Convex Hulls of Quadratically Parameterized Sets With Quadratic Constraints

Jiawang Nie*

October 10, 2011

Dedicated to Bill Helton on the occasion of his 65th birthday.

Abstract

Let V be a semialgebraic set parameterized as

$$\{(f_1(x), \dots, f_m(x)) : x \in T\}$$

for quadratic polynomials $f_0, \dots, f_m$ and a subset T of $\mathbb{R}^n$. This paper studies semidefinite representation of the convex hull $\mathrm{conv}(V)$ or its closure, i.e., describing $\mathrm{conv}(V)$ by projections of spectrahedra (defined by linear matrix inequalities). When T is defined by a single quadratic constraint, we prove that $\mathrm{conv}(V)$ is equal to the first order moment type semidefinite relaxation of V, up to taking closures. Similar results hold when every f_i is a quadratic form and T is defined by two homogeneous (modulo constants) quadratic constraints, or when all f_i are quadratic rational functions with a common denominator and T is defined by a single quadratic constraint, under some general conditions.

1 Introduction

A basic question in convex algebraic geometry is to find convex hulls of semialgebraic sets. A typical class of semialgebraic sets is parameterized by multivariate polynomial functions defined on some sets. Let $V \subset \mathbb{R}^m$ be a set parameterized as

$$V = \{(f_1(x), \dots, f_m(x)) : x \in T\} \tag{1.1}$$

with every $f_i(x)$ being a polynomial and T a semialgebraic set in $\mathbb{R}^n$. We are interested in finding a representation for the convex hull $\mathrm{conv}(V)$ of V or its closure, based on $f_1, \dots, f_m$ and T. Since V is semialgebraic, $\mathrm{conv}(V)$ is a convex semialgebraic set. Thus, one wonders whether $\mathrm{conv}(V)$ is representable by a spectrahedron or its projection, i.e., as a feasible set of *semidefinite programming (SDP)*. A *spectrahedron* of $\mathbb{R}^k$ is a set defined by a linear matrix inequality (LMI) like

$$L_0 + w_1 L_1 + \cdots + w_k L_k \succeq 0$$

for some constant symmetric matrices $L_0, \dots, L_k$. Here the notation $X \succeq 0$ (resp. $X \succ 0$) means the matrix X is positive semidefinite (resp. definite). Equivalently, a spectrahedron is

*Department of Mathematics, University of California, 9500 Gilman Drive, La Jolla, CA 92093. Email: njw@math.ucsd.edu. The research was partially supported by NSF grants DMS-0757212 and DMS-0844775.

the intersection of a positive semidefinite cone and an affine linear subspace. Not every convex semialgebraic set is a spectrahedron, as found by Helton and Vinnikov [7]. Actually, they [7] proved a necessary condition called *rigid convexity* for a set to be a spectrahedron. They also proved that rigid convexity is sufficient in the two dimensional case. Typically, projections of spectrahedra are required in representing convex sets (if so, they are also called *semidefinite representations*). It has been found that a very general class of convex sets are representable as projections of spectrahedra, as shown in [4, 5]. The proofs used sum of squares (SOS) type representations of polynomials that are positive on compact semialgebraic sets, as given by Putinar [15] or Schmüdgen [16]. More recent work about semidefinite representations of convex semialgebraic sets can be found in [6, 9, 10, 11, 12].

A natural semidefinite relaxation for the convex hull $\text{conv}(V)$ can be obtained by using the moment approach [9, 13]. To describe it briefly, we consider the simple case that $n = 1$, $T = \mathbb{R}$ and $(f_1(x), f_2(x), f_3(x)) = (x^2, x^3, x^4)$ with $m = 3$. The most basic moment type semidefinite relaxation of $\text{conv}(V)$ in this case is

$$R = \left\{ (y_2, y_3, y_4) : \begin{bmatrix} 1 & y_1 & y_2 \\ y_2 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{bmatrix} \succeq 0 \text{ for some } y_1 \in \mathbb{R} \right\}.$$

The underlying idea is to replace each monomial x^i by a lifting variable y_i and to pose the LMI in the definition of R, which is due to the fact that

$$\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}^T = \begin{bmatrix} 1 & x & x^2 \\ x & x^2 & x^3 \\ x^2 & x^3 & x^4 \end{bmatrix} \succeq 0 \qquad \forall\, x \in \mathbb{R}.$$

If $n = 1$, the sets R and $\text{conv}(V)$ (or their closures) are equal (cf. [13]). When $T = \mathbb{R}^n$ with $n > 1$, we have similar results if every f_i is quadratic or every f_i is quartic but $n = 2$ (cf. [8]). However, in more general cases, similar results typically do not exist anymore.

In this paper, we consider the special case that every f_i is quadratic and T is a quadratic set of $\mathbb{R}^n$. When T is defined by a single quadratic constraint, we will show that the first order moment type semidefinite relaxation represents $\text{conv}(V)$ or its closure as the projection of a spectrahedron (Section 2). This is also true when every f_i is a quadratic form and T is defined by two homogeneous (modulo constants) quadratic constraints (Section 3), or when all f_i are quadratic rational functions with a common denominator and T is defined by a single quadratic constraint (Section 4), under some general conditions.

