THE TRUNCATED MOMENT PROBLEM VIA HOMOGENIZATION AND FLAT EXTENSIONS

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Abstract. A truncated moment sequence (tms) y in n variables and of degree d is a finite real sequence $\{y_\alpha\}$ indexed by nonnegative integer vectors $\alpha := (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq d$. It *admits a measure* if there exists a positive Borel measure μ on $\mathbb{R}^n$ such that each y_α is the α-th moment of μ. The truncated moment problem (TMP) studies conditions for a tms y to admit a measure. A homogeneous tms (htms) is indexed by α with $|\alpha|$ constant. This paper proposes an approach to solving TMP via homogenization and flat extensions of moment matrices. We first transform TMP to a homogeneous TMP (HTMP), and then use semidefinite programming to solve HTMP. Assume the degree d is even. Our main results are: (i) a tms is the limit of a sequence of tms admitting measures on $\mathbb{R}^n$ if and only if its homogenized tms (htms) admits a measure supported on the unit sphere in $\mathbb{R}^{n+1}$; (ii) an htms admits a measure if and only if the optimal values of a sequence of SDP problems are nonnegative; (iii) under some conditions that are almost necessary and sufficient, by solving these SDP problems, a representing measure for an htms can be explicitly constructed if one exists.

1. Introduction

A truncated moment sequence (tms) y in n variables and of degree d is a finite sequence $\{y_\alpha\}$ indexed by nonnegative integer vectors $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ with $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq d$. We say that a tms y *admits a measure* if there exists a positive Borel measure on $\mathbb{R}^n$ such that

$$y_\alpha \quad = \quad \int_{\mathbb{R}^n} x^\alpha \, d\mu, \qquad \forall\, \alpha \in \mathbb{N}^n : |\alpha| \leq d.$$

Here $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The *truncated moment problem (TMP)* studies conditions for a tms to admit a measure. Let

$$\mathscr{M}_{n,d} := \big\{ y \equiv (y_\alpha) : \alpha \in \mathbb{N}^n, |\alpha| \leq d \big\}. \tag{1.1}$$

For a tms y, denote by $meas(y)$ the set of all measures admitted by y. If $\mu \in meas(y)$, we also say μ represents y, or μ is a *representing measure* of y. Define

$$\mathscr{R}_{n,d} := \big\{ y \in \mathscr{M}_{n,d} : meas(y) \neq \emptyset \big\}. \tag{1.2}$$

A measure is called *finitely atomic* if its support is finite, and is called r-atomic if its support has cardinality r. A fundamental result of Bayer and Teichmann [1] is that a tms $y \in \mathscr{M}_{n,d}$ admits a measure μ if and only if it admits an r-atomic measure with $r \leq \binom{n+d}{d}$. Several general necessary or sufficient conditions for the

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existence of representing measures, or for membership in the closure of $\mathscr{R}_{n,d}$, are known (cf. Theorem 1.1, and Theorem 2.1), but the conditions in these results are difficult to characterize concretely for general tms. In this manuscript, we present a semidefinite program (SDP) approach that can be used to check numerically whether or not a given tms belongs to the closure of $\mathscr{R}_{n,d}$, or, in some cases, to compute a representing measure.

Every tms $y \in \mathscr{M}_{n,d}$ defines a *Riesz functional* $\mathscr{L}_y$ acting on $\mathbb{R}[x]_d$ (the space of real polynomials in $(x_1, \dots, x_n)$ of degree at most d) as

$$\mathscr{L}_y\Big(\sum_{|\alpha|\le d} p_\alpha x^\alpha\Big) \quad := \quad \sum_{|\alpha|\le d} p_\alpha y_\alpha. \tag{1.3}$$

For convenience, sometimes we also denote $\langle p, y\rangle := \mathscr{L}_y(p)$. Let $P_{n,d}$ be the cone of all polynomials in $\mathbb{R}[x]_d$ that are nonnegative in $\mathbb{R}^n$. A necessary condition for $y \in \mathscr{M}_{n,d}$ to admit a measure μ is that $\mathscr{L}_y$ is *positive*, that is,

$$\mathscr{L}_y(p) \ge 0 \quad \forall\, p \in P_{n,d};$$

this is because $\mathscr{L}_y(p) = \int_{\mathbb{R}^n} p d\mu \ge 0$ whenever $p \in P_{n,d}$. A stronger condition is that $\mathscr{L}_y$ is *strictly positive*, that is,

$$\mathscr{L}_y(p) > 0 \quad \forall\, p \in P_{n,d},\, p \ne 0.$$

In general, it is very difficult to directly verify that $\mathscr{L}_y$ is positive or strictly positive. A weaker condition than $\mathscr{L}_y$ being positive, but one that is easier to check, is that the moment matrix associated to y is positive semidefinite.

For a tms $y \in \mathscr{M}_{n,2k}$, its *moment matrix*, denoted by $M_k(y)$, is the unique real symmetric matrix (linear in y) such that

$$\mathscr{L}_y(pq) \quad = \quad p^T M_k(y) q \quad \forall\, p, q \in \mathbb{R}[x]_k. \tag{1.4}$$

(Here p denotes the vector of coefficients of $p(x)$ with respect to graded lexicographical ordering.) If $y \in \mathscr{M}_{n,2k}$ admits a measure μ, then for every $p \in \mathbb{R}[x]_k$,

$$p^T M_k(y) p = \mathscr{L}_y(p^2) = \int_{\mathbb{R}^n} p^2 d\mu \ge 0, \qquad \text{so } M_k(y) \quad \succeq \quad 0. \tag{1.5}$$

(Here $X \succeq 0$ (resp. $X \succ 0$) means that X is a symmetric matrix that is positive semidefinite (resp. positive definite)). Thus, (1.5) is a necessary condition for $y \in \mathscr{R}_{n,2k}$. If $n = 1$ and $M_k(y) \succ 0$, or $n = 2$ and $M_2(y) \succ 0$ ($2k = 4$), then y admits a measure (see [10]). In general, (1.5) is not sufficient for $y \in \mathscr{R}_{n,2k}$. However, if y is *flat*, that is, it satisfies $M_k(y) \succeq 0$ and the rank condition:

$$\operatorname{rank} M_{k-1}(y) \quad = \quad \operatorname{rank} M_k(y), \tag{1.6}$$

then y admits a measure, i.e., $y \in \mathscr{R}_{n,2k}$. This is a result of Curto and Fialkow that we will utilize in Section 4.

Theorem 1.1 (Curto and Fialkow, [6]). *Let d be even. If a tms $y \in \mathscr{M}_{n,d}$ is flat, then y admits a unique, $\operatorname{rank} M_{d/2}(y)$-atomic, representing measure.*

A problem that is more general than TMP is the *truncated K-moment problem (TKMP)*. Let $K \subset \mathbb{R}^n$ be a closed set. TKMP studies whether a tms admits a representing measure that is supported in K. For K compact, it follows from Tchakaloff's Theorem [20] that y has a measure supported in K if and only if $\mathscr{L}_y$ is *K-positive*, i.e.,

$$p \in \mathbb{R}[x]_d, p|_K \ge 0 \Longrightarrow \mathscr{L}_y(p) \ge 0;$$

however, there is no known concrete characterization of K-positivity for a general compact set K. In [12], Helton and the second-named author addressed TKMP for K compact and semialgebraic. They obtained the following results: whether a tms admits a measure supported in K or not can be checked by solving a sequence of SDP problems; when y admits no such a measure, a certificate will be given; when y does, a representing measure for y will be obtained by solving the SDP under some almost necessary and sufficient conditions. Moreover, they also propose a practical SDP method that often finds a flat extension of a tms when it admits a representing measure. TMP can be considered as a special case of TKMP with $K = \mathbb{R}^n$, and thus it is tempting to apply the approach of [12] to TMP. However, $K = \mathbb{R}^n$ is not compact, so the results of [14] cannot be applied directly to TMP. In this paper, we discuss how to solve TMP by generalizing the approaches in [12] and introducing new techniques.

Every tms $y \in \mathscr{M}_{n,d}$ can be thought of as the subsequence $\tilde{y}$ of a tms in $\mathscr{M}_{n+1,d}$ indexed by homogeneous integer vectors, defined as $\tilde{y}_{(d-|\alpha|,\alpha)} := y_\alpha$ for every $|\alpha| \le d$. In other words, to define $\tilde{y}$, we homogenize the indices of y. For convenience, we identify $\tilde{y}$ with y and denote

$$\mathscr{M}^h_{n+1,d} = \left\{ y = (y_\beta) : \beta \in \mathbb{N}^{n+1}, |\beta| = d \right\}.$$

A sequence in $\mathscr{M}^h_{n+1,d}$ is also called a homogeneous tms (htms). Via homogenizing indices, $\mathscr{M}_{n,d}$ is isomorphic to $\mathscr{M}^h_{n+1,d}$, and we write $\mathscr{M}_{n,d} \cong \mathscr{M}^h_{n+1,d}$. Every tms $y \in \mathscr{M}_{n,d}$ can be identified as $\tilde{y} \in \mathscr{M}^h_{n+1,d}$, and vice versa. Throughout this paper, whenever $y \in \mathscr{M}_{n,d}$ (resp. $\tilde{y} \in \mathscr{M}^h_{n+1,d}$), one can automatically think of $\tilde{y} \in \mathscr{M}^h_{n+1,d}$ (resp. $y \in \mathscr{M}_{n,d}$). The correspondence between y and $\tilde{y}$ at the level of Riesz functionals and representing measures will be explored in detail in Section 3. In particular, Theorem 3.1 implies that for d even and $y \in \mathscr{M}_{n,d}$, $\mathscr{L}_y$ is positive (equivalently, $y \in cl(\mathscr{R}_{n,d})$), if and only if $\tilde{y}$ admits a representing measure supported in the sphere $\mathbb{S}^n$. In Section 4, we use semidefinite programming and Theorem 3.1 to associate to y a computable sequence

$$\eta_0 \ge \eta_1 \ge \cdots \ge \eta_\infty := \lim_{k\to\infty} \eta_k > -\infty$$

in such a way that $\tilde{y}$ admits a measure supported in $\mathbb{S}^n$ if and only if $\eta_\infty \ge 0$.

