq 0$. Then the identity (2.5) becomes (noting (2.2))

$$a^T x - b = \lambda f(x) + \sum_{i=0}^{m} (g_i(x)\mathfrak{m}_{d-d_i}(x)\mathfrak{m}_{d-d_i}(x)^T) \bullet W_i$$

$$= \lambda f(x) + \sum_{i=0}^{m} \left(\sum_{\alpha \in \mathbb{N}^n : |\alpha| \leq 2d} G_\alpha^{(i)} x^\alpha \right) \bullet W_i.$$

In the above identity, if we replace each x^α by $\hat{y}_\alpha$, then get the contradiction

$$a^T \hat{x} - b = \lambda \cdot f^T \hat{y} + \sum_{i=0}^{m} N_i(\hat{y}) \bullet W_i \geq 0.$$

(ii) The proof is almost the same as for (i). The only difference is that

$$a^T x - b = \lambda f(x) + \sum_{\nu \in \{0,1\}^m} g_1^{\nu_1}(x) \cdots g_m^{\nu_m}(x)\sigma_\nu(x)$$

for some sos polynomials $\sigma_\nu(x)$ with $\deg(g_1^{\nu_1} \cdots g_m^{\nu_m} \sigma_\nu) \leq 2d_{pre}^{(P)} = 2d$. Write

$$\sigma_\nu(x) = \mathfrak{m}_{d-d_\nu}(x)^T W_\nu \mathfrak{m}_{d-d_\nu}(x), \ W_\nu \succeq 0.$$

Then a similar contradiction argument can be applied to prove the claim. $\qquad \square$

2.3. Some special cases

Now we turn to some special cases about q-module or preordering convexity/concavity.

2.3.1 The q-module or preordering convexity certificate using Hessian

The q-module or preordering convexity of $f(x)$ over the domain $\mathcal{D}$ can be verified by solving some semidefinite programming problems. See [14,10] about the sos polynomials and semidefinite programming. However, in some special cases like $\mathcal{D} = \mathbb{R}^n_+$, a certificate for semidefiniteness of the Hessian $\nabla^2 f(x)$ can be applied to prove the q-module or preordering convexity of $f(x)$.

First, consider the case that $g_k(x)$ are concave over $\mathbb{R}^n$. By concavity, it holds

$$g_k(sx + (1-s)u) \geq sg_k(x) + (1-s)g_k(u), \quad \forall\, s \in [0,1], \, x,u \in \mathbb{R}^n.$$

Now we assume the following certificate for the above criteria

$$g_k(sx + (1-s)u) - sg_k(x) - (1-s)g_k(u) =$$
$$\sigma_0^{(k)}(x,u,s) + s\sigma_1^{(k)}(x,u,s) + (1-s)\sigma_2^{(k)}(x,u,s) + s(1-s)\sigma_3^{(k)}(x,u,s) \tag{2.6}$$

where $\sigma_0^{(k)}(x,u,s), \sigma_1^{(k)}(x,u,s), \sigma_2^{(k)}(x,u,s), \sigma_3^{(k)}(x,u,s)$ are sos polynomials in (x,u,s). Note (2.6) is always true when $\mathcal{D}$ is a polyhedron, i.e., every $\deg(g_k) = 1$.

Theorem 2.4 Suppose for every k the identity (2.6) holds. If $\nabla^2 f(x)$ belongs to the quadratic module (resp. preordering) generated by $g_1(x), \cdots, g_m(x)$, i.e.,

$$\nabla^2 f(x) = \sum_{i=0}^m g_i(x)H_i(x) \quad \left(resp. \quad \nabla^2 f(x) = \sum_{\nu \in \{0,1\}^m} g_1^{\nu_1}(x) \cdots g_m^{\nu_m}(x)H_\nu(x) \right)$$

for some sos matrices $H_i(x)$ (resp. $H_\nu(x)$), then $f(x)$ is q-module convex (resp. preordering convex) over the domain $\mathcal{D}$.

Proof First, if $\nabla^2 f(x) = \sum_{i=0}^m g_i(x)H_i(x)$ for some sos matrices $H_i(x)$, then

$$f(x) - f(u) - \nabla f(u)^T(x-u)$$
$$= (x-u)^T \left(\int_0^1 \int_0^t \nabla^2 f(u + s(x-u)) \, ds\, dt \right)(x-u)$$
$$= (x-u)^T \left(\sum_{k=1}^m \underbrace{\int_0^1 \int_0^t g_k(sx + (1-s)u)H_i(sx + (1-s)u)dsdt}_{I_k} \right)(x-u).$$

By identity (2.6), we have

$$I_k = \int_0^1 \int_0^t \left(\sigma^{(k)}(x,u,s)H_i(w)dsdt + g_k(x)sH_i(w)dsdt + g_k(u)(1-s)H_i(w) \right)dsdt$$
$$\left(\text{ here } \sigma^{(k)} = \sigma_0^{(k)} + \sigma_1^{(k)} + (1-s)\sigma_2^{(k)} + s(1-s)\sigma_3^{(k)}, \, w = sx + (1-s)u \right)$$
$$= H_0^{(k)}(x,u) + g_k(x)H_1^{(k)}(x,u) + g_k(u)H_2^{(k)}(x,u)$$

for some sos matrices $H_0^{(k)}(x,u), H_1^{(k)}(x,u), H_2^{(k)}(x,u)$ (see Lemma 7 in [6]). So $f(x)$ is q-module convex over $\mathcal{D}$.

Second, if $\nabla^2 f(x)$ is in the preordering generated by $g_1(x), \cdots, g_m(x)$, i.e.,

$$\nabla^2 f(x) = \sum_{\nu \in \{0,1\}^m} g_1^{\nu_1}(x) \cdots g_m^{\nu_m}(x) H_\nu(x)$$

for sos matrices $H_\nu(x)$, we can similarly prove $f(x)$ is preordering convex over $\mathcal{D}$. $\quad\square$

Second, consider the special case that $\mathcal{D} = \mathbb{R}_+^n$ and the polynomial $f(x)$ is cubic.

Theorem 2.5 Let $\mathbb{R}_+^n = \{x : x_1 \geq 0, \cdots, x_n \geq 0\}$ be the domain. If $f(x)$ is a cubic polynomial concave over $\mathbb{R}_+^n$, then $S_{\mathbb{R}_+^n}(f) = \mathcal{L}_{qmod}^{\mathbb{R}_+^n}(f)$.

Proof By Theorem 2.3, it suffices to prove $f(x)$ is q-module concave over $\mathbb{R}_+^n$. Since $f(x)$ is cubic, we have

$$-\nabla^2 f(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$$

for some symmetric matrices A_i. When $f(x)$ is concave over $\mathbb{R}_+^n$, $-\nabla^2 f(x) \succeq 0$ for all $x \geq 0$. Hence all A_i must be positive semidefinite. This means that $-\nabla^2 f(x)$ belongs to the quadratic module generated by $x_1, \cdots, x_n$. By Theorem 2.4, $f(x)$ is q-module concave over $\mathbb{R}_+^n$. $\quad\square$

Example 2.6 The convex set $S_{\mathbb{R}_+^2}(f)$ with $f(x) = 1 - (x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3)$ equals $\mathcal{L}_{qmod}^{\mathbb{R}_+^2}(f)$:

$$\left\{ (x_1, x_2) : \exists y_{ij}, s.t. \begin{array}{c} 1 \geq y_{30} + y_{21} + y_{12} + y_{03} \\ \begin{bmatrix} 1 & x_1 & x_2 & y_{20} & y_{11} & y_{02} \\ x_1 & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ x_2 & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \succeq 0, \begin{bmatrix} x_1 & y_{20} & y_{11} \\ y_{20} & y_{30} & y_{21} \\ y_{11} & y_{21} & y_{12} \end{bmatrix} \succeq 0 \\ \begin{bmatrix} x_2 & y_{11} & y_{02} \\ y_{11} & y_{21} & y_{12} \\ y_{02} & y_{12} & y_{03} \end{bmatrix} \succeq 0 \end{array} \right\}.$$

This is because $f(x)$ is q-module concave over $\mathbb{R}_+^2$ (see Example 2.1). $\quad\square$

Third, consider the special case of univariate polynomials. When $\mathcal{D} = \mathbb{R}$, a univariate polynomial is convex if and only if it is sos-convex, which holds if and only if it is first order sos convex. When $\mathcal{D} = I$ is an interval, we will see that a univariate polynomial is convex over I if and only if it is q-module convex over I.

Proposition 2.7 Let $f(x)$ be a univariate polynomial and I be an interval like $[a, b]$, $(-\infty, b]$ or $[a, \infty)$. Then $f(x)$ is convex on I if and only if it is q-module convex on I.

Proof First suppose $I = [a, b]$ is finite. $f(x)$ is convex over $[a, b]$ if and only if $f''(x) \geq 0$ for all $x \in [a, b]$, which is true if and only if

$$f''(x) = \sigma_0(x) + (x - a)\sigma_1(x) + (b - x)\sigma_2(x)$$

for some sos polynomials $\sigma_0, \sigma_1, \sigma_2$ with degrees at most $2\lceil deg(f)/2 \rceil$ (see Powers and Reznick [16]). In other words, $f(x)$ is convex over $[a, b]$ if and only if its Hessian belongs to the quadratic module generated by polynomials $x - a, b - x$. Then the conclusion follows from Theorem 2.4.

The proof is similar for the case $(-\infty, b]$ or $[a, \infty)$. $\quad\square$

2.3.2 Epigraph of polynomial functions

For a given convex domain $\mathcal{D} \subseteq \mathbb{R}^n$, $f(x)$ is convex over $\mathcal{D}$ if and only if its epigraph

$$\mathbf{epi}(f) := \{(x,t) \in \mathcal{D} \times \mathbb{R} : f(x) \le t\}$$

is convex. Note that $\mathbf{epi}(f)$ is defined by the inequality $t - f(x) \ge 0$. If we consider $t - f(x)$ as a polynomial in x with coefficients in t, then $\mathcal{L}^{\mathcal{D}}_{qmod}(t - f)$ and $\mathcal{L}^{\mathcal{D}}_{pre}(t - f)$ are both linear in (x,t). Therefore, if $f(x)$ is q-module (resp. preordering) convex over $\mathcal{D}$, $\mathcal{L}^{\mathcal{D}}_{qmod}(t - f)$ (resp. $\mathcal{L}^{\mathcal{D}}_{pre}(t - f)$) presents an SDP representation for $\mathbf{epi}(f)$.

By Proposition 2.7, when $f(x)$ is a univariate polynomial convex over an interval I, we know its epigraph epi(f) is SDP representable and $\mathcal{L}^{I}_{qmod}(t - f)$ is one SDP representation.