Notations The symbol $\mathbb{R}$ (resp. $\mathbb{R}_+$) denotes the set of (resp. nonnegative) real numbers. For a symmetric matrix, $X \prec 0$ means X is negative definite ($-X \succ 0$); $\bullet$ denotes the standard Frobenius inner product in matrix spaces; $\|\cdot\|_2$ denotes the standard 2-norm. The superscript ^{T} denotes the transpose of a matrix; $\overline{K}$ denotes the closure of a set K in a Euclidean space, and $\text{conv}(K)$ denotes the convex hull of K. Given a function $q(x)$ defined on $\mathbb{R}^n$, denote

$$S(q) = \{x \in \mathbb{R}^n : q(x) \geq 0\}, \quad E(q) = \{x \in \mathbb{R}^n : q(x) = 0\}.$$

2 A single quadratic constraint

Suppose $V \subset \mathbb{R}^m$ is a semialgebraic set parameterized as

$$V = \{(f_1(x), \dots, f_m(x)) : x \in T\} \tag{2.1}$$

where every $f_i(x) = a_i + b_i^T x + x^T F_i x$ is quadratic and $T \subseteq \mathbb{R}^n$ is defined by a single quadratic inequality $q(x) \geq 0$ or equality $q(x) = 0$. The a_i, b_i, F_i are vectors or symmetric matrices of proper dimensions. Similarly, write

$$q(x) = c + d^T x + x^T Q x.$$

For every $x \in T$, it always holds that for $X = xx^T$

$$f_i(x) = a_i + b_i^T x + F_i \bullet X, \quad q(x) = c + d^T x + Q \bullet X \geq 0, \quad \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0.$$

Clearly, when $T = S(q)$, the convex hull $\operatorname{conv}(V)$ of V is contained in the convex set

$$\mathcal{W}_{in} = \left\{ (a_1 + b_1^T x + F_1 \bullet X, \dots, a_m + b_m^T x + F_m \bullet X) \;\middle|\; \begin{array}{c} \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0, \\ c + d^T x + Q \bullet X \geq 0 \end{array} \right\}.$$

When $T = E(q)$, the convex hull $\operatorname{conv}(V)$ is then contained in the convex set

$$\mathcal{W}_{eq} = \left\{ (a_1 + b_1^T x + F_1 \bullet X, \dots, a_m + b_m^T x + F_m \bullet X) \;\middle|\; \begin{array}{c} \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0, \\ c + d^T x + Q \bullet X = 0 \end{array} \right\}.$$

Both $\mathcal{W}_{in}$ and $\mathcal{W}_{eq}$ are projections of spectrahedra. One wonders whether $\mathcal{W}_{in}$ or $\mathcal{W}_{eq}$ is equal to $\operatorname{conv}(V)$. Interestingly, this is almost always true, as given below.

Theorem 2.1. *Let V, T, W_{in}, W_{eq}, q be defined as above, and $T \neq \emptyset$.*

(i) Let $T = S(q)$. If T is compact, then $conv(V) = \mathcal{W}_{in}$; otherwise, $\overline{conv(V)} = \overline{\mathcal{W}_{in}}$.

(ii) Let $T = E(q)$. If T is compact, then $conv(V) = \mathcal{W}_{eq}$; otherwise, $\overline{conv(V)} = \overline{\mathcal{W}_{eq}}$.

To prove the above theorem, we need a result on quadratic moment problems. A *quadratic moment sequence* is a triple $(t, z, Z) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n \times n}$ with Z symmetric. We say (t, z, Z) admits a representing measure supported on T if there exists a positive Borel measure μ with its support $\operatorname{supp}(\mu) \subseteq T$ and

$$t = \int 1 \, d\mu, \quad z = \int x d\mu, \quad Z = \int xx^T d\mu.$$

Denote by $\mathscr{R}(T)$ the set of all such quadratic moment sequences (t, z, Z) satisfying the above.

Theorem 2.2. *([2, Theorems 4.7,4.8]) Let $q(x) = c + d^T x + x^T Q x$, $T = S(q)$ or $E(q)$ be nonempty, and (t, z, Z) be a quadratic moment sequence satisfying*

$$\begin{bmatrix} 1 & z^T \\ z & Z \end{bmatrix} \succeq 0, \quad \begin{cases} c + d^T z + Q \bullet Z \geq 0, & \text{if } T = S(q); \\ c + d^T z + Q \bullet Z = 0, & \text{if } T = E(q). \end{cases}$$

(i) If $S(q)$ is compact, then $(t, z, Z) \in \mathscr{R}(S(q))$; otherwise, $(t, z, Z) \in \overline{\mathscr{R}(S(q))}$.

(ii) If $E(q)$ is compact, then $(t, z, Z) \in \mathscr{R}(E(q))$; otherwise, $(t, z, Z) \in \overline{\mathscr{R}(E(q))}$.