2. Some basics

In this section, after introducing some notation, we discuss certain results concerning representing measures, positive Riesz functionals, and moment matrices that we will utilize in the sequel. We then introduce certain cones of positive polynomials, and connect these to an optimization problem that is the subject of Section 4.

Notation The symbol $\mathbb{N}$ (resp., $\mathbb{R}$) denotes the set of nonnegative integers (resp., real numbers). The symbol $[x]_d$ denotes the vector of all monomials of degrees $\le d$:

$$[x]_d^T = [\,1 \quad x_1 \quad \cdots \quad x_n \quad x_1^2 \quad x_1x_2 \quad \cdots\cdots \quad x_1^d \quad x_1^{d-1}x_2 \quad \cdots\cdots \quad x_n^d\,]^T,$$

and $[x^d]$ denotes the subvector of $[x]_d$ consisting of all monomials of degree d, i.e.,

$$[x^d]^T = [\,x_1^d \quad x_1^{d-1}x_2 \quad \cdots\cdots \quad x_n^d\,].$$

The symbol $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ (resp., $\mathbb{R}[\tilde{x}] = \mathbb{R}[x_0, x_1, \ldots, x_n]$) denotes the ring of polynomials (resp., the set of forms) in $x := (x_1, \ldots, x_n)$ (resp., in $\tilde{x} :=$

$(x_0, x_1, \ldots, x_n)$) with real coefficients. The notation $\mathbb{R}[x]_d$ (resp. $\mathbb{R}[x]_{=d}$) denotes the subspace of polynomials (resp. the set of forms) in $\mathbb{R}[x]$ whose degrees are at most d (resp., equal d). For a set $S \subseteq \mathbb{R}^n$, $|S|$ denotes its cardinality, $int(S)$ denotes its interior, and $\mathrm{cl}(S)$ denotes its closure. The superscript ^{T} denotes the transpose of a matrix. For $u \in \mathbb{R}^n$, define $\|u\|_2 := \sqrt{u^T u}$; $\mathbb{S}^n$ denotes the n-dimensional unit sphere in the space $\mathbb{R}^{n+1}$.

For a tms $w \in \mathscr{M}_{n,k}$ with $k \geq d$, denote by $w|_d$ the truncation of w up to degree d, i.e., $w|_d$ is the subvector of w with all indices of degree $\leq d$. Clearly, if $y = w|_d$ and w is flat, then y and w admit a common finitely atomic measure (cf. Theorem 1.1).

2.1. **Positive Riesz functionals, representing measures, and moment matrices.** Recall that $\mathscr{R}_{n,d}$ is the subset of tms in $\mathscr{M}_{n,d}$ that admit measures in $\mathbb{R}^n$. The set $\mathscr{R}_{n,d}$ is a convex cone with nonempty interior, but is not closed (cf. [10]). So it is more convenient to work on its closure $\mathrm{cl}\,(\mathscr{R}_{n,d})$.

Theorem 2.1. *Let $\mathscr{M}_{n,d}, \mathscr{R}_{n,d}$ be defined as before. Then we have:*

(i) *[[10]] $cl\,(\mathscr{R}_{n,d}) = \{y \in \mathscr{M}_{n,d} : \mathscr{L}_y$ is positive$\}$.*
(ii) *$int\,(\mathscr{R}_{n,d}) = \{y \in \mathscr{M}_{n,d} : \mathscr{L}_y$ is strictly positive$\}$.*
(iii) *[[7]] For $d = 2k$ or $d = 2k+1$, a tms $y \in \mathscr{R}_{n,d}$ if and only if y admits an extension $y' \in \mathscr{M}_{n,2k+2}$ for which $\mathscr{L}_{y'}$ is positive.*

Proof. (i) is implied by Theorem 2.2 of [10] and its proof.

(ii) "only if" direction: Suppose $\mathscr{L}_y$ is strictly positive. Then, for all z close enough to y, $\mathscr{L}_z$ is strictly positive, and thus z admits a measure (see Lemma 2.3 and Theorem 2.4 of [10]). This means that $y \in int\,(\mathscr{R}_{n,d})$.

"if" direction: Suppose $y \in int\,(\mathscr{R}_{n,d})$. Let $\xi \in \mathscr{M}_{n,d}$ be a tms represented by the measure whose density function is $\exp\{-\|x\|_2^2\}$. Then, for $\epsilon > 0$ small enough, $z := y - \epsilon\xi \in int\,(\mathscr{R}_{n,d})$. Clearly, $\mathscr{L}_\xi$ is strictly positive and $\mathscr{L}_z$ is positive. Then $\mathscr{L}_y = \mathscr{L}_z + \epsilon\mathscr{L}_\xi$ implies $\mathscr{L}_y$ is strictly positive.

Item (iii) is implied by Theorems 1.2 and 2.4 of [7]. □

In the sequel, we will work with representing measures for htms in $\mathscr{M}^h_{n+1,d}$. Let $\tilde{x} = (x_0, x_1, \ldots, x_n)$ and denote by $[\tilde{x}^d]$ the vector of monomials in $\tilde{x}$ whose degrees are equal to d with respect to graded lexicographical ordering. Define

$$\mathscr{R}^h_{n+1,d} = \left\{ y \in \mathscr{M}^h_{n+1,d} \,\middle|\, y = \int_{\mathbb{S}^n} [\tilde{x}^d]\, d\nu \quad \begin{array}{c}\text{for some measure } \nu \ \equiv \nu(y) \\ \text{with } \mathrm{supp}(\nu) \subseteq \mathbb{S}^n\end{array} \right\}.$$

We note that an htms $y \in \mathscr{M}^h_{n+1,d}$ admits a measure if and only if $y \in \mathscr{R}^h_{n+1,d}$. Indeed, suppose y admits a representing measure for its moments (of degree d). By a formal repetition of the proof of the Bayer-Teichmann theorem [1] as given in [16, Theorem 5.9], y admits a finitely atomic measure, say,

$$y = \lambda_1[u_1^d] + \cdots + \lambda_r[u_r^d],$$

for $u_1, \ldots, u_r \in \mathbb{R}^{n+1}$ and $\lambda_1, \ldots, \lambda_r > 0$. Generally, we can assume each $u_i \neq 0$ (because otherwise, say, if $u_1 = 0$, one could write $w := \lambda_2[u_2^d] + \cdots + \lambda_r[u_r^d]$, and $w_\alpha = y_\alpha$ for all $|\alpha| = d$). Then the measure ν on $\mathbb{S}^n$ defined by $supp\ \nu = \{u_1/||u_1||_2, \ldots, u_r/\|u_r\|_2\}$ and $\nu(u_i) := ||u_i||_2^d \mu(u_i)\,(1 \leq i \leq m)$ also represents the moments in y. This shows that if $y \in \mathscr{M}^h_{n+1,d}$ admits a measure on $\mathbb{R}^{n+1}$, then it admits a finitely atomic measure on $\mathbb{S}^n$.

In our later proofs, we need an auxiliary result, which may be of independent interest. Let $K \subseteq \mathbb{R}^{n+1}$. With d not necessarily even, we say that a linear subspace H of $\mathbb{R}[\tilde{x}]_d$ is *K-full* if there exists $p_0 \in H$ such that $p_0(\tilde{x}) > 0$ for every $\tilde{x} \in K$.

Theorem 2.2. *Suppose $K \subset \mathbb{R}^{n+1}$ is compact, H is K-full and $N = \dim H$. Let $\mathscr{L} : H \longrightarrow \mathbb{R}$ be a linear functional that is K-positive with respect to H, i.e.,*

$$p \in H,\ p|_K \geq 0 \quad \Longrightarrow \quad \mathscr{L}(p) \geq 0.$$

Then there exist $m \leq N$, $u_1, \ldots, u_m \in K$, and $a_1, \ldots, a_m > 0$, such that

$$\mathscr{L}(p) = \sum_{i=1}^{m} a_i p(u_i) \qquad (p \in H). \tag{2.1}$$

Remark 2.3. For the case when $H = \mathbb{R}[\tilde{x}]_d$ (with $p_0 \equiv 1$), μ denotes Lebesgue-Borel measure on K, and $\mathscr{L}(p) = \int_K p\,d\mu$, Theorem 2.2 is Tchakaloff's Theorem [20, Theoreme II], which is the fundamental existence theorem of cubature theory. The proof of Tchakaloff's Theorem in [20] in turn depends on an abstract result concerning representations of linear functionals satisfying certain positivity conditions [20, Theoreme I]. Our proof of Theorem 2.2 is based on ideas in the proof of [20, Theoreme I], but is more direct.