3 Convex sets defined by rational functions

In this section, we discuss the SDP representation of convex set $S_{\mathcal{D}}(f)$ when $f(x)$ is a rational function while the domain $\mathcal{D} = \{x \in \mathbb{R}^n : g_1(x) \ge 0, \cdots, g_m(x) \ge 0\}$ is still defined by polynomials g_i. Let $f(x)$ be a rational function of the form

$$f(x) = \frac{1}{f_{den}(x)} \sum_{\alpha \in \mathbb{N}^n : |\alpha| \le 2d} f_\alpha x^\alpha.$$

Here $f_{den}(x)$ is the denominator of $f(x)$. We assume that $f(x)$ is concave over the domain $\mathcal{D}$. So $f(x)$ can not have poles in the interior $int(\mathcal{D})$ of $\mathcal{D}$. Without loss of generality, assume $f_{den}(x)$ is positive over $int(\mathcal{D})$. Note that $f(x)$ is not defined on the boundary $\partial\mathcal{D}$ where $f_{den}(x)$ vanishes. If this happens, we think of $S_{\mathcal{D}}(f)$ as the closure of $\{x \in int(\mathcal{D}) : f(x) \ge 0\}$.

3.1. The q-module or preordering convexity of rational functions

We now introduce some definitions about convexity/concavity for rational functions. Let $p(x), q(x)$ be two given polynomials which are positive in $int(\mathcal{D})$. We say $f(x)$ is *q-module convex over $\mathcal{D}$ with respect to (p,q)* if the identity

$$p(x)q(u) \cdot R_f(x,u) = \sum_{i=0}^{m} g_i(x) \left(\sum_{j=0}^{m} g_j(u)\sigma_{ij}(x,u) \right) \tag{3.1}$$

holds for some sos polynomials $\sigma_{ij}(x,u)$. Then define $f(x)$ to be *q-module concave over $\mathcal{D}$ with respect to (p,q)* if $-f(x)$ is q-module convex over $\mathcal{D}$ with respect to (p,q). We say $f(x)$ is *preordering convex over $\mathcal{D}$ with respect to (p,q)* if the identity

$$p(x)q(u)\cdot R_f(x,u) = \sum_{\nu \in \{0,1\}^m} g_1^{\nu_1}(x) \cdots g_m^{\nu_m}(x) \left(\sum_{\mu \in \{0,1\}^m} g_1^{\mu_1}(u) \cdots g_m^{\mu_m}(u)\sigma_{\nu,\mu}(x,u) \right) \tag{3.2}$$

holds for some sos polynomials $\sigma_{\nu,\mu}(x,u)$. Similarly, $f(x)$ is called *preordering concave over $\mathcal{D}$ with respect to (p,q)* if $-f(x)$ is preordering convex over $\mathcal{D}$ with respect to (p,q). We point out the definition of q-module or preordering convexity/concavity over

$\mathcal{D}$ for rational functions assumes a certain set of defining polynomials $g_i(x)$ for $\mathcal{D}$ is clear in the context.

In identities (3.1) or (3.2), there is no information on how to find polynomials p, q. However, since $R_f(x, u)$ has denominator $f_{den}(x)f_{den}^2(u)$, a possible choice for (p, q) is

$$p(x) = f_{den}(x), \quad q(u) = f_{den}^2(u). \tag{3.3}$$

If the choice (p, q) in (3.3) makes the identity (3.1) (resp. (3.2)) hold, we say $f(x)$ is q-module (resp. preordering) convex over $\mathcal{D}$ with respect to $p(x)$, or just simply say $f(x)$ is q-module (resp. preordering) convex over $\mathcal{D}$ if the denominator $f_{den}(x)$ is clear in the context.

In the special case $\mathcal{D} = \mathbb{R}^n$, the definitions of q-module and preordering convexity over $\mathcal{D}$ coincide and they are called *first order sos convexity* when (p, q) is given by (3.3), as consistent with the definition of first order sos convexity in Section 2. First order sos concavity is defined similarly.

Example 3.1 (i) The rational function $\frac{x_2^2}{x_1}$ is convex over $\mathbb{R}_+ \times \mathbb{R}$. It is also q-module convex over $\mathbb{R}_+ \times \mathbb{R}$ with respect to x_1, which is due to the fact that

$$\frac{x_2^2}{x_1} - \frac{u_2^2}{u_1} - \left(-\frac{u_2^2}{u_1^2}(x_1 - u_1) + \frac{2u_2}{u_1}(x_2 - u_2) \right) = \frac{1}{x_1 u_1^2}(x_1 u_2 - x_2 u_1)^2.$$

(ii) The rational function $f(x) = \frac{x_1^4 + x_1^2 x_2^2 + x_2^4}{x_1^2 + x_2^2}$ is convex over $\mathbb{R}^2$. This is because

$$f(x) - f(u) - \nabla f(u)^T (x - u) = \frac{f_1^2 + f_2^2 + \frac{1}{2}(f_3^2 + f_4^2 + f_5^2 + f_6^2) + f_7^2 + f_8^2 + f_9^2}{(x_1^2 + x_2^2)(u_1^2 + u_2^2)^2}$$

where the polynomials f_i are given as below

$$
\begin{aligned}
&f_1 = -u_1 u_2 x_2^2 - u_1 u_2 x_1^2 + u_1 u_2^2 x_2 + u_1^2 u_2 x_1, && f_6 = -u_2^2 x_2^2 + u_2^3 x_2 - u_1^2 x_1^2 + u_1^3 x_1, \\
&f_2 = -u_1 u_2 x_2^2 + u_1 u_2 x_1^2 + u_1 u_2^2 x_2 - u_1^2 u_2 x_1, && f_7 = -2u_1 u_2 x_1 x_2 + u_1 u_2^2 x_1 + u_1^2 u_2 x_2, \\
&f_3 = -u_2^2 x_1 x_2 + u_2^3 x_1 - u_1^2 x_1 x_2 + u_1^3 x_2, && f_8 = u_2^2 x_1^2 - u_1^2 x_2^2, \\
&f_4 = u_2^2 x_1 x_2 - u_2^3 x_1 - u_1^2 x_1 x_2 + u_1^3 x_2, && f_9 = -u_1 u_2^2 x_1 + u_1^2 u_2 x_2. \\
&f_5 = u_2^2 x_2^2 - u_2^3 x_2 - u_1^2 x_1^2 + u_1^3 x_1,
\end{aligned}
$$

So the $f(x)$ given above is first order sos convex. $\qquad\square$

Obviously, the q-module convexity implies preordering convexity, which then implies the convexity, but the converse might not be true. For instance, $\frac{1}{x_1 x_2}$ is convex over $\mathbb{R}_+^2$, but it is neither q-module nor preordering convex over $\mathbb{R}_+^2$ with respect to the denominator $x_1 x_2$. Note that for $f(x) = \frac{1}{x_1 x_2}$ it holds

$$u_1^2 u_2^2 x_1 x_2 R_f(x, u) = u_1^2 u_2^2 + x_1^2 x_2 u_2 + x_1 x_2^2 u_1 - 3x_1 x_2 u_1 u_2.$$

There are no sos polynomials $\sigma_{\nu, \mu}(x, u)$ such that

$$u_1^2 u_2^2 x_1 x_2 R_f(x, u) = \sum_{\nu \in \{0,1\}^2, \mu \in \{0,1\}^2} x_1^{\nu_1} x_2^{\nu_2} u_1^{\mu_1} u_2^{\mu_2} \sigma_{\nu, \mu}(x, u).$$

Otherwise, if they exist, we replace (x_1, x_2, u_1, u_2) by $(x_1^2, x_2^2, 1, 1)$ and then get the dehomogenized Motzkin's polynomial $1 + x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2$ is sos, which is impossible (see Reznick [18]).

Proposition 3.2 Let $CQ_{p,q}(\mathcal{D})$ (resp. $CP_{p,q}(\mathcal{D})$) be the set of all q-module (resp. preordering) convex rational functions over $\mathcal{D}$ with respect to (p,q). Then they have the properties:

(i) Both $CQ_{p,q}(\mathcal{D})$ and $CP_{p,q}(\mathcal{D})$ are convex cones.

(ii) If $f(x) \in CQ_{p,q}(\mathcal{D})$ (resp. $f(x) \in CP_{p,q}(\mathcal{D})$), then $f(Az + b) \in CQ_{\tilde{p},\tilde{q}}(\tilde{\mathcal{D}})$ (resp. $f(Az + b) \in CP_{\tilde{p},\tilde{q}}(\tilde{\mathcal{D}})$), where $\tilde{\mathcal{D}} = \{z : Az + b \in \mathcal{D}\}$ and $\tilde{p}(z) = p(Az + b)$, $\tilde{q}(z) = q(Az + b)$. That is, the q-module convexity and preordering convexity are preserved under linear transformations.

Proof The item (i) can be verified explicitly, and item (ii) can be obtained by substituting $Az + b$ for x and noting the chain rule of derivatives. □

3.2. SDP representations

Now we turn to the construction of SDP representations for $S_{\mathcal{D}}(f)$. Recall that $g_0(x) \equiv 1$. Throughout this subsection, we assume $f(x)$ and $g_i(x)$ are either all q-module concave or all preordering concave over $\mathcal{D}$ with respect to (p, q). Thus for any $h(x)$ from $\{f(x), g_1(x), \cdots, g_m(x)\}$, we have either

$$-p(x)q(u)R_h(x,u) = \sum_{i=0}^{m} g_i(x) \left(\sum_{j=0}^{m} g_j(u)\sigma_{ij}^h(x,u) \right)$$

for some sos polynomials $\sigma_{ij}^h(x,u)$, or

$$-p(x)q(u)R_h(x,u) = \sum_{\nu \in \{0,1\}^m} g_1^{\nu_1}(x) \cdots g_m^{\nu_m}(x) \left(\sum_{\mu \in \{0,1\}^m} g_1^{\mu_1}(u) \cdots g_m^{\mu_m}(u)\sigma_{\nu,\mu}^h(x,u) \right)$$

for some sos polynomials $\sigma_{\nu,\mu}^h(x,u)$. Let

$$
\begin{aligned}
d_{qmod}^{(R)} &= \max_{h \in \{f,g_1,\cdots,g_m\}} \quad \max_{0 \le i,j \le m} \quad \lceil \tfrac{1}{2} \deg_x(g_i\sigma_{ij}^h) \rceil, \\
d_{pre}^{(R)} &= \max_{h \in \{f,g_1,\cdots,g_m\}} \quad \max_{\nu \in \{0,1\}^m, \mu \in \{0,1\}^m} \quad \lceil \tfrac{1}{2} \deg_x(g_1^{\nu_1}\cdots g_m^{\nu_m}\sigma_{\nu,\mu}^h) \rceil.
\end{aligned}
\tag{3.4}
$$

Then set $d = d_{qmod}^{(R)}$ (resp. $d = d_{pre}^{(R)}$) when $f(x)$ and $g_i(x)$ are all q-module (resp. preordering) concave over $\mathcal{D}$ with respect to (p, q).