Proof of Theorem 2.1 (i) We have already seen that $\text{conv}(V) \subseteq \mathcal{W}_{in}$, which clearly implies $\overline{\text{conv}(V)} \subseteq \overline{\mathcal{W}_{in}}$. Suppose (x, X) is a pair satisfying the conditions in $\mathcal{W}_{in}$.

If $T = S(q)$ is compact, by Theorem 2.2, the quadratic moment sequence $(1, x, X)$ admits a representing measure supported in T. By the Bayer-Teichmann Theorem [1], the triple $(1, x, X)$ also admits a measure having a finite support contained in T. So, there exist $u_1, \ldots, u_r \in T$ and scalars $\lambda_1 > 0, \ldots, \lambda_r > 0$ such that

$$\begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 & u_1^T \\ u_1 & u_1 u_1^T \end{bmatrix} + \cdots + \lambda_r \begin{bmatrix} 1 & u_r^T \\ u_r & u_r u_r^T \end{bmatrix}.$$

The above implies that

$$(a_1 + b_1^T x + F_1 \bullet X, \ldots, a_m + b_m^T x + F_m \bullet X) = \sum_{i=1}^{r} \lambda_i (f_1(u_i), \ldots, f_m(u_i)).$$

Clearly, $\lambda_1 + \cdots + \lambda_r = 1$. So, $\mathcal{W}_{in} \subseteq \text{conv}(V)$ and hence $\mathcal{W}_{in} = \text{conv}(V)$.

If $T = S(q)$ is noncompact, the quadratic moment sequence $(1, x, X) \in \overline{\mathscr{R}(T)}$, and

$$(1, x, X) = \lim_{k \to \infty} (1, x^{(k)}, X^{(k)}), \text{ with every } (1, x^{(k)}, X^{(k)}) \in \mathscr{R}(T).$$

As we have seen in (i), every

$$(a_1 + b_1^T x^{(k)} + F_1 \bullet X^{(k)}, \ldots, a_m + b_m^T x^{(k)} + F_m \bullet X^{(k)}) \in \text{conv}(V).$$

This implies

$$(a_1 + b_1^T x + F_1 \bullet X, \ldots, a_m + b_m^T x + F_m \bullet X) \in \overline{\text{conv}(V)}.$$

So, $\overline{\mathcal{W}_{in}} \subseteq \overline{\text{conv}(V)}$ and consequently $\overline{\mathcal{W}_{in}} = \overline{\text{conv}(V)}$.

(ii) can be proved in the same way as for (i). $\square$

Example 2.3. Consider the parametrization

$$V = \left\{ (3x_1 - 2x_2 - 4x_3, 5x_1x_2 + 7x_1x_3 - 9x_2x_3) : \|x\|_2 \leq 1 \right\}.$$

The set V is drawn in the dotted area of Figure 1. By Theorem 2.1, the convex hull $\text{conv}(V)$ is given by the semidefinite representation

$$\left\{ \begin{pmatrix} 3x_1 - 2x_2 - 4x_3 \\ 5X_{12} + 7X_{13} - 9X_{23} \end{pmatrix} \left| \begin{array}{c} \begin{bmatrix} 1 & x_1 & x_2 & x_3 \\ x_1 & X_{11} & X_{12} & X_{13} \\ x_2 & X_{12} & X_{22} & X_{23} \\ x_3 & X_{13} & X_{23} & X_{33} \end{bmatrix} \succeq 0, \\ 1 - X_{11} - X_{22} - X_{33} \geq 0 \end{array} \right. \right\}.$$

The boundary of the above set is the outer curve in Figure 1. One can easily see that $\text{conv}(V)$ is correctly given by the above semidefinite representation. $\square$