Proof. (Proof of Theorem 2.2.) For $p = \sum_{|\alpha| \leq d} a_\alpha \tilde{x}^\alpha$ and $q = \sum_{|\alpha| \leq d} b_\alpha \tilde{x}^\alpha$, we define the inner product $\langle p, q \rangle$ and its induced norm $||p||$ as

$$\langle p, q \rangle := \sum_\alpha a_\alpha b_\alpha, \qquad ||p|| := \langle p, p \rangle^{1/2}.$$

Since H is finite dimensional, all linear functionals on H are $||\cdot||$-continuous, and we have $H \cong H^* \cong H^{**}$ (the superscript * denotes the dual of a space); corresponding to $F \in H^{**}$, there is a unique $f \in H$ such that $F(L) = L(f)$ $(\forall\, L \in H^*)$.

For $u \in K$, define $L_u \in H^*$ as $L_u(h) = h(u)$ $(\forall\, h \in H)$. Let η denote the maximum number of linearly independent functionals in $\Lambda := \{L_u : u \in K\}$; thus $\eta \leq dim\ H^* = dim\ H = N$. Let

$$\mathscr{C} := \left\{ L \in H^* : L = \sum_{i=1}^{\eta} a_i L_{u_i}, a_i \geq 0, \text{ every } u_i \in K \right\}.$$

Our goal is to show that $\mathscr{C}$ is a closed convex cone in H^*.

The set $\mathscr{C}$ is clearly closed under multiplication by nonnegative scalars, so to show that it is a convex cone it suffices to show that it is closed under addition. For $L_1, L_2 \in \mathscr{C}$, it is clear that $L := L_1 + L_2$ admits a representation as $L = \sum_{i=1}^m a_i L_{u_i}$, with $m \leq 2\eta$, $a_i > 0$, $u_i \in K$ $(1 \leq i \leq m)$. We may assume that $m > \eta$, so there exist scalars $c_1, \ldots, c_m$, with some $c_i > 0$, such that $c_1 L_{u_1} + \cdots + c_m L_{u_m} = 0$. Setting

$$\mu = \frac{c_j}{a_j} \equiv \max_{1 \leq i \leq m} \frac{c_i}{a_i} > 0,$$

then $L = \sum_{i=1}^m \frac{\mu a_i - c_i}{\mu} L_{u_i}$. Since $\mu a_j - c_j = 0$, we may rewrite L as

$$L = \sum_{i=1}^{m-1} a_i' L_{u_i'} \qquad (a_i' \geq 0,\ u_i' \in K).$$

By repeating the preceding argument successively, we see that $L \in \mathscr{C}$.

To show that $\mathscr{C}$ is closed, suppose $\Gamma := \{L_s\}_{s=1}^{\infty} \subseteq \mathscr{C}$ with $\lim_{s\to\infty} L_s = L$ in H^*; since Γ is convergent, it is bounded, i.e., $\gamma := \sup_s \ ||L_s|| < +\infty$. We may write L_s as $L_s = \sum_{i=1}^{\eta} a_{s,i} L_{u_{s,i}}$, where each $a_{s,i} \geq 0$ and each $u_{s,i} \in K$. Since H is K-full, there exists $p_0 \in H$ such that $p_0(\tilde{x}) > 0$ for every $\tilde{x} \in K$. Fix j, $1 \leq j \leq \eta$. Then

$$L_s(p_0) = \sum_{i=1}^{\eta} a_{s,i} L_{u_{s,i}}(p_0) = \sum_{i=1}^{\eta} a_{s,i} p_0(u_{s_i}) \geq a_{s,j}\delta,$$

where $\delta := min\{p_0(u) : u \in K\} > 0$. Now,

$$0 \leq a_{s,j} \leq \frac{||L_s||||p_0||}{\delta} \leq \frac{\gamma||p_0||}{\delta}.$$

Thus, for each j, $\{a_{s,j}\}_{s=1}^{\infty}$ is bounded. By passing to appropriate subsequences (which we designate in the same way) and by using the compactness of K, we may assume that

$$\lim_{s\to\infty} a_{s,j} = a_j \geq 0, \qquad \lim_{s\to\infty} u_{s,j} = u_j \in K \ (1 \leq j \leq \eta).$$

Let $L_0 = \sum_{i=1}^{\eta} a_i L_{u_i} \in \mathscr{C}$. For $h \in H$ we have

$$L_0(h) = \sum_{i=1}^{\eta} a_i h(u_i) = \lim_{s\to\infty} \sum_{i=1}^{\eta} a_{s,i} h(u_{s,i}) = \lim_{s\to\infty} L_s(h) = L(h),$$

whence $L = L_0 \in \mathscr{C}$.

Now $\mathscr{C}$ is a closed convex cone, and it suffices to show that $\mathscr{L} \in \mathscr{C}$. If, to the contrary, $\mathscr{L} \notin \mathscr{C}$, then it follows from the Minkowski separation theorem that there exists $F \in H^{**}$ such that $F(L_u) \geq 0$ $(\forall u \in K)$ but $F(\mathscr{L}) < 0$. Corresponding to F there exists $f \in H$ such that $f(u) = L_u(f) = F(L_u) \geq 0$ $(\forall u \in K)$ and $\mathscr{L}(f) = F(\mathscr{L}) < 0$, which contradicts the hypothesis that $\mathscr{L}$ is K-positive with respect to H. □

Corollary 2.4. *$\tilde{y} \in \mathscr{R}_{n+1,d}^{h}$ if and only if $\mathscr{L}_{\tilde{y}}$ is $\mathbb{S}^n$-positive with respect to $\mathbb{R}[\tilde{x}]_{=d}$.*

Proof. The "only if" direction is clear. For the converse, let $K = \mathbb{S}^n$, let H denote the subspace of $\mathbb{R}[\tilde{x}]_d$ generated by $\mathbb{R}[\tilde{x}]_{=d}$, and let $\mathscr{L} = \mathscr{L}_{\tilde{y}} : H \longrightarrow \mathbb{R}$. Note that $p_0(\tilde{x}) := ||\tilde{x}||^d \in H$, and $p_0(\tilde{u}) = 1 > 0$ for all $\tilde{u} \in \mathbb{S}^n$. Theorem 2.2 now implies that if $\mathscr{L}$ is K-positive with respect to H, then $\mathscr{L}$ corresponds to a finitely atomic measure supported in K, so $\tilde{y} \in \mathscr{R}_{n+1,d}^{h}$. □

We conclude this subsection with a result which shows that the kernel of a moment matrix has an ideal-like property. In the sequel, every polynomial $p(x)$ will be identified with its vector of coefficients (with respect to graded lexicographical ordering), which we also denote by p for convenience. When $\deg(p) \leq k$, then $M_k(w)p$ is defined as the usual matrix-vector product. Note that $p \in \ker M_k(w)$ (i.e., $M_k(w)p = 0$) if and only if $\mathscr{L}_w(p(x) \cdot x^{\alpha}) = 0$ for every $|\alpha| \leq k$. The following result appears in [4] for the case of complex moment matrices; for the equivalence between real and complex moment matrices, see [6]; this result also appears in [16, Lemma 5.7].

Lemma 2.5. *(*[4, Theorem 7.5], [16, Lemma 5.7]*) Let $w \in \mathscr{M}_{n,2k}$, $p \in \mathbb{R}[x]$ be such that $M_k(w) \succeq 0$ and $p \in ker\ M_k(w)$, with* $\deg(p) < k$. *If $q \in \mathbb{R}[x]$ and* $\deg(pq) \leq k-1$, *then $pq \in \ker M_k(w)$.*

2.2. **Positive polynomials, sum of squares and semidefinite programming.** For a polynomial $f \in \mathbb{R}[x]$, f is said to be *sum of squares (SOS)* if there exist polynomials $f_1, \ldots, f_r$ such that $f = f_1^2 + \cdots + f_r^2$, and f is said to be *positive semidefinite (psd)* or *nonnegative* if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$. Similar terminology applies for forms (homogeneous polynomials). Clearly, if f is SOS, then f must be nonnegative everywhere; but the converse is not necessarily true. We refer to Reznick [17] for a survey about SOS and psd polynomials.

Theorem 2.6 (Reznick [17])**.** *Let $f \in \mathbb{R}[\tilde{x}]_{=d}$ be a form that is strictly positive on the unit sphere $\mathbb{S}^n$. Then for k sufficiently large, the form $(\tilde{x}^T\tilde{x})^k f$ is SOS.*

Denote by $\Sigma_{n,2k}$ the cone of SOS polynomials in n variables and of degree $2k$. As a complement to Theorem 2.6, we include the following result of de Klerk et al. [9], which we require in Section 4.