Define matrices $P_\alpha^{(i)}, P_\alpha^{(\nu)}, Q_\alpha^{(i)}, Q_\alpha^{(\nu)}$ such that

$$
\left.
\begin{aligned}
\frac{g_i(x)}{p(x)}\mathfrak{m}_{d-d_i}(x)\mathfrak{m}_{d-d_i}(x)^T &= \sum_{\alpha \in \mathbb{N}^n : |\alpha|+|LE(p)| \le 2d} Q_\alpha^{(i)} x^\alpha \\
&\quad + \sum_{\beta \in \mathbb{N}^n : \beta < LE(p)} P_\alpha^{(i)} \frac{x^\beta}{p(x)}, \quad 0 \le i \le m \\
\frac{g_1^{\nu_1}\cdots g_m^{\nu_m}}{p(x)}\mathfrak{m}_{d-d_\nu}(x)\mathfrak{m}_{d-d_\nu}(x)^T &= \sum_{\alpha \in \mathbb{N}^n : |\alpha|+|LE(p)| \le 2d} Q_\alpha^{(\nu)} x^\alpha \\
&\quad + \sum_{\beta \in \mathbb{N}^n : \beta < LE(p)} P_\alpha^{(\nu)} \frac{x^\beta}{p(x)}, \quad \nu \in \{0,1\}^m
\end{aligned}
\right\}.
\tag{3.5}
$$

Here the monomials are ordered in the lexicographical ordering $(x_1 > x_2 > \cdots > x_n)$ and $LE(p)$ denotes the exponent of the leading monomial of $p(x)$. Note the union

$$\left\{ x^\alpha : \alpha \in \mathbb{N}^n, |\alpha| + |LE(p)| \le 2d \right\} \bigcup \left\{ \frac{x^\beta}{p(x)} : \beta \in \mathbb{N}^n, \beta < LE(p) \right\}$$

is a set of polynomials and rational functions that are linearly independent.

Let y be a vector indexed by $\alpha \in \mathbb{N}^n$ such that $|\alpha| + |LE(p)| \leq 2d$, and z be a vector indexed by $\beta \in \mathbb{N}^n$ such that $\beta < LE(p)$. Then define

$$\left. \begin{array}{l} Q_i(y,z) = \sum\limits_{\alpha \in \mathbb{N}^n : |\alpha| + |LE(p)| \leq 2d} Q_\alpha^{(i)} y_\alpha + \sum\limits_{\beta \in \mathbb{N}^n : \beta < LE(p)} P_\alpha^{(i)} z_\beta, \, 0 \leq i \leq m \\[2em] Q_\nu(y,z) = \sum\limits_{\alpha \in \mathbb{N}^n : |\alpha| + |LE(p)| \leq 2d} Q_\alpha^{(\nu)} y_\alpha + \sum\limits_{\beta \in \mathbb{N}^n : \beta < LE(p)} P_\alpha^{(\nu)} z_\beta, \, \nu \in \{0,1\}^m \end{array} \right\} . \quad (3.6)$$

Suppose the rational function $f(x)$ is given in the form

$$f(x) = \sum_{\alpha \in \mathbb{N}^n : |\alpha| + |LE(p)| \leq 2d} f_\alpha^{(1)} x^\alpha + \sum_{\beta \in \mathbb{N}^n : \beta < LE(p)} f_\beta^{(2)} \frac{x^\beta}{p(x)},$$

then define vectors $f^{(1)}, f^{(2)}$ such that

$$(f^{(1)})^T y + (f^{(2)})^T z = \sum_{\alpha \in \mathbb{N}^n : |\alpha| + |LE(p)| \leq 2d} f_\alpha^{(1)} y_\alpha + \sum_{\beta \in \mathbb{N}^n : \beta < LE(p)} f_\beta^{(2)} z_\beta.$$

Define two SDP representable sets

$$\mathcal{R}_{qmod}^{\mathcal{D}}(f) = \left\{ (y,z) : y_0 = 1, \begin{bmatrix} f^{(1)} \\ f^{(2)} \end{bmatrix}^T \begin{bmatrix} y \\ z \end{bmatrix} \geq 0, \, Q_i(y,z) \succeq 0, \, \forall 0 \leq i \leq m \right\}, \quad (3.7)$$

$$\mathcal{R}_{pre}^{\mathcal{D}}(f) = \left\{ (y,z) : y_0 = 1, \begin{bmatrix} f^{(1)} \\ f^{(2)} \end{bmatrix}^T \begin{bmatrix} y \\ z \end{bmatrix} \geq 0, \, Q_\nu(y,z) \succeq 0, \, \forall \nu \in \{0,1\}^m \right\}. \quad (3.8)$$

We say $S_{\mathcal{D}}(f)$ equals $\mathcal{R}_{qmod}^{\mathcal{D}}(f)$ (resp. $\mathcal{R}_{pre}^{\mathcal{D}}(f)$) if $S_{\mathcal{D}}(f)$ equals the image of $\mathcal{R}_{qmod}^{\mathcal{D}}(f)$ (resp. $\mathcal{R}_{pre}^{\mathcal{D}}(f)$) of the projection $\rho(y,z) = (y_{10...0}, \, y_{010...0}, \, \ldots, \, y_{00...01})$. Similarly, we say $x \in \mathcal{R}_{qmod}^{\mathcal{D}}(f)$ (resp. $x \in \mathcal{R}_{pre}^{\mathcal{D}}(f)$) if there exists $(y,z) \in \mathcal{R}_{qmod}^{\mathcal{D}}(f)$ (resp. $(y,z) \in \mathcal{R}_{pre}^{\mathcal{D}}(f)$) such that $x = \rho(y,z)$.

Lemma 3.3 Assume $\mathcal{D}$ and $S_{\mathcal{D}}(f)$ are both convex and have nonempty interior. Let $\{x \in \mathbb{R}^n : a^T x = b\}$ be a supporting hyperplane of $S_{\mathcal{D}}(f)$ such that $a^T x \geq b$ for all $x \in S_{\mathcal{D}}(f)$ and $a^T u = b$ for some point $u \in S_{\mathcal{D}}(f)$ such that $q(u) > 0$, and either $f(u) > 0$ or $f_{den}(u) > 0$.

(i) If $f(x)$ and every $g_i(x)$ are q-module concave over $\mathcal{D}$ with respect to (p,q), then

$$p(x) \cdot (a^T x - b - \lambda f(x)) = \sum_{i=0}^{m} g_i(x)\sigma_i(x)$$

for some $\lambda \geq 0$ and sos polynomials $\sigma_i(x)$ with $\deg(g_i\sigma_i) \leq 2d_{qmod}^{(R)}$.

(ii) If $f(x)$ and every $g_i(x)$ are preordering concave over $\mathcal{D}$ with respect to (p,q), then

$$p(x) \cdot (a^T x - b - \lambda f(x)) = \sum_{\nu \in \{0,1\}^m} g_1^{\nu_1}(x) \cdots g_m^{\nu_m}(x)\sigma_\nu(x)$$

for some $\lambda \geq 0$ and sos polynomials $\sigma_i(x)$ with $\deg(g_1^{\nu_1} \cdots g_m^{\nu_m}\sigma_\nu) \leq 2d_{pre}^{(R)}$.

Proof Since $S_{\mathcal{D}}(f)$ has nonempty interior, the first order optimality condition holds at u for the convex optimization problem (u is one minimizer)

$$\min_x \quad a^T x \quad \text{subject to} \quad f(x) \geq 0, \ g_i(x) \geq 0, \ i = 1, \ldots, m.$$

If $f(u) > 0$, the constraint $f(x) \geq 0$ is inactive. If $f_{den}(u) > 0$, $f(x)$ is differential at u. Hence, in either case, there exist Lagrange multipliers $\lambda \geq 0, \lambda_1 \geq 0, \ldots, \lambda_m \geq 0$ such that

$$a = \lambda \nabla f(u) + \sum_{i=1}^m \lambda_i \nabla g_i(u), \quad \lambda f(u) = \lambda_1 g_1(u) = \cdots = \lambda_m g_m(u) = 0.$$

Hence, we get the identity

$$a^T x - b - \lambda f(x) - \sum_{i=1}^m \lambda_i g_i(x) =$$

$$\lambda \Big(f(u) + \nabla f(u)^T (x - u) - f(x) \Big) + \sum_{i=1}^m \lambda_i \Big(g_i(u) + \nabla g_i(u)^T (x - u) - g_i(x) \Big).$$

Therefore, the claims (i) and (ii) can be implied immediately by the definition of q-module concavity or preordering concavity of f and g_i, and plugging the value of u. $\square$

Theorem 3.4 Assume $\mathcal{D}$ and $S_{\mathcal{D}}(f)$ are both convex and have nonempty interior. Let (p, q) be given in (3.1) or (3.2). Suppose $\dim(\mathcal{Z}(f) \cap \mathcal{Z}(f_{den}) \cap \partial S_{\mathcal{D}}(f)) < n - 1$ and $\dim(\mathcal{Z}(q) \cap \partial S_{\mathcal{D}}(f)) < n - 1$.

(i) If $f(x)$ and every $g_i(x)$ are q-module concave over $\mathcal{D}$ with respect to (p, q), then $S_{\mathcal{D}}(f) = \mathcal{R}_{qmod}^{\mathcal{D}}(f)$.
(ii) If $f(x)$ and every $g_i(x)$ are preordering concave over $\mathcal{D}$ with respect to (p, q), then $S_{\mathcal{D}}(f) = \mathcal{R}_{pre}^{\mathcal{D}}(f)$.

Proof (i) Since $S_{\mathcal{D}}(f)$ is contained in the projection of $\mathcal{R}_{qmod}^{\mathcal{D}}(f)$, we only need to prove $S_{\mathcal{D}}(f) \supseteq \mathcal{R}_{qmod}^{\mathcal{D}}(f)$. For a contradiction, suppose there exists some $(\hat{y}, \hat{z}) \in \mathcal{R}_{qmod}^{\mathcal{D}}(f)$ such that $\hat{x} = \rho(\hat{y}) \notin S_{\mathcal{D}}(f)$. By the convexity of $S_{\mathcal{D}}(f)$, it holds

$$S_{\mathcal{D}}(f) = \bigcap_{\substack{\{a^T x = b\} \text{ is a} \\ \text{supporting hyperplane}}} \Big\{ x \in \mathbb{R}^n : a^T x \geq b \Big\}.$$