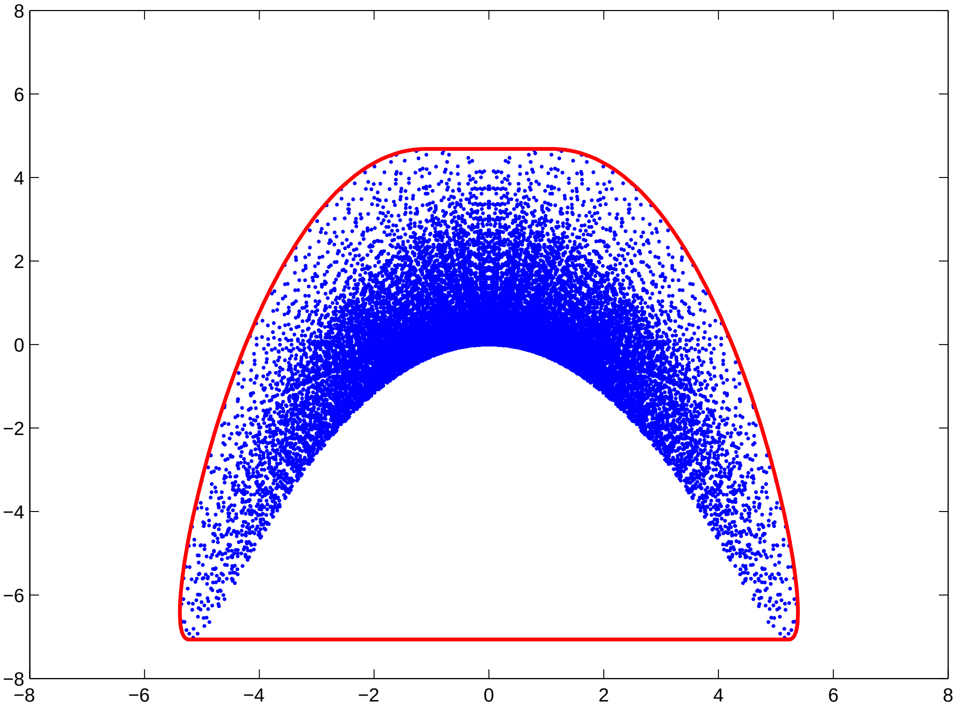

Figure 1: The dotted area is the set V in Example 2.3, and the outer curve is the boundary of the convex hull $\text{conv}(V)$.

3 Two homogeneous constraints

Suppose $V \subset \mathbb{R}^m$ is a semialgebraic set parameterized as

$$V = \{(x^T A_1 x, \ldots, x^T A_m x) : x \in T\}. \tag{3.1}$$

Here, every A_i is a symmetric matrix and T is defined by two homogeneous (modulo constants) inequalities/equalities $h_j(x) \geq 0$ or $h_j(x) = 0$, $j = 1, 2$. Write

$$h_1(x) = x^T B_1 x - c_1, \quad h_2(x) = x^T B_2 x - c_2,$$

for symmetric matrices B_1, B_2. The set T is one of the four cases:

$$E(h_1) \cap E(h_2), \quad S(h_1) \cap E(h_2), \quad E(h_1) \cap S(h_2), \quad S(h_1) \cap S(h_2).$$

Note the relations:

$$x^T A_i x = A_i \bullet (xx^T) \quad (1 \leq i \leq m), \quad xx^T \succeq 0,$$

$$x^T B_1 x = B_1 \bullet (xx^T), \quad x^T B_2 x = B_2 \bullet (xx^T).$$

If we replace xx^T by a symmetric matrix $X \succeq 0$, then V, as well as $\text{conv}(V)$, is contained respectively in the following projections of spectrahedra:

$$\begin{array}{rcl} \mathcal{H}_{e,e} & = & \{(A_1 \bullet X, \ldots, A_m \bullet X) : X \succeq 0,\ B_1 \bullet X = c_1, B_2 \bullet X = c_2\}, \\ \mathcal{H}_{i,e} & = & \{(A_1 \bullet X, \ldots, A_m \bullet X) : X \succeq 0,\ B_1 \bullet X \geq c_1, B_2 \bullet X = c_2\}, \\ \mathcal{H}_{e,i} & = & \{(A_1 \bullet X, \ldots, A_m \bullet X) : X \succeq 0,\ B_1 \bullet X = c_1, B_2 \bullet X \geq c_2\}, \\ \mathcal{H}_{i,i} & = & \{(A_1 \bullet X, \ldots, A_m \bullet X) : X \succeq 0,\ B_1 \bullet X \geq c_1, B_2 \bullet X \geq c_2\}. \end{array} \tag{3.2}$$

To analyze whether they represent $\text{conv}(V)$ respectively, we need the following conditions for the four cases:

$$\begin{cases} C_{e,e} : \exists(\mu_1, \mu_2) \in \mathbb{R} \times \mathbb{R},\ s.t. \quad \mu_1 B_1 + \mu_2 B_2 \prec 0, \\ C_{i,e} : \exists(\mu_1, \mu_2) \in \mathbb{R}_+ \times \mathbb{R},\ s.t. \quad \mu_1 B_1 + \mu_2 B_2 \prec 0, \\ C_{e,i} : \exists(\mu_1, \mu_2) \in \mathbb{R} \times \mathbb{R}_+,\ s.t. \quad \mu_1 B_1 + \mu_2 B_2 \prec 0, \\ C_{i,i} : \exists(\mu_1, \mu_2) \in \mathbb{R}_+ \times \mathbb{R}_+,\ s.t. \quad \mu_1 B_1 + \mu_2 B_2 \prec 0. \end{cases} \tag{3.3}$$

Theorem 3.1. *Let $V \neq \emptyset, \mathcal{H}_{e,e}, \mathcal{H}_{i,e}, \mathcal{H}_{e,i}, \mathcal{H}_{i,i}$ be defined as above. Then we have*

$$\mathit{conv}(V) = \begin{cases} \mathcal{H}_{e,e}, & \text{if } T = E(h_1) \cap E(h_2) \text{ and } C_{e,e} \text{ holds;} \\ \mathcal{H}_{i,e}, & \text{if } T = S(h_1) \cap E(h_2) \text{ and } C_{i,e} \text{ holds;} \\ \mathcal{H}_{e,i}, & \text{if } T = E(h_1) \cap S(h_2) \text{ and } C_{e,i} \text{ holds;} \\ \mathcal{H}_{i,i}, & \text{if } T = S(h_1) \cap S(h_2) \text{ and } C_{i,i} \text{ holds.} \end{cases} \tag{3.4}$$