Proposition 2.7. *(*[9, Proposition 2]*) Let d be even. For a form $f \in \mathbb{R}[\tilde{x}]_{=d}$, the product $(\tilde{x}^T\tilde{x})^k f$ is SOS if and only if there exist $\sigma \in \Sigma_{n+1,2k+d}$ and $h \in \mathbb{R}[\tilde{x}]_{2k+d-2}$ such that*

$$f \;=\; \sigma + (1 - \|\tilde{x}\|_2^2)h. \tag{2.2}$$

For each integer $k \geq 0$, define the cone

$$Q_k(n+1,d) := \left\{ f \in \mathbb{R}[\tilde{x}]_{=d} : \, (x_0^2 + \cdots + x_n^2)^k f \in \Sigma_{n+1,2k+d} \right\}. \tag{2.3}$$

By Proposition 2.7, this cone can also be equivalently defined as

$$Q_k(n+1,d) := \{ \sigma + \rho h \in \mathbb{R}[\tilde{x}]_{=d} : \, \sigma \in \Sigma_{n+1,2k+d}, h \in \mathbb{R}[\tilde{x}]_{2k+d-2} \}. \tag{2.4}$$

The union of all $Q_k(n+1,d)$ for fixed n, d is denoted as

$$Q(n+1,d) \;:=\; \bigcup_{k=0}^{\infty} Q_k(n+1,d). \tag{2.5}$$

The set $Q_k(n+1,d)$ is a convex cone. Its dual cone is defined as

$$Q_k(n+1,d)^* := \{ y \in \mathscr{M}_{n+1,d}^h : \, \langle f, y \rangle \geq 0 \;\; \forall f \in Q_k(n+1,d) \}.$$

For $\rho := \|\tilde{x}\|_2^2 - 1$ and each $k \geq 1$, define a linear operator $L_\rho^{(k)}$ acting on $\mathscr{M}_{2k+d}$ as

$$L_\rho^{(k)}(\tilde{y}) = \mathscr{L}_{\tilde{y}}\left(\rho(\tilde{x})[\tilde{x}]_{k+d/2-1}[\tilde{x}]_{k+d/2-1}^T \right), \; \forall \, \tilde{y} \in \mathscr{M}_{n+1,2k+d}.$$

Then, it can be shown that

$$Q_k(n+1,d)^* = \left\{ y \in \mathscr{M}_{n+1,d}^h \;\middle|\; \begin{array}{c} \exists \, \tilde{y} \in \mathscr{M}_{n+1,2k+d}, \; \tilde{y}|_d = y, \\ M_{k+d/2}(\tilde{y}) \succeq 0, \; L_\rho^{(k+d/2)}(\tilde{y}) = 0 \end{array} \right\}. \tag{2.6}$$

(cf. [13, §4] or [15, §4.2] or [16].) Note that the dual cone $Q_k(n+1,d)^*$ is the feasible set of a semidefinite program (SDP), i.e., it is defined by some linear scalar equalities and a linear matrix inequality.

Given $a, c \in \mathscr{M}_{n+1,d}^h$ and $b \in \mathbb{R}$, consider the linear conic optimization problem

$$\left\{ \begin{array}{ll} \min\limits_{p} & \langle p, c \rangle \\ s.t. & \langle p, a \rangle = b, \; p \in Q_k(n+1,d). \end{array} \right. \tag{2.7}$$

Its dual optimization problem is

$$\left\{ \begin{array}{ll} \max\limits_{\lambda, w} & b\lambda \\ s.t. & w + \lambda a = c, \; w \in Q_k(n+1,d)^*, \; \lambda \in \mathbb{R}. \end{array} \right. \tag{2.8}$$

We refer to [2, §2.4] for an introduction to linear conic optimization and its duality theory. The optimizations (2.7) and (2.8) are primal and dual semidefinite programming (SDP) problems. SDP is a generalization of linear programming, and is a class of linear convex optimization problems whose constraints have the cone of positive semidefinite matrices. SDP problems can be solved efficiently by numerical software (e.g., `SeDuMi` [18]). The optimal value of (2.7) (resp. (2.8)) is an upper bound (resp. lower bound) of the optimal value of the other one. This is called *weak duality*. If (2.7) (resp. (2.8)) has a feasible point that lies in the interior of $Q_k(n+1,d)$ (resp. $Q_k(n+1,d)^*$), then (2.8) (resp. (2.7)) has an optimizer. In either case, they have same optimal values. This is called *strong duality*. We refer to [13, 15, 16] for SDPs arising from moment problems and polynomial optimization.

3. Homogenizing TMP

As discussed in Section 1, there is a one to one mapping (i.e., a bijection) between a tms $y \in \mathscr{M}_{n,d}$ and its homogenization $\tilde{y} \in \mathscr{M}^h_{n+1,d}$ via homogenizing indices. In this section we show that under this mapping, y is in the closure of $\mathscr{R}_{n,d}$ if and only if $\tilde{y}$ admits a representing measure in $\mathbb{S}^n$ (relative to $(n+1)$-dimensional moments of degree d).

Theorem 3.1. *Let $y \in \mathscr{M}_{n,d}$ and d be even. If $y \in \mathscr{R}_{n,d}$, then $\tilde{y} \in \mathscr{R}^h_{n+1,d}$. Furthermore, $y \in cl(\mathscr{R}_{n,d})$ if and only if $\tilde{y} \in \mathscr{R}^h_{n+1,d}$.*

For the purposes of proving Theorem 3.1, we will distinguish notationally between $y \in \mathscr{M}_{n,d}$ and its homogenization $\tilde{y} \in \mathcal{M}^h_{n+1,d}$, defined by $\tilde{y}_{(d-|\alpha|,\alpha)} = y_\alpha$ ($|\alpha| \leq d$). Setting $x = (x_1, \ldots, x_n)$ and $\tilde{x} = (x_0, x) \equiv (x_0, x_1, \ldots, x_n)$, we define the Riesz functional of $\tilde{y}$ as

$$\mathscr{L}_{\tilde{y}} : \mathbb{R}[\tilde{x}]_{=d} \longrightarrow \mathbb{R}, \qquad \mathscr{L}_{\tilde{y}}(p(x_0, x)) = \mathscr{L}_y(p(1, x)).$$

Note for future reference that $\mathscr{L}_y$ is $\mathbb{R}^n$-positive if and only if $\mathscr{L}_{\tilde{y}}$ is $\mathbb{S}^n$-positive with respect to $\mathbb{R}[\tilde{x}]_{=d}$. Indeed, suppose that $\mathscr{L}_y$ is $\mathbb{R}^n$-positive and $p(x_0, x)$ is form of degree d with $p|_{\mathbb{S}^n} \geq 0$. Then for all $x \in \mathbb{R}^n$,

$$0 \leq p\left(\frac{(1, x)}{(1 + ||x||_2^2)^{1/2}}\right) = \frac{p(1, x)}{(1 + ||x||_2^2)^{d/2}} \implies p(1, x) \geq 0,$$

whence $\mathscr{L}_{\tilde{y}}(p) = \mathscr{L}_y(p(1, x)) \geq 0$; thus $\mathscr{L}_{\tilde{y}}$ is $\mathbb{S}^n$-positive with respect to $\mathbb{R}[\tilde{x}]_{=d}$. Conversely, suppose $\mathscr{L}_{\tilde{y}}$ is $\mathbb{S}^n$-positive with respect to $\mathbb{R}[\tilde{x}]_{=d}$. If $p \in \mathbb{R}[x]_d$ is psd, then its homogenization $\tilde{p}(x_0, x) := x_0^d p(x/x_0)$ is also psd, so

$$\mathscr{L}_y(p) = \mathscr{L}_y(\tilde{p}(1, x)) = \mathscr{L}_{\tilde{y}}(\tilde{p}) \geq 0;$$

thus $\mathscr{L}_y$ is positive. We also note that a minor modification of the preceding argument shows that $\mathscr{L}_y$ is strictly $\mathbb{R}^n$-positive if and only if $\mathscr{L}_{\tilde{y}}$ is strictly $\mathbb{S}^n$-positive with respect to $\mathbb{R}[\tilde{x}]_{=d}$; we will use this fact in Section 4.

Proof. (Proof of Theorem 3.1.) First, suppose $y \in \mathscr{R}_{n,d}$. By the result of Bayer and Teichmann [1], y also admits a finitely atomic measure μ, say,

$$\mu = c_1 \delta_{u_1} + \cdots + c_r \delta_{u_r},$$

where $c_1 > 0, \ldots, c_r > 0$ and δ_{u_i} denotes the Dirac measure supported on the point $u_i \in \mathbb{R}^n$. Then

$$y = c_1 [u_1]_d + \cdots + c_r [u_r]_d.$$

To each point u_i we correspond a point in $\mathbb{S}^n$ by

$$\tilde{u}_i \quad = \quad (1+\|u_i\|_2^2)^{-1/2}(1,u_i) \in \mathbb{S}^n.$$

Considered as an htms in $\mathscr{M}^h_{n+1,d}$, the homogenization $\tilde{y}$ has the representation

$$\tilde{y} \quad = \quad \sum_{i=1}^r c_i \cdot (1+\|u_i\|_2^2)^{d/2} \cdot [\tilde{u}_i^d],$$

so $\tilde{y} \in \mathscr{R}^h_{n+1,d}$.