If $\hat{x} \notin S_{\mathcal{D}}(f)$, then there exists a supporting hyperplane $\{a^T x = b\}$ of $S_{\mathcal{D}}(f)$ with tangent point $u \in \partial S_{\mathcal{D}}(f)$ such that $a^T \hat{x} < b$. Since $\dim\big(\mathcal{Z}(f) \cap \mathcal{Z}(p) \cap \partial S_{\mathcal{D}}(f)\big) < n-1$ and $\dim(\mathcal{Z}(q) \cap \partial S_{\mathcal{D}}(f)) < n - 1$, by continuity, we can choose $\{a^T x = b\}$ such that either $f(u) > 0$ or $p(u) > 0$, and $q(u) > 0$. By Lemma 3.3, we have

$$a^T x - b = \lambda f(x) + \sum_{i=0}^m \frac{g_i(x)}{p(x)} \sigma_i(x) \tag{3.9}$$

for some $\lambda \geq 0$ and sos polynomials $\sigma_i(x)$ such that $\deg(g_i \sigma_i) \leq 2d_{qmod}^{(R)}$. Note that $d = d_{qmod}^{(R)}$. Then write $\sigma_i(x)$ as

$$\sigma_i(x) = \mathfrak{m}_{d-d_i}(x)^T W_i \mathfrak{m}_{d-d_i}(x), \ i = 0, 1, \ldots, m$$
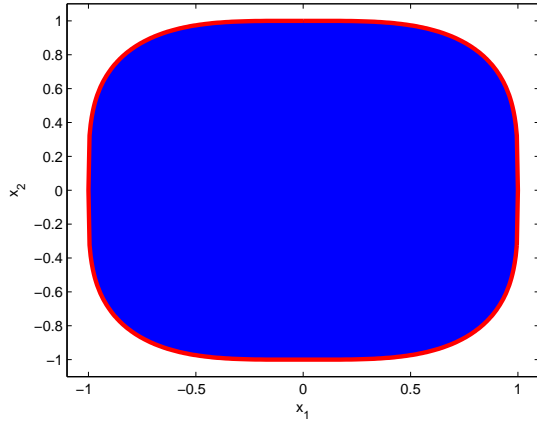
Fig. 1 The convex set defined by $x_1^2 + x_2^2 \geq x_1^4 + x_1^2 x_2^2 + x_2^4$.

for some symmetric matrices $W_i \succeq 0$. Then identity (3.9) becomes (noting (3.5))

$$a^T x - b = \lambda f(x) + \sum_{i=0}^{m} \left(\frac{g_i(x)}{p(x)} \mathfrak{m}_{d-d_i}(x) \mathfrak{m}_{d-d_i}(x)^T \right) \bullet W_i$$

$$= \lambda f(x) + \sum_{i=0}^{m} \left(\sum_{\alpha \in \mathbb{N}^n : |\alpha| + |LE(p)| \leq 2d} Q_\alpha^{(i)} x^\alpha + \sum_{\beta \in \mathbb{N}^n : \beta < LE(p)} P_\alpha^{(i)} \frac{x^\beta}{p(x)} \right) \bullet W_i.$$

In the above identity, if we replace each x^α by $\hat{y}_\alpha$ and $\frac{x^\beta}{p(x)}$ by $\hat{z}_\beta$, then we get the contradiction

$$a^T \hat{x} - b = \lambda \cdot \left((f^{(1)})^T \hat{y} + (f^{(2)})^T \hat{z} \right) + \sum_{i=0}^{m} Q_i(\hat{y}, \hat{z}) \bullet W_i \geq 0.$$

(ii) The proof is almost the same as for (i). The only difference is that

$$a^T x - \gamma = \lambda f(x) + \sum_{\nu \in \{0,1\}^m} \frac{g_1^{\nu_1} \cdots g_m^{\nu_m}}{p(x)} \sigma_\nu(x)$$

for some sos polynomials $\sigma_\nu(x)$ such that $\deg(g_i \sigma_i) \leq 2d_{pre}^{(R)}$. Note that $d = d_{pre}^{(R)}$. A similar contradiction argument like in (i) can be applied to prove the claim. $\quad\square$

Example 3.5 The convex set $\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq x_1^4 + x_1^2 x_2^2 + x_2^4\}$ can be defined as $S_{\mathbb{R}^2}(f)$ with rational function $f(x) = 1 - \frac{x_1^4 + x_1^2 x_2^2 + x_2^4}{x_1^2 + x_2^2}$. The set $S_{\mathbb{R}^2}(f)$ is the shaded area bounded by a thick curve in Figure 1. We have already seen that $f(x)$ is first order

sos concave. So $S_{\mathbb{R}^2}(f) = \mathcal{R}_{qmod}^{\mathbb{R}^2}(f)$, and by construction (3.7) it equals

$$
\left\{
\begin{array}{l}
(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : \quad \exists \ y_{ij}, z_{ij}, \quad s.t. \quad 1 \geq y_{20} + z_{04}, \\
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & x_1 & x_2 & 0 \\
0 & 0 & 0 & x_2 & 0 & 0 \\
1 & x_1 & x_2 & y_{20} - y_{02} & y_{11} & y_{02} \\
0 & x_2 & 0 & y_{11} & y_{02} & 0 \\
0 & 0 & 0 & y_{02} & 0 & 0
\end{bmatrix}
+
\begin{bmatrix}
z_{00} & z_{10} & z_{01} & -z_{02} & z_{11} & z_{02} \\
z_{10} & -z_{02} & z_{11} & -z_{12} & -z_{03} & z_{12} \\
z_{01} & z_{11} & z_{02} & -z_{03} & z_{12} & z_{03} \\
-z_{02} & -z_{12} & -z_{03} & z_{04} & -z_{13} & -z_{04} \\
z_{11} & -z_{03} & z_{12} & -z_{13} & -z_{04} & z_{13} \\
z_{02} & z_{12} & z_{03} & -z_{04} & z_{13} & z_{04}
\end{bmatrix}
\succeq 0
\end{array}
\right\}.
$$

The plot of the projection of the above coincides with the shaded area in Figure 1. $\quad\square$

3.3. Some special cases

3.3.1 Epigraph of rational functions

A function $f(x)$ is convex over a convex domain $\mathcal{D}$ if and only if its epigraph

$$\mathbf{epi}(f) := \{(x, t) \in \mathcal{D} \times \mathbb{R} : f(x) \leq t\}$$

is convex. The LMI $\mathcal{R}_{qmod}^{\mathcal{D}}(t - f)$ and $\mathcal{R}_{pre}^{\mathcal{D}}(t - f)$ can be constructed by thinking of $t - f(x)$ as a polynomial in x with coefficients in t. So, if $f(x)$ is q-module (resp. preordering) convex over $\mathcal{D}$, $\mathcal{R}_{qmod}^{\mathcal{D}}(t-f)$ (resp. $\mathcal{R}_{pre}^{\mathcal{D}}(t-f)$) is an SDP representation for $\mathbf{epi}(f)$.

Consider the special case that $f(x) = \frac{q(x)}{p(x)}$ is a univariate rational function which is convex on an interval $I = [a, b]$ and $p(x)$ is positive in (a, b). Note $I = \{x \in \mathbb{R} : x - a \geq 0, b - x \geq 0\}$ ($x - a \geq 0$ is not required if $a = -\infty$, and similarly for $b - x \geq 0$).

Theorem 3.6 Let $f(x) = \frac{q(x)}{p(x)}$ be a univariate rational function and $p(x)$ be a polynomial nonnegative on an interval I. If $f(x)$ is convex on I, then $\mathbf{epi}(f) = \mathcal{R}_{qmod}^{I}(f)$.

Proof First assume $I = [a, b]$ is finite. Since $\mathcal{R}_{qmod}^{[a,b]}(f)$ is linear in t, it suffices to show for any fixed t

$$\left\{x \in [a, b] : \exists y, z, s.t., x = y_1, (y, z, t) \in \mathcal{R}_{qmod}^{[a,b]}(f)\right\} = \{x \in [a, b] : f(x) \leq t\}.$$

In the above the right hand side is contained in the left hand side. Now we prove the reverse containment. For a contradiction, suppose there exists a tuple $(\hat{y}, \hat{z}, t) \in \mathcal{R}_{qmod}^{[a,b]}(f)$ such that $\hat{y}_1$ does not belong to the convex set $\{x \in [a, b] : f(x) \leq t\}$. Then there exists an affine function $c_0 + c_1 x$ such that

$$c_0 + c_1 \hat{x} < 0, \quad c_0 + c_1 x \geq 0, \quad \forall\, x \in [a, b] : f(x) \leq t.$$

Since $f(x)$ is convex over [a,b], by optimality condition, there exists $\lambda \geq 0$ such that

$$c_0 + c_1 x - \lambda(t - f(x)) = \frac{(c_0 + c_1 x - \lambda t)p(x) + \lambda q(x)}{p(x)} \geq 0, \quad \forall\, x \in [a, b].$$

Then we can see that $(c_0 + c_1 x - \lambda t)p(x) + \lambda q(x)$ is a univariate polynomial nonnegative on $[a, b]$. So there exist sos polynomials $s_0(x), s_1(x), s_2(x)$ of degrees at most $2d, 2d -$

$2, 2d - 2$ respectively (assume $\deg(p) + 1, \deg(q) \leq 2d$ and see Powers and Reznick [16]) such that

$$(c_0 + c_1 x - \lambda t)p(x) + \lambda q(x) = s_0(x) + (x - a)s_1(x) + (b - a)s_2(x)$$

and hence

$$c_0 + c_1 x = \lambda(t - f(x)) + \frac{s_0(x) + (x - a)s_1(x) + (b - a)s_2(x)}{p(x)}.$$

Write $s_0(x), s_1(x), s_2(x)$ as

$$s_i(x) = \mathfrak{m}_{d - d_i}(x)^T W_i \mathfrak{m}_{d - d_i}(x), \ i = 0, 1, 2, \ d_0 = 0, d_1 = d_2 = 1$$

for some symmetric $W_0, W_1, W_2 \succeq 0$. Then it holds

$$c_0 + c_1 x = \lambda(t - f(x)) + \sum_{i=0}^{2} \left(\frac{g_i(x)}{p(x)} \mathfrak{m}_{d - d_i}(x) \mathfrak{m}_{d - d_i}(x)^T \right) \bullet W_i \quad (d_0 = 0, d_1 = d_2 = 1)$$

$$= \lambda(t - f(x)) + \sum_{i=0}^{2} \left(\sum_{\alpha \in \mathbb{N}: |\alpha| + |LE(p)| \leq 2d} Q_\alpha^{(i)} x^\alpha + \sum_{\beta \in \mathbb{N}: \beta < LE(p)} P_\alpha^{(i)} \frac{x^\beta}{p(x)} \right) \bullet W_i$$

where $g_0(x) = 1, g_1(x) = x - a, g_2(x) = b - x$ and (3.5) is used. In the above identity, if we replace each x^α by $\hat{y}_\alpha$ and $\frac{x^\beta}{p(x)}$ by $\hat{z}_\beta$, then we get the contradiction

$$c_0 + c_1 \hat{x} = \lambda \cdot \left(t - (f^{(1)})^T \hat{y} - (f^{(2)})^T \hat{z} \right) + \sum_{i=0}^{2} Q_i(\hat{y}, \hat{z}) \bullet W_i \geq 0.$$

Therefore $\mathbf{epi}(f) = \mathcal{R}_{qmod}^{[a,b]}(f)$.

When I is an infinite interval of the form $[a, \infty) = \{x : x - a \geq 0\}$, $(-\infty, b] = \{x : b - x \geq 0\}$ or $(-\infty, \infty) = \{x : 1 \geq 0\}$, a similar argument can be applied to prove $\mathbf{epi}(f) = \mathcal{R}_{qmod}^{I}(f)$. □

3.3.2 Convex sets defined by structured rational functions

For the convenience of discussion, we define some basic convex sets ($r \leq s$)

$$K_{r,s} = \{(w, v) \in \mathbb{R}_+^r \times \mathbb{R}_+ : w_1 \cdots w_r \geq v^s\} \tag{3.10}$$

which are all SDP representable (see §3.3 in [1]).