Proof. We just prove for the case that $T = S(h_1) \cap S(h_2)$ and condition $C_{i,i}$ holds. The proof is similar for the other three cases. The condition $C_{i,i}$ implies that for some $\mu_1 \geq 0, \mu_2 \geq 0, \epsilon > 0$

$$-\mu_1 c_1 - \mu_2 c_2 \geq x^T(-\mu_1 B_1 - \mu_2 B_2)x \geq \epsilon \|x\|_2^2.$$

So, T and $\text{conv}(V)$ are compact. Clearly, $\text{conv}(V) \subseteq \mathcal{H}_{i,i}$. We need to show $\mathcal{H}_{i,i} \subseteq \text{conv}(V)$. Suppose otherwise it is false, then there exists a symmetric matrix Z satisfying

$$(A_1 \bullet Z, \ldots, A_m \bullet Z) \notin \text{conv}(V), \quad B_1 \bullet Z \geq c_1, \quad B_2 \bullet Z \geq c_2, \quad Z \succeq 0.$$

Because $\text{conv}(V)$ is a closed convex set, by the Hahn-Banach theorem, there exists a vector $(\ell_0, \ell_1, \ldots, \ell_m) \neq 0$ satisfying

$$\begin{array}{c} \ell_1 x^T A_1 x + \cdots + \ell_m x^T A_m x \geq \ell_0 \quad \forall x \in T, \\ \ell_1 A_1 \bullet Z + \cdots + \ell_m A_m \bullet Z < \ell_0. \end{array}$$

Consider the SDP problem

$$\begin{array}{rl} p^* := \min & \ell_1 A_1 \bullet X + \cdots + \ell_m A_m \bullet X \\ s.t. & X \succeq 0,\ B_1 \bullet X \geq c_1,\ B_2 \bullet X \geq c_2. \end{array} \tag{3.5}$$

Its dual optimization problem is

$$\begin{array}{rl} \max & c_1 \lambda_1 + c_2 \lambda_2 \\ s.t. & \sum_i \ell_i A_i - \lambda_1 B_1 - \lambda_2 B_2 \succeq 0,\ \lambda_1 \geq 0, \lambda_2 \geq 0. \end{array} \tag{3.6}$$

The condition $C_{i,i}$ implies that the dual problem (3.6) has nonempty interior. So, the primal problem (3.5) has an optimizer. Define $\tilde{A}_0, \tilde{B}_1, \tilde{B}_2$ and a new variable Y as:

$$\tilde{A}_0 = \begin{bmatrix} \sum_{i=1}^m \ell_i A_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tilde{B}_1 = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tilde{B}_2 = \begin{bmatrix} B_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, Y = \begin{bmatrix} X & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}.$$

They are all $(n+2) \times (n+2)$ symmetric matrices. Clearly, the primal problem (3.5) is equivalent to

$$\begin{aligned} p^* := \min \quad & \tilde{A}_0 \bullet Y \\ s.t. \quad & Y \succeq 0,\, \tilde{B}_1 \bullet Y = c_1,\, \tilde{B}_1 \bullet Y = c_2. \end{aligned} \tag{3.7}$$

It must also have an optimizer. By Theorem 2.1 of Pataki [14], (3.7) has an extremal solution U of rank r satisfying

$$\frac{1}{2} r(r+1) \leq 2.$$

So, we must have $r = 1$ and can write $Y = vv^T$. Let $u = v(1:n)$. Then $u \in T$ and

$$p^* = \ell_1 u^T A_1 u + \cdots + \ell_m u^T A_m u \geq \ell_0.$$

However, Z is also a feasible solution of (3.5), and we get the contradiction

$$p^* \leq \ell_1 A_1 \bullet Z + \cdots + \ell_m A_m \bullet Z < p^*.$$

Therefore, $\mathcal{H}_{i,i} \subseteq \text{conv}(V)$ and they must be equal. □

Example 3.2. Consider the parameterization

$$V = \left\{ \begin{pmatrix} 2x_1^2 - 3x_2^2 - 4x_3^2 \\ 5x_1x_2 - 7x_1x_3 - 9x_2x_3 \end{pmatrix} \middle| \begin{array}{c} x_1^2 - x_2^2 - x_3^2 = 0, \\ 1 - x^T x \geq 0 \end{array} \right\}.$$

The set V is drawn in the dotted area of Figure 2. By Theorem 3.1, the convex hull conv(V) is given by the following semidefinite representation

$$\left\{ \begin{pmatrix} 2X_{11} - 3X_{22} - 4X_{33} \\ 5X_{12} - 7X_{13} - 9X_{23} \end{pmatrix} \middle| \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12} & X_{22} & X_{23} \\ X_{13} & X_{23} & X_{33} \end{bmatrix} \succeq 0, \begin{array}{c} X_{11} - X_{22} - X_{33} = 0, \\ 1 - X_{11} - X_{22} - X_{33} \geq 0 \end{array} \right\}.$$

The convex region described above is surrounded by the outer curve in Figure 2, which is clearly the convex hull of the dotted area. □

The conditions like $C_{i,i}$ can not be removed in Theorem 3.1. We show this by a counterexample.