Next, suppose $y \in \mathrm{cl}\,(\mathscr{R}_{n,d})$. It follows from Theorem 2.1(i) that $\mathscr{L}_y$ is $\mathbb{R}^n$-positive, so, from above, $\mathscr{L}_{\tilde{y}}$ is $\mathbb{S}^n$-positive with respect to $\mathbb{R}[\tilde{x}]_{=d}$. It now follows from Corollary 2.4 that $\tilde{y} \in \mathscr{R}^h_{n+1,d}$.

Conversely, suppose $\tilde{y} \in \mathscr{R}^h_{n+1,d}$, and let μ denote a measure for $\tilde{y}$ supported in $\mathbb{S}^n$. Note that μ can be chosen as a finitely atomic measure. Indeed, $||\tilde{x}||_2^d$ is homogeneous of degree d, so $\int_{\mathbb{S}^n} 1 d\mu = \int_{\mathbb{S}^n} ||\tilde{x}||_2^d d\mu < +\infty$. It follows that $\tilde{y}$ can be extended to a tms $\hat{y} \in \mathscr{R}_{n+1,d}$ that admits a measure (namely, μ) supported in $\mathbb{S}^n$. Now, by the result of Bayer and Teichmann [1], $\hat{y}$ (and thus also $\tilde{y}$) admits an atomic measure supported in $\mathbb{S}^n$, say,

$$\tilde{y} = \tilde{c}_1[v_1^d] + \cdots + \tilde{c}_r[v_r^d], \quad v_i \in \mathbb{S}^n,\, \tilde{c}_i > 0,\, i = 1, \ldots, r.$$

Write $v_i = (v_{i,0}, v_{i,1}, \ldots, v_{i,n}) \in \mathbb{S}^n$. Since d is even, if every $v_{i,0} \neq 0$, then

$$y = \sum_{i=1}^r c_i[u_i]_d, \qquad \text{where each } \; c_i = \tilde{c}_i \cdot v_{i,0}^d > 0,\, u_i = \frac{1}{v_{i,0}}(v_{i,1}, \ldots, v_{i,n}) \in \mathbb{R}^n.$$

That is, y admits a measure in $\mathbb{R}^n$. If some $v_{i,0} = 0$, then $y \in \mathrm{cl}\,(\mathscr{R}_{n,d})$, because

$$y = \lim_{\epsilon \to 0} \sum_{i=1}^r \tilde{c}_i (v_{i,0} + \epsilon)^d [u_i(\epsilon)]_d,$$

where

$$(v_{i,0}+\epsilon)^d [u_i(\epsilon)]_d \to [v_i^d] \quad \text{as} \quad \epsilon \to 0, \quad u_i(\epsilon) := \frac{(v_{i,1}, \ldots, v_{i,n})}{v_{i,0}+\epsilon} \in \mathbb{R}^n.$$

(Note that all $v_{i,0} + \epsilon$ will be nonzero if $\epsilon > 0$ is sufficiently small.) Thus, y is the limit of a sequence of tms that admit measures. □

For every tms $y \in \mathscr{M}_{n,d}$ (or equivalently, $y \in \mathscr{M}^h_{n+1,d}$), its Riesz functional $\mathscr{L}_y$ is a linear functional acting in $\mathbb{R}[x]_d$ (or equivalently in $\mathbb{R}[\tilde{x}]_{=d}$). In the following, we characterize the membership of $\mathrm{cl}\,(\mathscr{R}_{n,d})$ and $\mathscr{R}^h_{n+1,d}$ by using $\mathscr{L}_y$.

Theorem 3.2. *Let $y \equiv \tilde{y} \in \mathscr{M}_{n,d} \cong \mathscr{M}^h_{n+1,d}$, and $d > 0$ be even. Then we have:*

(i) *The cone $\mathscr{R}^h_{n+1,d}$ is a closed convex set with nonempty interior.*

(ii) *The tms $y \in \mathscr{R}^h_{n+1,d}$ if and only if $\mathscr{L}_y(f) \geq 0$ for every psd form $f \in \mathbb{R}[\tilde{x}]_{=d}$.*

(iii) *The tms $y \notin \mathscr{R}^h_{n+1,d}$ if and only if there exists p satisfying*

$$\langle p, y \rangle < 0, \quad p \in Q(n+1,d).$$

(iv) *When $n = 1$ or $d = 2$ or $(n,d) = (2,4)$, $y \in \mathscr{R}^h_{n+1,d}$ if and only if $M_{d/2}(y) \succeq 0$.*

Proof. (i) Consider the convex cone $\mathscr{C} := \{y \in \mathscr{M}_{n,d} : \mathscr{L}_y \succeq 0\}$. Clearly, $\mathscr{C}$ is closed. For the measure $\nu := e^{-||x||^2}dx$, the sequence y, of ν-moments up to degree d, has a Riesz functional that is strictly positive, so $y \in int(\mathscr{C})$ by Theorem 2.1(ii). Under the bi-continuous mapping $y \longleftrightarrow \tilde{y}$, $\mathscr{C}$ corresponds to

$$\tilde{\mathscr{C}} := \left\{\tilde{y} \in \mathscr{M}^h_{n+1,d} : \mathscr{L}_{\tilde{y}} \text{ is } \mathbb{S}^n - \text{positive w.r.t. } \mathbb{R}[\tilde{x}]_{=d}\right\}$$

(see the remarks preceding the proof of Theorem 3.1). It follows that $\tilde{\mathscr{C}}$ is a closed convex cone with nonempty interior, and Corollary 2.4 shows that $\tilde{\mathscr{C}} = \mathscr{R}^h_{n+1,d}$.

(ii) Corollary 2.4 shows that $y \equiv \tilde{y} \in \mathscr{R}^h_{n+1,d}$ if and only if $\mathscr{L}_y$ is $\mathbb{S}^n$-positive w.r.t. $\mathbb{R}[\tilde{x}]_{=d}$. To complete the proof, note that a form $f \in \mathbb{R}[\tilde{x}]_{=d}$ is psd if and only if $f|_{\mathbb{S}^n} \geq 0$.

(iii) The "if" part is trivial. For the "only if" part, suppose $y \notin \mathscr{R}^h_{n+1,d}$. By (ii), there exists a psd form $f \in \mathbb{R}[\tilde{x}]_{=d}$ such that $\mathscr{L}_y(f) < 0$. Then for $\epsilon > 0$ small enough, the form $p := f + \epsilon(\tilde{x}^T\tilde{x})^{d/2}$ is positive on $\mathbb{S}^n$ and $\mathscr{L}_y(p) < 0$. By Theorem 2.6, $p \in Q_k(n+1,d)$ for some k. So (iii) is proved.

(iv) From Corollary 2.4, $\tilde{y} \in \mathscr{R}^h_{n+1,d}$ if and only if $\mathscr{L}_{\tilde{y}}$ is $\mathbb{S}^n$-positive w.r.t. $\mathbb{R}[\tilde{x}]_{=d}$, and by the remarks preceding the proof of Theorem 3.1, this is equivalent to $\mathscr{L}_y \succeq 0$. Thus, it suffices to show that if $M_{d/2}(y) \succeq 0$, then $\mathscr{L}_y \succeq 0$. In the listed cases, each psd polynomial is SOS (see [17]). Thus, if $M_{d/2}(y) \succeq 0$ and $p = \sum p_i^2$, then $\mathscr{L}_y(p) = \sum\langle M_{d/2}(y)p_i, p_i\rangle \geq 0$. □

4. Homogeneous TMP

In this section, let y be a tms in the space $\mathscr{M}^h_{n+1,d}$ (or, equivalently, in $\mathscr{M}_{n,d}$, via dehomogenizing indices). Assume the degree d is even. Let $\zeta \in \mathscr{R}^h_{n+1,d}$ be a fixed tms whose Riesz functional $\mathscr{L}_\zeta$ is strictly positive.

In view of Theorem 3.2 (iii), consider the sequence of semidefinite optimization problems (for $k = 0, 1, 2, \ldots$):

$$\left\{\begin{array}{ll} \min\limits_{p} & \langle p, y\rangle(\equiv \mathscr{L}_y(p)) \\ s.t. & \langle p, \zeta\rangle = 1, p \in Q_k(n+1,d). \end{array}\right. \tag{4.1}$$

The primal-dual relation between (2.7) and (2.8) implies that the dual optimization problem of (4.1) is

$$\left\{\begin{array}{rl} \eta_k := \max\limits_{w,\eta} & \eta \\ s.t. & w|_{=d} \;=\; y - \eta\zeta,\; w \in \mathscr{M}_{n+1,2k+d}, \\ & M_{k+d/2}(w) \succeq 0,\; L_\rho^{(k+d/2)}(w) = 0. \end{array}\right. \tag{4.2}$$

Here $\rho(\tilde{x}) := \tilde{x}^T\tilde{x} - 1$ is the defining polynomial for the unit sphere $\mathbb{S}^n$.