First, consider epigraphs of rational functions of the form

$$f(x) = \frac{\sum_{i=1}^{P} (q_i(x))^{s_i}}{p_1(x) \cdots p_r(x)} \tag{3.11}$$

where $q_i(x), p_j(x)$ are polynomials nonnegative over $\mathcal{D}$ and the integer $s_i \geq r + 1$. Then

$$\mathbf{epi}(f) = \left\{ (x, t) \in \mathcal{D} \times \mathbb{R} : \sum_{i=1}^{P} (q_i(x))^{s_i} \leq p_1(x) \cdots p_r(x) t \right\}.$$

The epigraph $\mathbf{epi}(f)$ can be equivalently presented as

$$\mathbf{epi}(f) = \left\{ (x,t) \in \mathcal{D} \times \mathbb{R} : \exists u_i, w_i, \sum_{i=1}^{P} \frac{u_i^{s_i}}{w_1 \cdots w_r} \leq t, q_i(x) \leq u_i, w_i \leq p_i(x), i = 1, \ldots, P \right\}.$$

Note that

$$q_i(x) \leq u_i \Longleftrightarrow (x, u_i) \in \mathbf{epi}(q_i), \qquad w_i \leq p_i(x) \Longleftrightarrow (x, -w_i) \in \mathbf{epi}(-p_i),$$

$$\sum_{i=1}^{P} \frac{u_i^{s_i}}{w_1 \cdots w_r} \leq t \Longleftrightarrow \exists v_1, \cdots, v_P \geq 0, \ v_1 + \cdots + v_P \leq t, (w_1, \cdots, w_r, v_i, u_i) \in K_{r+1, s_i}.$$

Hence we obtain that

$$\mathbf{epi}(f) = \Big\{ (x,t) \in \mathcal{D} \times \mathbb{R} : \exists u_i, w_i, v_i, \sum_{i=1}^{P} v_i \leq t, (w_1, \cdots, w_r, v_i, u_i) \in K_{r+1, s_i},$$

$$(x, u_i) \in \mathbf{epi}(q_i), (x, -w_i) \in \mathbf{epi}(-p_i), i = 1, \ldots, P \Big\}.$$

When $q_i(x)$ are q-module (resp. preordering) convex over $\mathcal{D}$, we know $\mathbf{epi}(q_i) = \mathcal{L}^{\mathcal{D}}_{qmod}(u_i - q_i)$ (resp. $\mathbf{epi}(q_i) = \mathcal{L}^{\mathcal{D}}_{pre}(u_i - q_i)$), and similarly for $-p_i(x)$. Thus, we get the theorem:

Theorem 3.7 Suppose $f(x)$ is given in the form (3.11) and all $q_i(x), p_j(x)$ there are nonnegative over $\mathcal{D}$.

(i) If all $q_i(x), -p_j(x)$ are q-module convex over $\mathcal{D}$, then

$$\mathbf{epi}(f) = \left\{ (x,t) : \exists u_i, w_i, v_i, x \in \left(\bigcap_{i=1}^{P} \mathcal{L}^{\mathcal{D}}_{qmod}(u_i - q_i) \right) \bigcap \left(\bigcap_{j=1}^{P} \mathcal{L}^{\mathcal{D}}_{qmod}(p_j - w_j) \right), \right.$$

$$\left. \sum_{i=1}^{P} v_i \leq t, (w_1, \cdots, w_r, v_i, u_i) \in K_{r+1, s_i} \right\}.$$

(ii) If all $q_i(x), -p_j(x)$ are preordering convex over $\mathcal{D}$, then

$$\mathbf{epi}(f) = \left\{ (x,t) : \exists u_i, w_i, v_i, x \in \left(\bigcap_{i=1}^{P} \mathcal{L}^{\mathcal{D}}_{pre}(u_i - q_i) \right) \bigcap \left(\bigcap_{j=1}^{P} \mathcal{L}^{\mathcal{D}}_{pre}(p_j - w_j) \right), \right.$$

$$\left. \sum_{i=1}^{P} v_i \leq t, (w_1, \cdots, w_r, v_i, u_i) \in K_{r+1, s_i} \right\}.$$

Example 3.8 (i) Consider the epigraph

$$\left\{ (x_1, x_2, t) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} : t \geq \frac{(1 + x_2^2)^2}{x_1} \right\}.$$

The rational function here is in the form (3.11). By Theorem 3.7, it equals

$$\left\{ (x_1, x_2, t) : \exists u, \begin{bmatrix} t & u \\ u & x_1 \end{bmatrix} \succeq 0, \begin{bmatrix} u - 1 & x_2 \\ x_2 & 1 \end{bmatrix} \succeq 0 \right\}.$$

(ii) Consider the epigraph

$$\left\{ (x, t) \in B(0, 1) \times \mathbb{R} : t \geq \frac{(\sum_i x_i^2)^{n+1}}{(1 - x_1^2) \cdots (1 - x_n^2)} \right\}.$$

The rational function here is given in the form (3.11). By Theorem 3.7, it equals

$$\left\{ (x, t) : \exists\, u, v_i, w_i, \ s.t., \ \sum_{i=1}^{n} v_i \leq t, \begin{bmatrix} 1 - w_i & x_i \\ x_i & 1 \end{bmatrix} \succeq 0 \,, \right.$$
$$\left. \begin{bmatrix} u & x^T \\ x & I_n \end{bmatrix} \succeq 0, (u, w_1, \ldots, w_n, u) \in K_{n+1,n+1} \right\}. \qquad \square$$

Second, consider epigraphs of rational functions of the form

$$h(x) = \sum_{k=1}^{L} (f_k(x))^{b_k} \tag{3.12}$$

where $f_k(x)$ are rational functions given in form (3.11) and $b_k \geq 1$. Then the epigraph $\mathbf{epi}(h) = \{(x, t) \in \mathcal{D} \times \mathbb{R} : h(x) \leq t\}$ can be presented as

$$\mathbf{epi}(h) = \left\{ (x, t) : \exists \eta_k, \tau_k, \ \sum_{k=1}^{L} \tau_k \leq t, (x, \eta_k) \in \mathbf{epi}(f_k), (\tau_k, \eta_k) \in K_{1,b_k}, 1 \leq k \leq L \right\}.$$

Once SDP representations for all $\mathbf{epi}(f_k)$ are available, an SDP representation for $\mathbf{epi}(h)$ can be obtained consequently from the above.

Example 3.9 Consider the epigraph

$$\left\{ (x_1, x_2, t) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} : t \geq \left(\frac{1 + x_2^2}{x_1} \right)^2 \right\}.$$

From the above discussion, we know it equals

$$\left\{ (x_1, x_2, t) : \exists\, u, \begin{bmatrix} t & u \\ u & 1 \end{bmatrix} \succeq 0, \begin{bmatrix} x_1 & 0 & 1 \\ 0 & x_1 & x_2 \\ 1 & x_2 & u \end{bmatrix} \succeq 0 \right\}. \qquad \square$$

Third, consider the convex sets given in the form

$$T = \left\{ x \in \mathcal{D} : a(x) \geq \sum_{k=1}^{J} h_k(x) \right\} \tag{3.13}$$

where $a(x)$ is a polynomial and every $h_k(x)$ is given in the form (3.12). Then

$$T = \left\{ x : \exists t, \theta_1, \cdots, \theta_J, t \geq \sum_{k=1}^{J} \theta_k, (x, -t) \in \mathbf{epi}(-a(x)), \begin{matrix} (x, \theta_k) \in \mathbf{epi}(h_k(x)) \\ k = 1, \ldots, J \end{matrix} \right\}.$$

When $a(x)$ is q-module or preordering concave over $\mathcal{D}$, $\mathbf{epi}(-a(x))$ is representable by $\mathcal{L}_{qmod}^{\mathcal{D}}(a(x) - t)$ or $\mathcal{L}_{pre}^{\mathcal{D}}(a(x) - t)$. Once the SDP representations for $\mathbf{epi}(-a(x))$ and all $\mathbf{epi}(h_k(x))$ are available, an SDP representation for T can be obtained consequently.

Example 3.10 (i) Consider the convex set

$$\left\{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R} : 1 - 3x_2^2 \geq \frac{(1 + x_2^2)^2}{x_1}\right\}.$$

It is given in the form (3.13) with $\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}$. From the discussion above, it equals

$$\left\{(x_1, x_2) : \exists\, u, w, \begin{bmatrix} 1 - u & x_2 \\ x_2 & \frac{1}{3} \end{bmatrix} \succeq 0, \begin{bmatrix} w - 1 & x_2 \\ x_2 & 1 \end{bmatrix} \succeq 0, (u, x_1, w) \in K_{2,2}\right\}.$$

Note that $(u, x_1, w) \in K_{2,2}$ is equivalent to $\begin{bmatrix} u & w \\ w & x_1 \end{bmatrix} \succeq 0$.

(ii) Consider the convex set

$$\left\{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R} : 1 - x_2^2 \geq \frac{(1 + x_2^2)^2}{x_1^2}\right\}.$$

It is given in the form (3.13) with $\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}$. From the discussion above, it equals

$$\left\{(x_1, x_2) : \exists\, u, \begin{bmatrix} 1 + u & x_2 \\ x_2 & 1 - u \end{bmatrix} \succeq 0, \begin{bmatrix} x_1 & 0 & 1 \\ 0 & x_1 & x_2 \\ 1 & x_2 & u \end{bmatrix} \succeq 0\right\}. \qquad \square$$

4 Convex sets with singularities

Let $S_{\mathcal{D}}(f) = \{x \in \mathcal{D} : f(x) \geq 0\}$ be a convex set defined by a polynomial or rational function $f(x)$. Here $\mathcal{D} = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$ is still a convex domain defined by polynomials. Suppose the origin belongs to the boundary $\partial S_{\mathcal{D}}(f)$ and is a singular point of the hypersurface $\mathcal{Z}(f) = \{x \in \mathbb{C}^n : f(x) = 0\}$, i.e.,

$$f(0, 0) = 0, \ \nabla f(0, 0) = 0.$$

We are interested in finding SDP representability conditions for $S_{\mathcal{D}}(f)$.