Example 3.3. Consider the quadratically parameterized set

$$V = \{(x_1x_2, x_1^2) : 1 - x_1x_2 \geq 0, 1 + x_2^2 - x_1^2 \geq 0\},$$

which is motivated by Example 4.4 of [3]. The condition $C_{i,i}$ is clearly not satisfied. The semidefinite relaxation $\mathscr{H}_{i,i}$ for conv(V) is

$$\{(X_{12}, X_{11}) : X \succeq 0, 1 - X_{12} \geq 0, 1 + X_{22} - X_{11} \geq 0\}.$$

They are not equal, and neither are their closures. This is because V is bounded above in the direction $(1,1)$, while $\mathscr{H}_{i,i}$ is unbounded (cf. [3, Example 4.4]). So, $\overline{\text{conv}(V)} \neq \overline{\mathscr{H}_{i,i}}$ for this example, which is due to the failure of the condition $C_{i,i}$. □

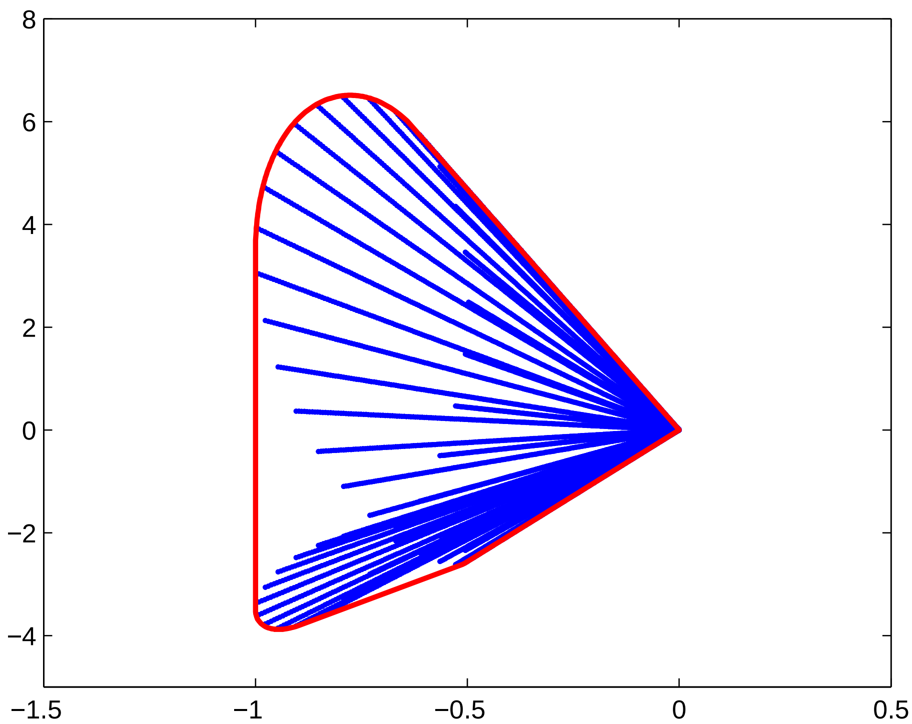

Figure 2: The dotted area is the set V in Example 3.2, and the outer curve surrounds its convex hull.

4 Rational parametrization

Consider the rationally parameterized set

$$U \quad = \quad \left\{ \left(\frac{f_1(x)}{f_0(x)}, \dots, \frac{f_m(x)}{f_0(x)} \right) : \, x \in T \right\} \tag{4.1}$$

with all $f_0, \dots, f_m$ being polynomials and T a semialgebraic set in $\mathbb{R}^n$. Assume $f_0(x)$ is nonnegative on T and every f_i/f_0 is well defined on T, i.e., the limit $\lim_{x \to z} f_i(x)/f_0(x)$ exists whenever f_0 vanishes at $z \in T$. The convex hull $\mathrm{conv}(U)$ would be investigated through considering the polynomial parameterization

$$P \quad = \quad \left\{ \left(f_1^h(x^h), \dots, f_m^h(x^h) \right) : \, f_0^h(x^h) = 1, \, x^h \in T^h \right\}. \tag{4.2}$$

Here $x^h = (x_0, x_1, \dots, x_n)$ is an augmentation of x and

$$f_i^h(x^h) \quad = \quad x_0^d f_i(x/x_0) \qquad (d = \max_i \deg(f_i))$$

is a homogenization of $f_i(x)$, and T^h is the homogenization of T defined as

$$T^h = \overline{\{x^h : \, x_0 > 0, x/x_0 \in T\}}. \tag{4.3}$$

The relation between conv(V) and conv(P) is given as below.