To analyze the properties of (4.1) and (4.2), we classify measures by their supports. Let $\mathcal{Z}(f)$ denote the zero set of a polynomial f. A Borel measure μ on $\mathbb{R}^{n+1}$ is said to be $(\mathbb{S}^n, d)$*-semialgebraic* if $\text{supp}(\mu) \subseteq \mathbb{S}^n \cap \mathcal{Z}(q)$ for some *nonzero* $q \in Q(n+1,d)$. Not every measure supported in $\mathbb{S}^n$ is $(\mathbb{S}^n, d)$-semialgebraic.

Theorem 4.1. *Assume $d > 0$ is even. Let $y \in \mathscr{M}^h_{n+1,d}$, η_k be defined in (4.2), and $\zeta \in \mathscr{R}^h_{n+1,d}$ be such that $\mathscr{L}_\zeta$ is strictly $\mathbb{S}^n$-positive in the space $\mathbb{R}[\tilde{x}]_{=d}$. Then we have:*

(i) *The sequence $\{\eta_k\}$ is monotonically decreasing and $\eta_\infty := \lim\limits_{k\to\infty} \eta_k > -\infty$. Both (4.1) and (4.2) have optimizers, and their optimal values are equal.*

(ii) *For the htms* $y \in \mathscr{M}^h_{n+1,d}$, $y \in \mathscr{R}^h_{n+1,d}$ *if and only if* $\eta_k \geq 0$ *for every* k, *and* $y \notin \mathscr{R}^h_{n+1,d}$ *if and only if* $\eta_k < 0$ *for some* k. *The shifted tms* $\hat{y} := y - \eta_\infty \zeta \in \mathscr{R}^h_{n+1,d}$.

(iii) *For each* $\mu \in meas(\hat{y})$, μ *is* $(\mathbb{S}^n, d)$*-semialgebraic if and only if* $\eta_k = \eta_\infty$ *for some* k, *and* μ *is not* $(\mathbb{S}^n, d)$*-semialgebraic if and only if every* $\eta_k > \eta_\infty$.

(iv) *Let* (w^*, η_k) *be an optimal pair for (4.2). If for some* ℓ *with* $d \leq 2\ell \leq 2k+d$ *the truncation* $\omega := w^*|_{2\ell}$ *is flat, then* $\eta_k = \eta_\infty$ *and* $\hat{y} \in \mathscr{R}^h_{n+1,d}$.

Proof. (i) The decreasing monotonicity of $\{\eta_k\}$ is obvious. Since $\mathscr{L}_\zeta$ is strictly $\mathbb{S}^n$-positive, $\mathscr{L}_{\zeta+z}$ is $\mathbb{S}^n$-positive for all $z \in \mathscr{M}^h_{n+1,d}$ sufficiently small (cf. Theorem 2.1 (ii) and the remarks just preceding the proof of Theorem 3.1). By Theorem 3.2 (ii), we know $\zeta + z$ belongs to $\mathscr{R}^h_{n+1,d}$ for all z small enough, which means that ζ lies in the interior of $\mathscr{R}^h_{n+1,d}$. So, there exists $r > 0$ such that $y - (-r)\zeta \in \mathscr{R}^h_{n+1,d}$, whence $\eta_k \geq -r$ for every k and thus the limit η_∞ is finite. The feasible set of (4.1) intersects the interior of $Q_k(n+1, d)$ (e.g., a positive scaling of $\hat{p} := \|\tilde{x}\|^d$ satisfies $\langle \hat{p}, \zeta \rangle = 1$ and is in the interior of $Q_k(n+1, d)$; indeed, as a vector in the space $\mathbb{R}[\tilde{x}]_{=d}$, $\hat{p}$ is in the interior of the cone $\Sigma_{n+1,d} \cap \mathbb{R}[\tilde{x}]_{=d}$, i.e., the set of SOS forms in $n+1$ variables and of degree d). Thus, (4.1) and (4.2) share the same optimal value, and (4.2) has an optimizer. (This is implied by Theorem 2.4.I of [2]; cf. Section 2.)

To complete the proof of (i), it remains to show that (4.1) has an optimizer, and for this it suffices to show that the feasible set of (4.1) is compact. Denote this set by F. First, we show F is closed. Suppose $\{f_i\}_{i=1}^\infty \subset F$ is convergent to $f \in \mathbb{R}[\tilde{x}]_d$. Clearly, $f \in \mathbb{R}[\tilde{x}]_{=d}$ and $\langle f, \zeta \rangle = 1$. Since each $||\tilde{x}||^{2k} f_i$ is SOS, the limit $||\tilde{x}||^{2k} f$ is also SOS because the SOS cone $\Sigma_{n+1,2k+d}$ is closed (cf. [16, Corollary 3.50]). So $f \in Q_k(n+1, d)$, and thus F is closed. Second, we show F is bounded. Since $\mathscr{L}_\zeta$ is strictly positive, the Riesz functional $\mathscr{L}_\zeta$ attains a strictly positive minimum, say $\epsilon > 0$, on the compact set $\{p \in \mathbb{R}[\tilde{x}]_{=d} : p \in P_{n+1,d}, \|p\|_2 = 1\}$ (here $\|p\|_2$ denotes the 2-norm of the coefficient vector of p). Thus, for every $f \in F$, $\|f\|_2 \leq \langle f, \zeta \rangle / \epsilon = 1/\epsilon$. So, F is bounded and hence compact.

(ii) is implied by i) above and items (ii), (iii) of Theorem 3.2. Note that if we consider the shifted $\hat{y}$ as a new y, then its corresponding $\eta_\infty = 0$; so $\hat{y}$ admits a measure.

(iii) "only if" direction: Suppose $\mu \in meas(\hat{y})$ is $(\mathbb{S}^n, d)$-semialgebraic. Then there exists $0 \neq \hat{q} = s + \rho h \in \mathbb{R}[\tilde{x}]_{=d}$, with $h \in \mathbb{R}[\tilde{x}]_{2k+d-2}$ and $s \in \Sigma_{n+1,2k+d}$ (for some $k \geq 0$), such that $\text{supp}(\mu) \subseteq \mathbb{S}^n \cap \mathcal{Z}(\hat{q})$. Since $\mathscr{L}_\zeta$ is strictly $\mathbb{S}^n$-positive, we can scale $\hat{q}$ as $\langle \hat{q}, \zeta \rangle = 1$. So $\hat{q}$ is feasible for (4.1) and we have (using item (i))

$$0 = \int_{\mathbb{S}^n} \hat{q} d\mu = \langle \hat{q}, \hat{y} \rangle = \langle \hat{q}, y \rangle - \eta_\infty \langle \hat{q}, \zeta \rangle \geq \eta_k - \eta_\infty.$$

Since $\eta_k \geq \eta_\infty$, it follows that $\eta_k = \eta_\infty$.

"if" direction: Suppose $\eta_k = \eta_\infty$. Since (4.1) has a minimizer, let $f \in \mathbb{R}[\tilde{x}]_{=d} \cap Q_k(n+1, d)$ be one such. Clearly, $0 \neq f \in Q(n+1, d)$ and

$$\int_{\mathbb{S}^n} f d\mu = \langle f, \hat{y} \rangle = \langle f, y - \eta_\infty \zeta \rangle = \eta_k - \eta_\infty = 0.$$

The nonnegativity of f on $\mathbb{S}^n$ implies $\text{supp}(\mu) \subseteq \mathbb{S}^n \cap \mathcal{Z}(f)$. So μ is $(\mathbb{S}^n, d)$-semialgebraic.

(iv) From [6], we know that for every $j \geq \ell$, ω can be extended to a flat tms z satisfying

$$z \in \mathscr{M}_{n+1,2j+d},\ M_{j+d/2}(z) \succeq 0,\ L_\rho^{(j+d/2)}(z) = 0.$$

Since $z|_{2\ell} = \omega$, it follows that $\eta_j \geq \eta_k$; thus, by the decreasing monotonicity of the sequence $\{\eta_i\}$, we must have $\eta_j = \eta_k$ for every $j \geq \ell$, so $\eta_k = \eta_\infty$. The membership $\hat{y} \in \mathscr{R}^h_{n+1,d}$ is clear from the flatness of ω. □

Item (ii) of Theorem 4.1 is very useful in certifying $y \notin \mathscr{R}^h_{n+1,d}$, while item (iv) is practical in certifying $\hat{y} \in \mathscr{R}^h_{n+1,d}$ and $y \in \mathscr{R}^h_{n+1,d}$ (if $\eta_k \geq 0$). This is because of the decomposition

$$y = \eta_k \zeta + \omega|_d.$$

If $\nu \in meas(\zeta)$ and $\mu \in meas(\omega)$, then

$$\eta_k \cdot \nu + \mu$$

is a representing measure for y. So, it is the most interesting case if the sequence $\{\eta_i\}$ has finite convergence (this is equivalent to the condition that a measure representing $\hat{y}$ is ($\mathbb{S}^n$,d)-semialgebraic) and ω is flat. Indeed, under some reasonable assumptions, the flatness of ω in item (iv) of Theorem 4.1 is also guaranteed, as the next result shows.