As we have seen in Introduction, one natural approach to getting an SDP representation for $S_{\mathcal{D}}(f)$ is to find a "nicer" defining function (possibly a concave rational function). Let $p(x)$ be a polynomial or rational function positive in $int(\mathcal{D})$. Then we can see $S_{\mathcal{D}}(f)$ is the closure of the set

$$\left\{x \in int(\mathcal{D}) : \frac{f(x)}{p(x)} \geq 0\right\}.$$

If $\frac{f(x)}{p(x)}$ has nice properties, e.g., $\frac{f(x)}{p(x)}$ has special structures discussed in Section 3, or it is q-module or preordering concave over $\mathcal{D}$, then an explicit SDP representation for $S_{\mathcal{D}}(f)$ can be obtained. For instance, consider the convex set

$$\{(x_1, x_2) \in \mathbb{R}_+^2 : -x_1^3 + 3x_1 x_2^2 - (x_1^2 + x_2^2)^2 \geq 0\}.$$

The origin is a singular point on its boundary. If we choose $p(x) = x_2^2$, then it equals

$$\left\{(x_1, x_2) \in \mathbb{R}_+^2 : 3x_1 \geq 2x_1^2 + x_2^2 + \frac{x_1^3}{x_2^2} + \frac{x_1^4}{x_2^2}\right\}.$$

Then this set can be represented as

$$\Big\{(x_1, x_2) \in \mathbb{R}^2_+ : \exists\, u_1, u_2, u_3, u_4,\ s.t.,\ 3x_1 \geq 2u_1 + u_2 + u_3 + u_4,\ (u_1, x_1) \in K_{1,2},$$

$$(u_2, x_2) \in K_{1,2}, (u_3, x_2, x_2, x_1) \in K_{3,3}, (u_4, x_2, x_2, x_1) \in K_{3,4}\Big\}.$$

However, there is no general rule to find such a nice function $p(x)$. In convex analysis, there is a technique called *perspective transformation* which might be very useful here. Generally, we can assume

$$S_{\mathcal{D}}(f) \subset \mathbb{R}_+ \times \mathbb{R}^{n-1}, \qquad int(S_{\mathcal{D}}(f)) \neq \emptyset.$$

Define the perspective transformation $\mathfrak{p}$ as

$$\mathfrak{p}(x_1, x_2, \cdots, x_n) = (1/x_1, x_2/x_1, \cdots, x_n/x_1).$$

The image of $S_{\mathcal{D}}(f)$ of the perspective transformation $\mathfrak{p}$ is

$$\Big\{ (1/x_1, x_2/x_1, \cdots, x_n/x_1) : x \in S_{\mathcal{D}}(f) \Big\} \subset \mathbb{R}_+ \times \mathbb{R}^{n-1},$$

which is also convex (see §2.3 in [2]). Define new coordinates

$$\tilde{x}_1 = \frac{1}{x_1}, \quad \tilde{x}_2 = \frac{x_2}{x_1}, \quad \ldots, \quad \tilde{x}_n = \frac{x_n}{x_1}.$$

Denote $\tilde{\tilde{x}} = (\tilde{x}_2, \cdots, \tilde{x}_n)$. Suppose $f(x)$ has Laurent expansion around the origin

$$f(x) = f_b(x) - f_{b+1}(x) - \cdots - f_d(x)$$

where every $f_k(x)$ is a homogeneous part of degree k. Let

$$\tilde{f}(\tilde{x}) := \tilde{f}_0(\tilde{\tilde{x}}) - \frac{\tilde{f}_1(\tilde{\tilde{x}})}{\tilde{x}_1} - \cdots - \frac{\tilde{f}_{d-b}(\tilde{\tilde{x}})}{\tilde{x}_1^{d-b}}$$

where $\tilde{f}_i(\tilde{\tilde{x}}) := x_1^{b+i} f_{b+i}(\tilde{x})$. Define a new domain $\tilde{\mathcal{D}}$ as

$$\tilde{\mathcal{D}} = \{\tilde{x} \in \mathbb{R}^n : \tilde{g}_1(\tilde{x}) \geq 0, \cdots, \tilde{g}_m(\tilde{x}) \geq 0\}$$

where $\tilde{g}_i(\tilde{x}) = \frac{g_i(x)}{x_1^{\deg(g_i)}}$. Note that $\tilde{\mathcal{D}}$ is convex if and only if $\mathcal{D}$ is convex (see §2.3 in [2]). Therefore, under $\mathfrak{p}$, the set $S_{\mathcal{D}}(f)$ can be equivalently defined as

$$S_{\tilde{\mathcal{D}}}(\tilde{f}) = \{\tilde{x} \in \tilde{\mathcal{D}} : \tilde{f}(\tilde{x}) \geq 0\}.$$

Proposition 4.1 If $S_{\mathcal{D}}(f)$ is convex and the origin is a singular point on its boundary $\partial S_{\mathcal{D}}(f)$, then $\tilde{f}_0(\tilde{\tilde{x}}) \geq 0$ for any $\tilde{x} = (\tilde{x}_1, \tilde{\tilde{x}}) \in S_{\tilde{\mathcal{D}}}(\tilde{f})$.

Proof Fix $x = (x_1, \cdots, x_n) \in S_{\mathcal{D}}(f)$ and $\tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_n) \in S_{\tilde{\mathcal{D}}}(\tilde{f})$ such that $\tilde{x} = \mathfrak{p}(x_1, x_2, \cdots, x_n)$. By the convexity of $S_{\mathcal{D}}(f)$, the line segment $\{tx : 0 \leq t \leq 1\}$ belongs to $S_{\mathcal{D}}(f)$. Thus its image

$$\mathfrak{p}(tx_1, tx_2, \cdots, tx_n) = (\frac{1}{t}\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$$

belongs to $S_{\tilde{\mathcal{D}}}(\tilde{f})$ and $\tilde{f}(\frac{1}{t}\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) \geq 0$. Let $t \to 0$, then we get $\tilde{f}_0(\tilde{\tilde{x}}) \geq 0$. □

4.1. The case of structured $\tilde{f}_k(\tilde{\tilde{x}})$

In this subsection, we assume $\tilde{f}_k(\tilde{\tilde{x}})$ have special structures. Then the methods in Subsection 3.3.2 can be applied to construct SDP representations for $S_{\tilde{\mathcal{D}}}(\tilde{f})$.

Theorem 4.2 Suppose every $\tilde{f}_k(\tilde{\tilde{x}})\,(k = 1, \cdots, d - b)$ is given in the form

$$\tilde{f}_k(\tilde{\tilde{x}}) = \sum_{i=1}^{L_k} (q_{k,i}(\tilde{\tilde{x}}))^{r_{k,i}} + \sum_{j=1}^{R_k} \Big(\sum_{\ell=1}^{Q_{k,j}} (p_{k,j,\ell}(\tilde{\tilde{x}}))^{s_{k,j,\ell}} \Big)^k$$

for some polynomials $q_{k,i}(\tilde{\tilde{x}}), p_{k,j,\ell}(\tilde{\tilde{x}})$ which are nonnegative over $\tilde{\mathcal{D}}$ and integers $r_{k,i} \ge k+1$, $s_{k,j,\ell} \ge 2$ ($p_{k,j}(\tilde{\tilde{x}})$ can be any affine polynomial when $s_{k,j,\ell}$ is even). Then $S_{\tilde{\mathcal{D}}}(\tilde{f})$ can be represented as

$$\left\{ \tilde{x} : \exists\, u, u_{k,i}, v_{k,j,\ell},\ u \ge \sum_{k=1}^{d-b} \left(\sum_{i=1}^{L_k} \frac{(u_{k,i})^{r_{k,i}}}{\tilde{x}_1^k} + \sum_{j=1}^{R_k} \left(\sum_{\ell=1}^{Q_{k,j}} \frac{(v_{k,j,\ell})^{s_{k,j,\ell}}}{\tilde{x}_1} \right)^k \right), \right.$$
$$\left. (\tilde{\tilde{x}}, -u) \in \mathbf{epi}(-\tilde{f}_0),\ (\tilde{\tilde{x}}, u_{k,i}) \in \mathbf{epi}(q_{k,i}),\ (\tilde{\tilde{x}}, v_{k,j,\ell}) \in \mathbf{epi}(p_{k,j,\ell}) \right\}.$$

Furthermore, if $-\tilde{f}_0(\tilde{\tilde{x}})$ and all $q_{k,j}(\tilde{\tilde{x}}), p_{k,j,\ell}(\tilde{\tilde{x}})$ are q-module or preordering convex over $\tilde{\mathcal{D}}$, then $S_{\tilde{\mathcal{D}}}(\tilde{f})$ is SDP representable.

Proof The first conclusion is obvious by introducing new variables $u, u_{k,i}, v_{k,j,\ell}$. Note

$$u \ge \sum_{k=1}^{d-b} \Big(\sum_{i=1}^{L_k} \frac{(u_{k,i})^{r_{k,i}}}{\tilde{x}_1^k} + \sum_{j=1}^{R_k} \Big(\sum_{\ell=1}^{Q_{k,j}} \frac{(v_{k,j,\ell})^{s_{k,j,\ell}}}{\tilde{x}_1} \Big)^k \Big)$$

is equivalent to

$$u \ge \sum_{k=1}^{d-b} \Big(\sum_{i=1}^{L_k} \eta_{k,i} + \sum_{j=1}^{R_k} \xi_k \Big),\, (\tilde{x}_1, \cdots, \tilde{x}_1,\ \eta_{k,i}, u_{k,i}) \in K_{k+1, r_{k,i}},$$
$$(\xi_k, \zeta_k) \in K_{1,k},\ (\zeta_k, \tilde{x}_1, v_{k,j,\ell}) \in K_{2, s_{k,j,\ell}}.$$

When $-\tilde{f}_0(\tilde{\tilde{x}})$ is q-module (resp. preordering) convex over $\tilde{\mathcal{D}}$, $(\tilde{\tilde{x}}, -u) \in \mathbf{epi}(-\tilde{f}_0)$ is representable by $\mathcal{L}_{qmod}^{\tilde{\mathcal{D}}}(u - \tilde{f}_0)$ (resp. $\mathcal{L}_{pre}^{\tilde{\mathcal{D}}}(u - \tilde{f}_0)$). Similar results hold for $q_{k,j}(\tilde{\tilde{x}}), p_{k,j,\ell}(\tilde{\tilde{x}})$. Once the SDP representations for epigraphs of $-\tilde{f}_0(\tilde{\tilde{x}})$, $q_{k,j}(\tilde{\tilde{x}})$ and $p_{k,j,\ell}(\tilde{\tilde{x}})$ are all available, we can get an SDP representation for $S_{\tilde{\mathcal{D}}}(\tilde{f})$ consequently. $\square$

Now we show some examples on how to apply Theorem 4.2.