Proposition 4.1. *Suppose $f_0(x)$ is nonnegative on T and does not vanish on a dense subset of T, and every f_i/f_0 is well defined on T. Then*

$$\overline{conv(U)} \quad = \quad \overline{conv(P)}. \tag{4.4}$$

Moreover, if $T^h \cap \{f_0^h(x^h) = 1\}$ and T are compact and $f_0(x)$ is positive on T, then

$$conv(U) \quad = \quad conv(P). \tag{4.5}$$

Proof. Let T_1 be a dense subset of T such that $f_0(x) > 0$ for all $x \in T_1$. Clearly,

$$\overline{\mathrm{conv}(U)} = \overline{\mathrm{conv}\left\{\left(\frac{f_1^h(x^h)}{f_0^h(x^h)}, \dots, \frac{f_m^h(x^h)}{f_0^h(x^h)}\right) : x^h \in T_1^h\right\}}.$$

Since every f_i^h is homogeneous, we can assume that $f_0^h(x^h) = 1$. Then,

$$\overline{\mathrm{conv}(U)} = \overline{\mathrm{conv}\left\{\left(f_1^h(x^h), \dots, f_m^h(x^h)\right) : f_0^h(x^h) = 1,\, x^h \in T_1^h\right\}}.$$

The density of T_1 in T and the above imply (4.4).

When T is compact and $f_0(x)$ is positive on T, conv(U) is compact. The conv(P) is also compact when $T^h \cap \{f_0^h(x^h) = 1\}$ is compact. Thus, (4.5) follows from (4.4). □

Remark: If $d = \max_i \deg(f_i)$ is even and T is defined by polynomials of even degrees, then we can remove the condition $x_0 > 0$ in the definition of T^h in (4.3) and Proposition 4.1 still holds.

If every f_i in (4.1) is quadratic, T is defined by a single quadratic inequality, and f_0 is nonnegative on T, then a semidefinite representation for the convex hull conv(U) or its closure can be obtained by applying Proposition 4.1 and Theorem 3.1. Suppose $T = \{x : g(x) \geq 0\}$, with $g(x)$ being quadratic. Write every $f_i^h(x^h) = (x^h)^T F_i\, x^h$ and $g^h(x^h) = (x^h)^T G\, x^h$. Then

$$\overline{\mathrm{conv}(P)} \quad = \quad \overline{\mathrm{conv}\left\{\left((x^h)^T F_1\, x^h, \dots, (x^h)^T F_m\, x^h\right) : \begin{array}{c} (x^h)^T F_0\, x^h = 1, \\ x_0 > 0, (x^h)^T G\, x^h \geq 0 \end{array}\right\}}. \tag{4.6}$$

Since the forms f_i^h and g^h are all quadratic, the condition $x_0 > 0$ can be removed from the right hand side of (4.6), and we get

$$\overline{\mathrm{conv}(P)} \quad = \quad \overline{\mathrm{conv}\left\{\left((x^h)^T F_1\, x^h, \dots, (x^h)^T F_m\, x^h\right) : \begin{array}{c} (x^h)^T F_0\, x^h = 1, \\ (x^h)^T G\, x^h \geq 0 \end{array}\right\}}. \tag{4.7}$$

If there are numbers $\mu_1 \in \mathbb{R}$ and $\mu_2 \in \mathbb{R}_+$ satisfying $\mu_1 F_0 + \mu_2 G \prec 0$, then a semidefinite representation for $\overline{\mathrm{conv}(P)}$ can be obtained by applying Theorem 3.1. The case $T = \{x : g(x) = 0\}$ is defined by a single quadratic equality is similar.

Example 4.2. Consider the quadratically rational parametrization:

$$U = \left\{\left(\frac{x_1^2 + x_2^2 + x_3^2 + x_1 + x_2 + x_3}{1 + x^T x}, \frac{x_1 x_2 + x_1 x_3 + x_2 x_3}{1 + x^T x}\right) : x_1^2 + x_2^2 + x_3^2 \leq 1\right\}.$$

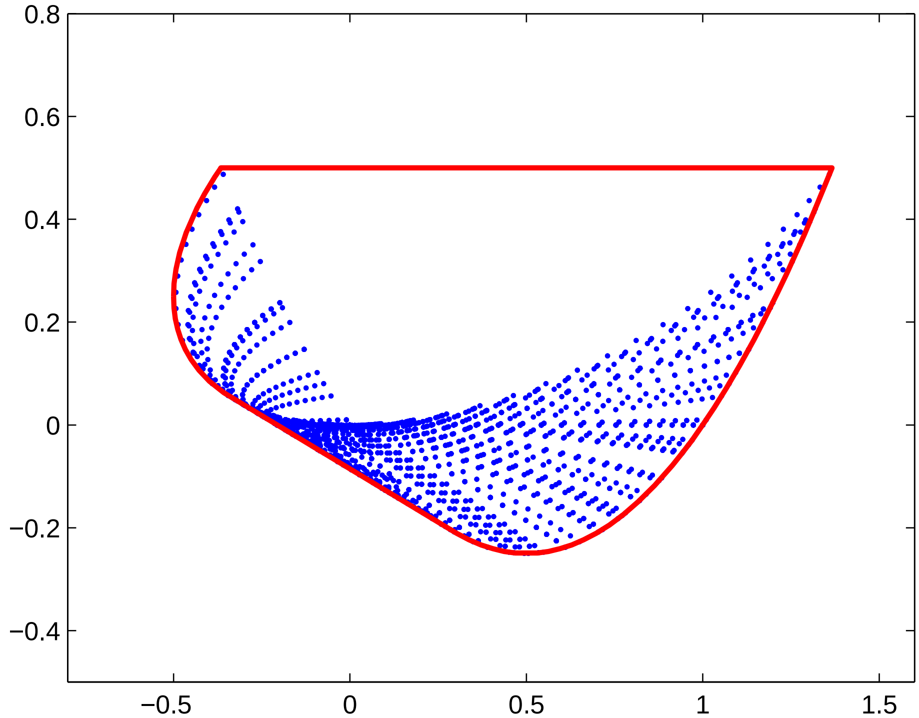

Figure 3: The dotted area is the set U in Example 4.2, and the outer curve is the boundary of its convex hull.