Theorem 4.2. *Let $d, y, \zeta, \eta_k, \eta_\infty, \hat{y}$ be the same as in Theorem 4.1. Suppose $\mu \in meas(\hat{y})$ is ($\mathbb{S}^n$,d)-semialgebraic and satisfies*

$$supp(\mu) \quad \subseteq \quad U := \mathbb{S}^n \cap \mathcal{Z}(q), \quad 0 \neq q \in Q(n+1, d).$$

Let w be optimal for(4.2). If $|U| < \infty$, then there exists $2\ell \in [d, 2k+d]$ such that $w|_{2\ell}$ is flat for k sufficiently large.

Proof. Since $q \in Q(n+1, d)$, there exist polynomials s (being SOS) and h from $\mathbb{R}[\tilde{x}]$ such that $q = s + \rho \cdot h$ (see Section 2). Write $s = f_1^2 + \cdots + f_r^2$. For every $k + d/2 > \frac{1}{2}\max(\deg(q), \deg(s))$, we have

$$0 = \int_{\mathbb{S}^n} q d\mu = \langle q, \hat{y} \rangle = \sum_j f_j^T M_{k+d/2}(w) f_j.$$

Since each $M_{k+d/2}(w) \succeq 0$, we have

$$M_{k+d/2}(w) f_j = 0,\ j = 1, \ldots, r.$$

Let $I(U) := \{p(x) \in \mathbb{R}[x] : p|_U = 0\}$ be the vanishing ideal of the set U. Since $|U| < \infty$, the quotient ideal $\mathbb{R}[x]/I(U)$ is finite dimensional (cf. [19]). Let $\{b_1, \ldots, b_l\}$ be a basis of $\mathbb{R}[x]/I(U)$, and let $\{g_1, \ldots, g_m\}$ be a Grobner basis for the ideal $I(U)$ in a total degree ordering. Then, each g_i vanishes on the variety

$$U = \{\tilde{x} \in \mathbb{R}^{n+1} : \rho(\tilde{x}) = f_1(\tilde{x}) = \cdots = f_r(\tilde{x}) = 0\}.$$

By Positivstellensatz (cf. Proposition 4.4.6 of [3]), for each g_i, there exist an integer $t > 0$, polynomials $\phi_0, \ldots, \phi_r$, and σ (which is SOS), such that

$$g_i^{2t} + \phi_0 \rho + \phi_1 f_1 + \cdots + \phi_r f_r + \sigma = 0. \tag{4.3}$$

Since $f_j \in \ker M_{k+d/2}(w)$, by Lemma 2.5, every $f_j \phi_j \in \ker M_{k+d/2}(w)$ if $\deg(f_j \phi_j) \leq k + d/2 - 1$. So

$$\langle \phi_j f_j, w \rangle = 0,\ j = 1, \ldots, r.$$

The condition $L_\rho^{(k+d/2)}(w) = 0$ implies $\langle \phi_0\rho, w\rangle = 0$ if $\deg(\phi_0\rho) \le 2k+d$. Since σ is SOS, then $\langle \sigma, w\rangle \ge 0$. Clearly, $\langle g_i^{2t}, w\rangle \ge 0$. Thus (4.3) implies $\langle M_{k+d/2}(w)g_i^t, g_i^t\rangle = 0$. Since $M_{k+d/2}(w) \succeq 0$, we must have $M_{k+d/2}(w)g_i^t = 0$ and hence $g_i \in \ker M_{k+d/2}(w)$ (by an induction on t, or see Lemma 3.9 of [14]), when k is sufficiently large.

For every exponent α, we can write

$$x^\alpha = r(\alpha) + \sum p_i g_i, \quad \deg(p_i g_i) \le |\alpha|, \quad r(\alpha) \in span\{b_1, \dots, b_l\}.$$

We know each $g_i \in \ker M_{k+d/2}(w)$ from above, and $p_i g_i \in \ker M_{k+d/2}(w)$ if $|\alpha| \le k + d/2 - 1$, by Lemma 2.5. Thus, $x^\alpha - r(\alpha) \in \ker M_{k+d/2}(w)$ whenever $|\alpha| = k + d/2 - 1$. So, if

$$k + d/2 - 1 \quad > \quad d_b := \max\{\deg(b_1), \dots, \deg(b_t)\},$$

every α-th ($|\alpha| = k + d/2 - 1$) column of $M_{k+d/2}(w)$ is a linear combination of the β-columns of $M_{k+d/2}(w)$ with $|\beta| \le d_b$. This means the truncated tms $w|_{2k+d-2}$ is flat for k big enough.

The proof is complete by choosing $\ell = k + d/2 - 1$ (of course, a smaller choice for ℓ might be possible). □

Now we present some examples which illustrate Theorems 4.1 and 4.2. The semidefinite optimization problem (4.2) and its dual are solved by the software `SeDuMi` [18]. In discussing our conclusions, we realize that they are made modulo the imprecision that is inherent in numerical calculations (due to computer round-off errors, etc.). For this reason, we have mostly chosen examples from the literature for which our conclusions can be independently verified through alternate approaches. Throughout these examples, choose $\zeta \in \mathscr{R}_{n,d}$ to be the tms admitting the standard Gaussian measure, i.e.,

$$\zeta_\alpha := \frac{1}{\sqrt{2\pi}^n} \int x^\alpha \exp\{-\|x\|_2^2/2\} dx \qquad \big(\alpha \in \mathbb{N}^n\big).$$

Since its associated Riesz functional $\mathscr{L}_\zeta$ is strictly positive, its homogenization, which will play the role of ζ in (4.2) and Theorem 4.1, has a strictly $\mathbb{S}^n$-positive Riesz functional, as required in these results (see the remarks following the statement of Theorem 3.1). We note that in some cases we are able to get a certificate for nonexistence of a representing measure, or in other cases to construct a representing measure, for a tms y where the moment matrix $M_{d/2}(y)$ is positive definite (where techniques based on moment matrix extensions have the most trouble.) This is shown in the following examples.

Example 4.3. ([12, Example 3.4]) Consider the tms $y \in \mathscr{M}_{2,6}$ below:

$$(28, 0, 0, 1.1, 0, 3.4, 0, 0, 0, 0, 1.1, 0, 1.2, 0, 1.6, 0, 0, 0, 0, 0, 0, 28, 0, 3.4, 0, 1.6, 0, 1.2).^1$$

Its 3rd order moment matrix $M_3(y)$ is positive definite. It can also be thought of as an htms in $\mathscr{M}_{3,6}^h$. Solving (4.2) for $k = 1$, we get its optimal value $\eta_1 \approx -0.0208 < 0$. Thus, Theorem 4.1 (ii) shows that $y \notin \mathscr{R}_{3,6}^h$, whence the tms y does not admit a measure on $\mathbb{R}^n$, i.e., $y \notin \mathscr{R}_{2,6}$. This fact can also be shown non-numerically as follows. Let $M(x) := x_1^2 + x_1^4 - 3x_1^2x_2^2 + x_2^6$. Its homogenization is the Motzkin

[1] Throughout the paper, the entries of a tms are listed in graded lexicographical ordering.

polynomial, which is psd but not SOS (cf. [17]). So M is also psd but not SOS. Applying the Riesz functional $\mathcal{L}_y$ to M, we get

$$\mathcal{L}_y(M) = 1.1 + 1.1 - 3 \cdot 1.2 + 1.2 = -0.2 < 0.$$

This implies that $\mathcal{L}_y$ is not positive, and hence $y \notin \mathscr{R}_{2,6}$. □

By Theorem 4.1 (iv), it is always possible to construct a representing measure for a tms y when $\eta_k \geq 0$ and a truncation ω of an optimal w^* is flat, even if the moment matrix of y is positive definite. This is because, from the decomposition $y = \eta_k \zeta + w^*|_d$, we know $\eta_k \nu + \mu$ is a representing measure for y if $\nu \in meas(\zeta)$ and $\mu \in meas(\omega)$. We illustrate this as follows.

Example 4.4. Consider the tms $y \in \mathscr{M}_{2,4}$:

$$(5, 0, 0, 5, 0, 5, 0, 0, 0, 0, 7, 0, 5, 0, 7).$$

Its 2nd order moment matrix $M_2(y)$ is positive definite. The existence of a representing measure for this tms is shown in [10], but no methods were given there for constructing such a measure. We apply Theorem 4.1 to construct a representing measure for this tms. Solving (4.2) for $k = 3$, we get its optimal value $\eta_3 = 1 > 0$ and an optimal w^*. Its truncation $w^*|_8$ is flat, and admits an 8-atomic measure supported on the points:

$$\sqrt{1/3}(\pm 1, \quad \pm 1, \quad \pm 1).$$

Thus, $y \in \mathscr{R}^h_{3,4}$ as an htms in $\mathscr{M}^h_{3,4}$. The x_0-coordinates of the above points are nonzero. By the dehomogenization technique described in the proof of Theorem 3.1, as a tms in $\mathscr{M}_{2,4}$, y is represented by the standard Gaussian measure ν^* plus a 4-atomic measure supported on the points $(\pm 1, \pm 1)$ (the weights are all ones), i.e.,

$$\nu^* + \delta_{(-1,-1)} + \delta_{(-1,1)} + \delta_{(1,-1)} + \delta_{(1,1)}$$

is a representing measure for the tms y above. □

We conclude this section with some examples from the literature.