Example 4.3 (a) Consider convex set $S = \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R} : x_1^2 - x_1^3 - x_2^2 \ge 0\}$. Its boundary is a cubic curve and the origin is a singular node. This convex set is the shaded area bounded by a thick curve in Figure 2(a). The thin curves are other branches of this cubic curve. The perspective transformation $\mathfrak{p}$ gives

$$\tilde{S} = \left\{ (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}_+ \times \mathbb{R} : 1 - \tilde{x}_2^2 - \frac{1}{\tilde{x}_1} \ge 0 \right\}.$$
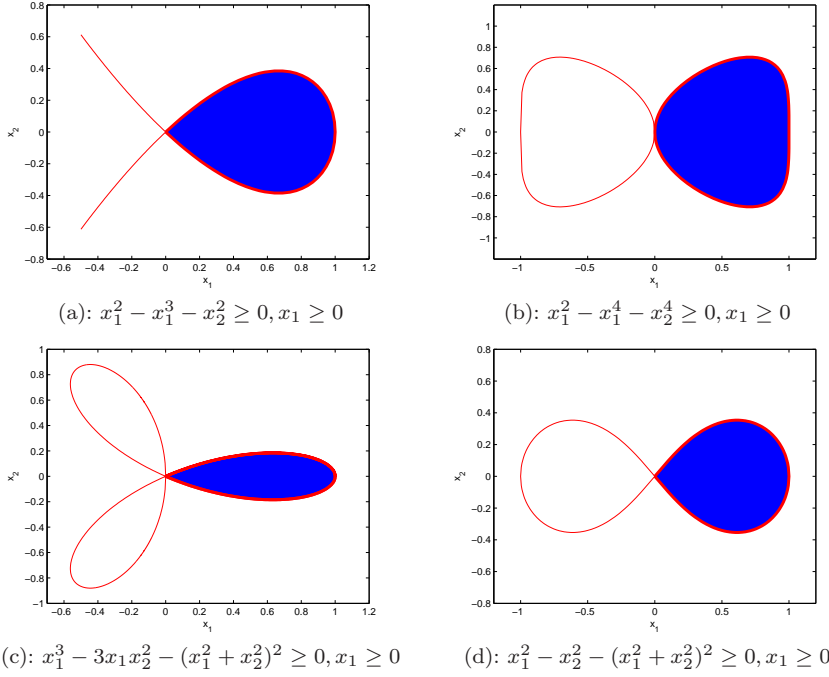
(a): $x_1^2 - x_1^3 - x_2^2 \geq 0, x_1 \geq 0$

(b): $x_1^2 - x_1^4 - x_2^4 \geq 0, x_1 \geq 0$

(c): $x_1^3 - 3x_1x_2^2 - (x_1^2 + x_2^2)^2 \geq 0, x_1 \geq 0$

(d): $x_1^2 - x_2^2 - (x_1^2 + x_2^2)^2 \geq 0, x_1 \geq 0$

Fig. 2 Four singular convex sets discussed in Example 4.3

.

From the lifted LMI of $\tilde{S}$ and $S = \mathfrak{p}^{-1}(\tilde{S})$, we get

$$S = \left\{ (x_1, x_2) : \exists\, u, \ s.t., \begin{bmatrix} x_1 - u & x_2 \\ x_2 & x_1 1 \end{bmatrix} \succeq 0, \begin{bmatrix} u & x_1 \\ x_1 & x_1 \end{bmatrix} \succeq 0 \right\}.$$

The plot of the above coincides with the shaded area in Figure 2(a).

(b) Consider convex set $S = \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R} : x_1^2 - x_2^4 - x_2^4 \geq 0\}$. The origin is a singular tacnode on the boundary. This convex set is the shaded area bounded by a thick curve in Figure 2(b). The thin curve is the other branch of the singular curve $x_1^2 - x_2^4 - x_2^4 = 0$. The perspective transformation $\mathfrak{p}$ gives

$$\tilde{S} = \left\{ (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}_+ \times \mathbb{R} : 1 \geq \frac{1 + \tilde{x}_2^4}{\tilde{x}_1^2} \right\}.$$

From the lifted LMI of $\tilde{S}$ and $S = \mathfrak{p}^{-1}(\tilde{S})$, we get

$$S = \left\{ (x_1, x_2) : \exists\, u_1, u_2, \ s.t., \begin{bmatrix} x_1 - u_1 & u_2 \\ u_2 & x_1 + u_1 \end{bmatrix} \succeq 0, \begin{bmatrix} u_1 & x_1 \\ x_1 & 1 \end{bmatrix} \succeq 0, \begin{bmatrix} u_2 & x_2 \\ x_2 & 1 \end{bmatrix} \succeq 0 \right\}.$$

The plot of the above coincides with the shaded area in Figure 2(b).

(c) Consider convex set $S = \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R} : x_1^3 - 3x_1x_2^2 - (x_1^2 + x_2^2)^2 \geq 0\}$. The origin is a singular triple point on the boundary. This convex set is the shaded area

bounded by a thick curve in Figure 2(c). The thin curves are other branches of the singular curve $x_1^3 - 3x_1 x_2^2 - (x_1^2 + x_2^2)^2 = 0$. The perspective transformation $\mathfrak{p}$ gives

$$\tilde{S} = \left\{ (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}_+ \times \mathbb{R} : 1 - 3\tilde{x}_2^2 - \frac{(1 + \tilde{x}_2^2)^2}{\tilde{x}_1} \geq 0 \right\}.$$

From the lifted LMI of $\tilde{S}$ and $S = \mathfrak{p}^{-1}(\tilde{S})$, we get

$$S = \left\{ (x_1, x_2) : \exists\, u, w, \; s.t., \; \begin{bmatrix} x_1 - u & x_2 \\ x_2 & \frac{1}{3}x_1 \end{bmatrix} \succeq 0, \; \begin{bmatrix} u & w \\ w & 1 \end{bmatrix} \succeq 0, \; \begin{bmatrix} w - x_1 & x_2 \\ x_2 & x_1 \end{bmatrix} \succeq 0 \right\}.$$

The plot of the above coincides with the shaded area in Figure 2(c).

(d) Consider convex set $S = \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R} : x_1^2 - x_2^2 - (x_1^2 + x_2^2)^2 \geq 0\}$. The origin is a singular node on the boundary which is one branch of the lemniscate curve $x_1^2 - x_2^2 - (x_1^2 + x_2^2)^2 = 0$. This convex set is the shaded area bounded a thick lemniscate curve in Figure 2(d). The thin curve is the other branch of the lemniscate curve. The perspective transformation $\mathfrak{p}$ gives

$$\tilde{S} = \left\{ (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}_+ \times \mathbb{R} : (1 - \tilde{x}_2^2) \geq \left(\frac{1 + \tilde{x}_2^2}{\tilde{x}_1} \right)^2 \right\}.$$

From the lifted LMI of $\tilde{S}$ and $S = \mathfrak{p}^{-1}(\tilde{S})$, we get

$$S = \left\{ (x_1, x_2) : \exists\, u, \; s.t. \; \begin{bmatrix} x_1 + u & x_2 \\ x_2 & x_1 - u \end{bmatrix} \succeq 0, \; \begin{bmatrix} u & x_1 & x_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{bmatrix} \succeq 0 \right\}.$$

The plot of the above coincides with the shaded area in Figure 2(d). □

4.2. The case of two consecutive homogeneous parts

In this subsection, we consider the special case that $f(x) = f_b(x) - f_{b+1}(x)$ having two consecutive homogeneous parts. Then, after perspective transformation, we get

$$S_{\tilde{\mathcal{D}}}(\tilde{f}) = \left\{ \tilde{x} \in \tilde{\mathcal{D}} : \tilde{f}_0(\tilde{x}) - \frac{\tilde{f}_1(\tilde{x})}{\tilde{x}_1} \geq 0 \right\}.$$

By Proposition 4.1, for any $\tilde{x} \in \tilde{\mathcal{D}}$, we have $f_0(\tilde{x}) \geq 0$. Let $\mathcal{D}'$ be the intersection of $\{\tilde{x} : f_0(\tilde{x}) \geq 0\}$ and the projection of $\mathcal{D} \subset \mathbb{R}_+ \times \mathbb{R}^{n-1}$ into $\mathbb{R}^{n-1}$, which is also convex. Then we get

$$S_{\tilde{\mathcal{D}}}(\tilde{f}) = \left\{ (\tilde{x}, \tilde{x}_1) \in \mathcal{D}' \times \mathbb{R}_+ : \tilde{x}_1 \geq h(\tilde{x}) \right\}, \quad h(\tilde{x}) := \frac{\tilde{f}_1(\tilde{x})}{\tilde{f}_0(\tilde{x})}.$$

Then $S_{\tilde{\mathcal{D}}}(\tilde{f})$ is the epigraph $\mathbf{epi}(h)$ of the rational function $h(\tilde{x})$ over the convex domain $\mathcal{D}'$. Note that $h(\tilde{x})$ is convex over $\mathcal{D}'$ if and only if $S_{\tilde{\mathcal{D}}}(\tilde{f})$ is convex, which then holds if and only if $S_{\mathcal{D}}(f)$ is convex. So, if $h(\tilde{x})$ is q-module or preordering convex over $\mathcal{D}'$, then $\mathcal{R}_{qmod}^{\mathcal{D}'}(\tilde{x}_1 - h)$ or $\mathcal{R}_{pre}^{\mathcal{D}'}(\tilde{x}_1 - h)$ is an SDP representation for $\mathbf{epi}(h)$, and then one can be obtained for $S_{\mathcal{D}}(h)$ after the inverse perspective transformation.

A very interesting case is $n = 2$. Then $\mathcal{D}'$ must be an interval I of the real line.

Theorem 4.4 Let $n = 2$ and $\mathcal{D}' = I$ be an interval as above. If $S_{\mathcal{D}}(f)$ is convex, then $S_{\tilde{\mathcal{D}}}(\tilde{f}) = \mathcal{R}_{qmod}^I(\tilde{x}_1 - h)$, and hence $S_{\mathcal{D}}(f) = \mathfrak{p}^{-1}(\mathcal{R}_{qmod}^I(\tilde{x}_1 - h))$.

Proof When $n = 2$, $f(\tilde{x})$ is a univariate rational function. When $S_\mathcal{D}(f)$ is convex, $S_{\tilde{\mathcal{D}}}(\tilde{f})$ is also convex. Since $S_{\tilde{\mathcal{D}}}(\tilde{f})$ is the epigraph of $f(\tilde{x})$, $f(\tilde{x})$ must be a univariate rational function convex over the interval I. By Theorem 3.6, its epigraph is representable by $\mathcal{R}^I_{qmod}(\tilde{x}_1 - h)$. Thus $S_{\tilde{\mathcal{D}}}(\tilde{f}) = \mathcal{R}^I_{qmod}(\tilde{x}_1 - h)$. After the inverse perspective transformation, we can get an SDP representation for $S_\mathcal{D}(f)$. $\square$

Now we see some examples on how to find SDP representations for singular convex sets by applying Theorem 4.4.

Example 4.5 (i) We revisit the singular convex set (a) in Example 4.3. After the perspective transformation $\mathfrak{p}$, we get $\tilde{S} = \left\{ (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R} \times [-1, 1] : \tilde{x}_1 \geq \frac{1}{1 - \tilde{x}_2^2} \right\}$. From the lifted LMI of $\tilde{S}$ and $S = \mathfrak{p}^{-1}(\tilde{S})$, we know S equals

$$
\left\{
(x_1, x_2) :
\begin{array}{l}
\exists\, y_2, y_3, y_4, z_0, z_1, \quad 1 \geq z_0, \quad
\begin{bmatrix} x_1 - y_2 & x_2 - y_3 \\ x_2 - y_3 & y_2 - y_4 \end{bmatrix} \succeq 0, \\[6pt]
\begin{bmatrix} x_1 & x_2 & y_2 \\ x_2 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix} \succeq 0,
\begin{bmatrix} z_0 & z_1 & -x_1 + z_0 \\ z_1 & -x_1 + z_0 & -x_2 + z_1 \\ -x_1 + z_0 & -x_2 + z_1 & -y_2 - x_1 + z_0 \end{bmatrix} \succeq 0
\end{array}
\right\}.
$$

The plot of the above coincides with the shaded area in Figure 2(a).