The dotted area in Figure 2 is the set U above. The set P in (4.2) is

$$P = \left\{ \begin{pmatrix} x_1^2 + x_2^2 + x_3^2 + x_0(x_1 + x_2 + x_3) \\ x_1x_2 + x_1x_3 + x_2x_3 \end{pmatrix} \middle| \begin{array}{c} x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1, \\ x_0^2 - x_1^2 - x_2^2 - x_3^2 \geq 0 \end{array} \right\}.$$

By Theorem 3.1, the convex hull $\text{conv}(P)$ is given by the semidefinite representation

$$\left\{ \begin{pmatrix} X_{11} + X_{22} + X_{33} + X_{01} + X_{02} + X_{03} \\ X_{12} + X_{13} + X_{23} \end{pmatrix} \middle| \begin{array}{c} \begin{bmatrix} X_{00} & X_{01} & X_{02} & X_{03} \\ X_{01} & X_{11} & X_{12} & X_{13} \\ X_{02} & X_{12} & X_{22} & X_{23} \\ X_{03} & X_{13} & X_{23} & X_{33} \end{bmatrix} \succeq 0, \\ X_{00} + X_{11} + X_{22} + X_{33} = 1, \\ X_{00} - X_{11} - X_{22} - X_{33} \geq 0 \end{array} \right\}.$$

The convex region described above is surrounded by the outer curve in Figure 3, which also surrounds the convex hull of the dotted area. Since T is compact and the denominator $1+x^Tx$ is strictly positive, $\text{conv}(U) = \text{conv}(P)$ by Proposition 4.1. $\square$

References

[1] C. Bayer and J. Teichmann. The proof of Tchakaloff's Theorem. *Proc. Amer. Math. Soc.*, 134(2006), 3035-3040.

[2] L. Fialkow and J. Nie. Positivity of Riesz functionals and solutions of quadratic and quartic moment problems. *J. Functional Analysis*, Vol. 258, No. 1, pp. 328–356, 2010.

[3] S. He, Z. Luo, J. Nie and S. Zhang. Semidefinite Relaxation Bounds for Indefinite Homogeneous Quadratic Optimization. *SIAM Journal on Optimization*, Vol. 19, No. 2, pp. 503–523, 2008.

[4] J.W. Helton and J. Nie. Semidefinite representation of convex sets. *Mathematical Programming*, Ser. A, Vol. 122, No.1, pp.21–64, 2010.

[5] J.W. Helton and J. Nie. Sufficient and necessary conditions for semidefinite representability of convex hulls and sets. *SIAM Journal on Optimization*, Vol. 20, No.2, pp. 759-791, 2009.

[6] J.W. Helton and J. Nie. Structured semidefinite representation of some convex sets. *Proceedings of 47th IEEE Conference on Decision and Control*, pp. 4797 - 4800, Cancun, Mexico, Dec. 9-11, 2008.

[7] W. Helton and V. Vinnikov. Linear matrix inequality representation of sets. *Comm. Pure Appl. Math.* 60 (2007), No. 5, pp. 654-674.

[8] D. Henrion. Semidefinite representation of convex hulls of rational varieties. LAAS-CNRS Research Report No. 09001, January 2009.

[9] J. Lasserre. Convex sets with semidefinite representation. *Mathematical Programming*, Vol. 120, No. 2, pp. 457–477, 2009.

[10] J. Lasserre. Convexity in semi-algebraic geometry and polynomial optimization. *SIAM Journal on Optimization*, Vol. 19, No. 4, pp. 1995 – 2014, 2009.

[11] J. Nie. First order conditions for semidefinite representations of convex sets defined by rational or singular polynomials. *Mathematical Programming*, to appear.

[12] J. Nie. Polynomial matrix inequality and semidefinite representation. *Mathematics of Operations Research*, Vol. 36, No. 3, pp. 398–415, 2011.

[13] P. Parrilo. Exact semidefinite representation for genus zero curves. Talk at the Banff workshop "Positive Polynomials and Optimization", Banff, Canada, October 8-12, 2006.

[14] G. Pataki. On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues. *Mathematics of Operations Research*, 23 (2), 339–358, 1998.

[15] M. Putinar. Positive polynomials on compact semi-algebraic sets, *Ind. Univ. Math. J.* 42 (1993) 203–206.

[16] K. Schmüdgen. The K-moment problem for compact semialgebraic sets. *Math. Ann.* **289** (1991), 203–206.