Example 4.5. ([11, Example 5.2]) Consider the tms $y \in \mathscr{M}_{2,6}$:

$$(1, 1, 0, 1, 0, 1, 1, 0, 1, c, 1, 0, 1, c, 1 + c^2, 1, 0, 1, c, 1 + c^2, 2c + c^3,$$
$$1, 0, 1, c, 1 + c^2, 2c + c^3, 1 + 3c^2 + c^4 + t),$$

with parameters $c, t \in \mathbb{R}$. When $t = 0$, y is flat and admits a measure; if $t > 0$, then y does not admit a measure, but is the limit of flat tms [11]. Consider the basic case $c = 0, t = 1$. For $k = 0, 1, 2$, solving (4.2), we get all optimal values $\eta_k = 0$. When $k = 2$, the truncation $w^*|_8$ (w^* being optimal for (4.2)) is flat and rank $M_4(w^*) = 6$. As a tms in $\mathscr{M}_{3,8}$, $w^*|_8$ admits, as well as does $\hat{y}$, a 6-atomic measure with support on $\mathbb{S}^2$:

$$\left\{ \pm\sqrt{1/3}(1, 1, 1), \quad \pm\sqrt{1/3}(1, 1, -1), \quad \pm(0, 0, 1) \right\}.$$

Since the x_0-coordinates of the last two points are zero, this measure does not yield a representing measure for $y \in \mathscr{M}_{2,6}$. However, following the proof of Theorem 3.1, we can use this measure to approximate y arbitrarily closely by tms in $\mathscr{R}_{2,6}$, i.e., $y \in \mathrm{cl}\,(\mathscr{R}_{2,6})$ (in agreement with [11]). In the following, we illustrate how to construct approximations by using the dehomogenization technique described in the proof of Theorem 3.1. As an htms in $\mathscr{M}^h_{3,6}$, we can decompose y as

$$y = 13.5[(3^{-\frac{1}{2}}, 3^{-\frac{1}{2}}, 3^{-\frac{1}{2}})^6] + 13.5[(3^{-\frac{1}{2}}, 3^{-\frac{1}{2}}, -3^{-\frac{1}{2}})^6] + [(0, 0, 1)^6].$$

Let $y(\epsilon)$ be the tms in $\mathscr{M}_{2,6}$ defined as

$$y(\epsilon) = \frac{1}{2}[(1,1)]_6 + \frac{1}{2}[(1,-1)]_6 + \epsilon^6[(0,\epsilon^{-1})]_6.$$

For any $\epsilon > 0$, $y(\epsilon)$ is a tms admitting a 3-atomic measure. The difference between $y(\epsilon)$ and y is the tms

$$(\epsilon^6, 0, \epsilon^5, 0, 0, \epsilon^4, 0, 0, 0, \epsilon^3, 0, 0, 0, 0, \epsilon^2, 0, 0, 0, 0, 0, \epsilon, 0, 0, 0, 0, 0, 0, 0).$$

Clearly, $\|y(\epsilon) - y\|_2 \to 0$ as $\epsilon \to 0$. □

Example 4.6 ([8])**.** Consider the tms $y \in \mathscr{M}_{2,6}$ below:

$$(1, 0, 0, 1, 2, 5, 0, 0, 0, 0, 2, 5, 14, 42, 132,$$
$$0, 0, 0, 0, 0, 0, 5, 14, 42, 132, 429, c, 2026881 - 2844c + c^2)$$

where c is a parameter. It is shown in [8] that this tms admits no measure if $c < 1429$, but it does when $c \geq 1429$. Here, we use (4.2) to solve this TMP.

For $c = 1428$, we have $\eta_1 \approx -0.0013 < 0$, which implies $y \notin \mathscr{R}^h_{3,6}$ and $y \notin \mathscr{R}_{2,6}$, modulo some numerical imprecision. Indeed, this assertion is proved in [8].

For $c = 1429$, we get $\eta_3 \approx -7 \cdot 10^{-8}$ and $w^*|_{10}$ is flat (w^* being optimal for (4.2)); rank $M_5(w^*) = 16$; as a tms in $\mathscr{M}_{3,6}$, w^* admits a 16-atomic measure supported on $\mathbb{S}^2$ (the x_0-coordinates are all nonzero); by using the techniques used in the proof of Theorem 3.1, as a tms in $\mathscr{M}_{2,6}$, $y \in \mathscr{R}_{2,6}$ and admits an 8-atomic measure. Its support consists of eight points $u_1, \ldots, u_8$ with weights $\rho_1, \ldots, \rho_8$ respectively. They are listed in Table 1 [2]. Let $z := \sum_{i=1}^{8} \rho_i [u_i]_6$ be the tms recovered from

u_i	$\pm\begin{pmatrix}1.8794\\6.6382\end{pmatrix}$	$\pm\begin{pmatrix}1.5321\\3.5963\end{pmatrix}$	$\pm\begin{pmatrix}1.0000\\1.0000\end{pmatrix}$	$\pm\begin{pmatrix}0.3473\\0.0419\end{pmatrix}$
ρ_i	0.0260	0.0918	0.1667	0.2155

TABLE 1. The listing of points u_i and weights ρ_i.

this 8-atomic measure. The tms y and z are almost same, modulo some numerical imprecision. Indeed, the existence of an 8-atomic measure representing y is shown in [8]. □

5. Some Extensions

Here we discuss two possible extensions of the results in this manuscript.

Noncompact TKMP Recall that TKMP for K compact and semialgebraic was addressed extensively in [12], and in the preceding sections we have treated the case $K = \mathbb{R}^n$. The more general TKMP for K noncompact and semialgebraic can also be solved using the techniques of homogenization and flat extension. Suppose $K = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$ with every g_i a polynomial. We may homogenize K as

$$\tilde{K} = \{\tilde{x} \in \mathbb{R}^n : \tilde{g}_1(\tilde{x}) \geq 0, \ldots, \tilde{g}_m(\tilde{x}) \geq 0, \|\tilde{x}\|_2^2 - 1 = 0\},$$

[2]For convenience, only four decimal digits are shown.

where each $\tilde{g}_i(\tilde{x}) = x_0^{\deg(g_i)} g_i(x/x_0)$ is the homogenization of $g_i(x)$. Note that $\tilde{K}$ is always compact. Hence, the approaches in [12] can be applied. As in (4.2), we can consider the sequence of semidefinite optimization problems:

$$(5.1) \qquad \begin{cases} \max\limits_{w,\eta} & \eta \\ s.t. & w|_{=d} \,=\, y - \eta\zeta,\, w \in \mathscr{M}_{n+1,2k+d}, \\ & M_{k+d/2}(w) \succeq 0,\, L_{\rho}^{(k+d/2)}(w) = 0, \\ & L_{\tilde{g}_1}^{(k+d/2)}(w) \succeq 0, \ldots, L_{\tilde{g}_m}^{(k+d/2)}(w) \succeq 0. \end{cases}$$

Here each $L_{\tilde{g}_i}^{(k+d/2)}(w)$ denotes a localizing matrix associated with $\tilde{g}_i$ and tms w (cf. [12]). Using this SDP, it is possible to obtain natural analogues of Theorems 3.2 and 4.1.

Odd TMP An interesting case of TMP is when the degree is odd. How can we check wether or not a tms $y \in \mathscr{M}_{n,d}$ of odd degree d admits a representing measure? In such situations, approaches similar to those in this paper can be applied. Every tms in $\mathscr{M}_{n,d}$ can be extended to a tms in $\mathscr{M}_{n,d+1}$, or equivalently, $\mathscr{M}_{n,d}$ is a projection of $\mathscr{M}_{n,d+1}$. Every $y \in \mathscr{M}_{n,d}$ can be thought of as a subvector of a tms $\tilde{y} \in \mathscr{M}_{n,d+1}$. We say $\tilde{y}$ is an extension of y if $\tilde{y}_\alpha = y_\alpha$ for every $|\alpha| \leq d$. Denote by $extend(y)$ the set of such extensions $\tilde{y}$ of y. Clearly, $y \in \mathscr{R}_{n,d}$ if and only if $extend(y) \cap \mathscr{R}_{n,d+1} \neq \emptyset$. Therefore, the results for the even degree case can be applied here. By analogy with (4.2), we can consider the sequence of semidefinite optimization problems:

$$(5.2) \qquad \begin{cases} \max\limits_{w,\theta,\tilde{y}} & \theta \\ s.t. & w|_{=d+1} \,=\, \tilde{y} - \theta\zeta,\, L_{\rho}^{(k+\frac{1}{2}(d+1))}(w) = 0, \\ & M_{k+\frac{1}{2}(d+1)}(w) \succeq 0,\, \tilde{y} \in extend(y),\, w \in \mathscr{M}_{n+1,2k+d+1}. \end{cases}$$

Results similar to Theorem 4.1 can be obtained in a natural way.

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