(ii) Revisit the singular convex set (c) in Example 4.3. After the perspective transformation $\mathfrak{p}$, we get $\tilde{S} = \left\{ (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R} \times \frac{1}{\sqrt{3}}[-1, 1] : \tilde{x}_1 \geq -\frac{x_2^2}{3} - \frac{7}{9} + \frac{16}{9(1 - 3x_2^2)} \right\}$. From the lifted LMI of $\tilde{S}$ and $S = \mathfrak{p}^{-1}(\tilde{S})$, we know S equals

$$
\left\{
(x_1, x_2) :
\begin{array}{l}
\exists\, y_2, y_3, y_4, z_0, z_1, \ 1 \geq -\frac{1}{3}y_2 - \frac{7}{9}x_1 + \frac{16}{9}z_0,
\begin{bmatrix} x_1 - 3y_2 & x_2 - 3y_3 \\ x_2 - 3y_3 & y_2 - 3y_4 \end{bmatrix} \succeq 0, \\[6pt]
\begin{bmatrix} x_1 & x_2 & y_2 \\ x_2 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix} \succeq 0,
\begin{bmatrix} z_0 & z_1 & \frac{1}{3}(z_0 - x_1) \\ z_1 & \frac{1}{3}(z_0 - x_1) & \frac{1}{3}(z_1 - x_2) \\ \frac{1}{3}(z_0 - x_1) & \frac{1}{3}(z_1 - x_2) & -\frac{1}{3}y_2 + \frac{1}{9}(z_0 - x_1) \end{bmatrix} \succeq 0
\end{array}
\right\}.
$$

The plot of the above also coincides with the shaded area in Figure 2(c). $\square$

Now let us conclude this subsection with an example such that Theorem 4.4 can be applied to get an SDP representation for $S_\mathcal{D}(f)$ while Theorem 4.2 can not.

Example 4.6 Consider the convex set $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1(x_1^2 + x_2^2) - x_1^4 - x_1^2 x_2^2 - x_2^4 \geq 0\}$. The origin is a singular point on the boundary ∂S which is a quartic bean curve. The picture of this convex set is the shaded area bounded by the thick bean curve in Figure 3. This example was from Henrion [9], where an SDP representation based on a rational parametrization of the bean curve is given. After the perspective transformation $\mathfrak{p}$, we get $\tilde{S} = \left\{ (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2 : \tilde{x}_1 \geq f(\tilde{x}_2) := \tilde{x}_2^2 + \frac{1}{1 + \tilde{x}_2^2} \right\}$. The function $f(\tilde{x}_2)$ does not have structures required by Theorem 4.2. Obviously $\mathcal{D}' = (-\infty, \infty)$. We can check that $f(\tilde{x}_2)$ is convex over $(-\infty, \infty)$, so its epigraph $\mathbf{epi}(f) = \mathcal{R}^{(-\infty, \infty)}_{qmod}(\tilde{x}_1 - f)$. From $S = \mathfrak{p}^{-1}(\tilde{S})$, we know S equals

$$
\left\{
(x_1, x_2) : \exists\, y_2, z_0, z_1, \ 1 \geq y_2 + z_0,
\begin{bmatrix} z_0 & z_1 & x_1 - z_0 \\ z_1 & x_1 - z_0 & x_2 - z_1 \\ x_1 - z_0 & x_2 - z_1 & y_2 - x_1 + z_0 \end{bmatrix} \succeq 0
\right\}.
$$

In the above, $y_2 + z_0$ can be placed by one parameter. The plot of the above is the shaded area in Figure 3. $\square$
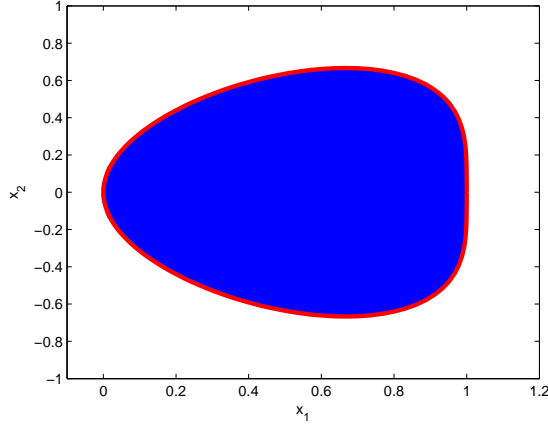
Fig. 3 The singular convex set defined by $x_1(x_1^2 + x_2^2) - x_1^4 - x_1^2 x_2^2 - x_2^4 \geq 0$.

4.3. General case

For the general case, we have the following result by applying Theorem 3.4.

Theorem 4.7 Assume $\tilde{\mathcal{D}}$ and $S_{\tilde{\mathcal{D}}}(\tilde{f})$ are both convex and have nonempty interior, and $\dim\left(\{\tilde{x}_1 = 0\} \cap \partial S_{\tilde{\mathcal{D}}}(\tilde{f})\right) < n - 1$.

(i) If $\tilde{f}(\tilde{x})$ and every $\tilde{g}_i(\tilde{x})$ are q-module concave over $\tilde{\mathcal{D}}$ with respect to $\tilde{x}_1^{d-b}$, then
$S_{\tilde{\mathcal{D}}}(\tilde{f}) = \mathcal{R}_{qmod}^{\tilde{\mathcal{D}}}(\tilde{f})$.
(ii) If $\tilde{f}(\tilde{x})$ and every $\tilde{g}_i(\tilde{x})$ are preordering concave over $\tilde{\mathcal{D}}$ with respect to $\tilde{x}_1^{d-b}$, then
$S_{\tilde{\mathcal{D}}}(\tilde{f}) = \mathcal{R}_{pre}^{\tilde{\mathcal{D}}}(\tilde{f})$.

After a perspective transformation $\mathfrak{p}$, the origin, if it is a singular point on the boundary $\partial S_{\mathcal{D}}(f)$, is mapped to one point at infinity of $S_{\tilde{\mathcal{D}}}(\tilde{f})$, i.e., $S_{\tilde{\mathcal{D}}}(\tilde{f})$ itself does not have a point which is the image $\mathfrak{p}(0)$. The mapping $\mathfrak{p}$ is smooth when $x_1 > 0$. At any point $x \in S_{\mathcal{D}}(f)$ with $x_1 > 0$, the mapping $\mathfrak{p}$ will preserve the singularity or nonsingularity at x. In this sense, the perspective transformation $\mathfrak{p}$ will remove one or more singular points. Of course, the new convex set $S_{\tilde{\mathcal{D}}}(\tilde{f})$ might have singularity somewhere else. In this case, we can apply some coordinate transformation to shift one singular point to the origin and then apply the perspective transformation again. So a sequence of perspective transformations might be applied. If there are finitely many singular points on the boundary, a finite number of perspective transformations can be applied to remove all the singularities. However, this approach might not work if there are infinitely many singular points, i.e., the singular locus is positively dimensional. For instance, the convex set

$$\{x \in \mathbb{R}^3 : \left(1 - (x_1 - 1)^2 - x_2^2\right)^3 - x_3^4 \geq 0\},$$

has a singular locus of dimension one. In this case, a finite number of perspective transformations is usually not able to remove all the singularies.

5 Some discussions

We conclude this paper with some discussions and open questions.

More general convex sets It is very natural to consider general convex sets of the form

$$S_{\mathcal{D}}(f_1, \cdots, f_k) = \{x \in \mathcal{D}: f_1(x) \geq 0, \cdots, f_k(x) \geq 0\},$$

where $f_1(x), \ldots, f_k(x)$ are given polynomials or rational functions concave over the convex domain $\mathcal{D} = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$ defined by polynomials $g_1(x), \cdots, g_m(x)$. Note that

$$S_{\mathcal{D}}(f_1, \cdots, f_k) = S_{\mathcal{D}}(f_1) \cap \cdots \cap S_{\mathcal{D}}(f_k).$$

So it suffices to consider each individual $S_{\mathcal{D}}(f_i)$ separately.

One interesting but unaddressed case is that the defining polynomials f_i are concave in some neighborhood of $S_{\mathcal{D}}(f_1, \cdots, f_k)$ but neither q-module nor preordering concave over $\mathcal{D}$. In this situation, is it always possible to find another domain $\mathcal{D}' \supset S_{\mathcal{D}}(f_1, \cdots, f_k)$ such that the f_i is q-module or preordering concave over $\mathcal{D}'$ with respect to some other (p', q')? Or is it always possible to find a different set of defining polynomials for $\mathcal{D} = \{x \in \mathbb{R}^n : \hat{g}_1(x) \geq 0, \cdots, \hat{g}_{m'}(x) \geq 0\}$ such that f_i is q-module or preordering concave over $\mathcal{D}$ using new defining polynomials with respect to some different $(\hat{p}, \hat{q})$? This is an interesting future research topic.

The separability in Positivestellensatz The rational function $f(x)$ is concave over $\mathcal{D}$ if and only if

$$f(u) + \nabla f(u)^T (x - u) - f(x) \geq 0, \, \forall \, x, u \in \mathcal{D}.$$

By Positivestellensatz of Stengle [21], the above is true if and only if

$$\eta(x, u) \cdot f_{den}(x) f_{den}^2(u) \cdot (f(u) + \nabla f(u)^T (x - u) - f(x)) =$$

$$\sum_{\nu \in \{0,1\}^m} g_1^{\nu_1}(x) \cdots g_m^{\nu_m}(x) \left(\sum_{\mu \in \{0,1\}^m} g_1^{\mu_1}(u) \cdots g_m^{\mu_m}(u) \sigma_{\nu,\mu}(x, u) \right)$$

for some sos polynomials $\eta(x, u), \sigma_{\nu,\mu}(x, u)$. Here f_{den} is the denominator of $f(x)$ which is nonnegative over $\mathcal{D}$. When $\eta(x, u) = \eta_1(x)\eta_2(u)$ is separable, we can choose $p(x) = \eta_1(x)f_{den}(x)$ and $q = \eta_2(u)f_{den}^2(u)$, and then get an SDP representation for $S_{\mathcal{D}}(f)$ by following the approach in Section 3. However, in general case, is it always possible to find a factor $\eta(x, u)$ that is separable? Or what conditions make the factor $\eta(x, u)$ to be separable? This is an interesting future research topic.

Resolution of singularities In algebraic geometry [5], a well known result is that any singular algebraic variety (over a ground field with characteristic zero) is birational to a nonsingular algebraic variety. But the convexity might not be preserved by this birational transformation. Given a convex semialgebraic set in $\mathbb{R}^n$ with singular boundary, is it birational to a convex semialgebraic set with nonsingular boundary? Or is every convex semialgebraic set in $\mathbb{R}^n$ equal to the projection of some higher dimensional convex semialgebraic set with nonsingular boundary? To the best knowledge of the author, all such questions are open. An interesting future work is to discuss how to remove the singular locus of convex semilagebraic sets while preserving the convexity.

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