Positivity of Riesz Functionals and Solutions of Quadratic and Quartic Moment Problems

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Abstract

We employ positivity of Riesz functionals to establish representing measures (or approximate representing measures) for truncated multivariate moment sequences. For a truncated moment sequence y, we show that y lies in the closure of truncated moment sequences admitting representing measures supported in a prescribed closed set $K \subseteq \mathbb{R}^n$ if and only if the associated Riesz functional L_y is K-positive. For a determining set K, we prove that if L_y is strictly K-positive, then y admits a representing measure supported in K. As a consequence, we are able to solve the truncated K-moment problem of degree k in the cases: (i) $(n, k) = (2, 4)$ and $K = \mathbb{R}^2$; (ii) $n \geq 1$, $k = 2$, and K is defined by one quadratic equality or inequality. In particular, these results solve the truncated moment problem in the remaining open cases of Hilbert's theorem on sums of squares.

Keywords: truncated moment sequence, Riesz functional, (strict) K-positivity, determining set, moment matrix, representing measure

AMS subject classifications: 47A57, 44A60, 47N40, 47A20

1 Introduction

Denote by $\mathbb{Z}_+$ the set of nonnegative integers and let $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha \equiv (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$. Let $y = (y_\alpha)_{\alpha \in \mathbb{Z}_+^n, |\alpha| \leq k}$ be a real multisequence of degree k in n variables (also referred to as a *truncated moment sequence*), and let $K \subseteq \mathbb{R}^n$ be a closed set. The *truncated K-moment problem* of degree k concerns conditions on y such that it has a *K-representing measure*, i.e., a positive Borel measure μ on $\mathbb{R}^n$, supported in K, such that

$$y_\alpha = \int_{\mathbb{R}^n} x^\alpha d\mu(x), \quad \forall\, \alpha \in \mathbb{Z}_+^n : |\alpha| \leq k. \tag{1.1}$$

(Here, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $x \equiv (x_1, \ldots, x_n) \in \mathbb{R}^n$.) For $K = \mathbb{R}^n$, we refer to (1.1) simply as the *truncated moment problem* and to μ as a *representing measure*. Let $\mathcal{P}_k \subset \mathbb{R}[x_1, \ldots, x_n]$

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denote the polynomials of degree at most k. Corresponding to the sequence y of degree k is the *Riesz functional* $L_y : \mathcal{P}_k \longrightarrow \mathbb{R}$ defined by

$$L_y(p) = \sum_{\alpha \in \mathbb{Z}_+^n : |\alpha| \le k} p_\alpha y_\alpha, \qquad \forall\, p \equiv \sum_{\alpha \in \mathbb{Z}_+^n : |\alpha| \le k} p_\alpha x^\alpha \in \mathcal{P}_k.$$

L_y is said to be *K-positive* if

$$L_y(p) \ge 0, \quad \forall\, p \in \mathcal{P}_k,\ p|_K \ge 0.$$

Further, L_y is *strictly K-positive* if L_y is K-positive and

$$L_y(p) > 0, \quad \forall\, p \in \mathcal{P}_k,\ p|_K \ge 0,\ p|K \not\equiv 0.$$

For $K = \mathbb{R}^n$ we say simply that L_y is *positive* or *strictly positive*. K-positivity is a necessary condition for K-representing measures, for if μ is a K-representing measure and $p \in \mathcal{P}_k$ with $p|_K \ge 0$, then $L_y(p) = \int_K p\ d\mu \ge 0$. The proof of Tchakaloff's Theorem [20] shows that if K is compact, then K-positivity is actually sufficient for K-representing measures, but this is not so in general (see below). Nevertheless, in [10] R.E. Curto and the first-named author obtained the following solution to the truncated K-moment problem expressed in terms of K-positivity.

Theorem 1.1 (Theorem 1.2, [10])**.** *A multisequence y of degree $2d$ or $2d+1$ admits a K-representing measure if and only if y can be extended to a sequence $\tilde{y}$ of degree $2d+2$ such that $L_{\tilde{y}}$ is K-positive.*

A significant issue associated with Theorem 1.1 is that in general it is quite difficult to establish that L_y or $L_{\tilde{y}}$ is K-positive. We show in Section 2 (Theorem 2.2) that L_y is K-positive if and only if $\lim_{m\to\infty} \|y - y^{(m)}\| = 0$ for a sequence $\{y^{(m)}\}$ in which each truncated moment sequence $y^{(m)}$ has a K-representing measure $\mu^{(m)}$. In this case, for each α, we have $y_\alpha = \lim_{m\to\infty} \int_K x^\alpha d\mu^{(m)}(x)$, and we say that $\{\mu^{(m)}\}$ is a sequence of *approximate representing measures* for y. This leads us to identify some cases of interest, including certain multivariate quadratic and quartic moment problems, in which we can utilize such approximating sequences to establish K-representing measures for y or K-positivity for L_y. To explain our results further, consider $K = \mathbb{R}^n$. For $k = 2d$, the moment sequence y is associated with the d-th order *moment matrix* $M_d(y)$ defined by

$$M_d(y) = (y_{\alpha+\beta})_{(\alpha,\beta) \in \mathbb{Z}_+^n \times \mathbb{Z}_+^n : |\alpha|,|\beta| \le d}.$$

(We sometimes refer to a representing measure for y as a representing measure for $M_d(y)$.) A basic necessary condition for positivity of L_y (and hence for the existence of a representing measure) is that M_y be positive semidefinite ($M_d(y) \succeq 0$). To see this, observe that $M_d(y)$ is uniquely determined by the relation

$$\langle M_d(y)\hat{p}, \hat{q} \rangle = L_y(pq) \quad p, q \in \mathcal{P}_d, \tag{1.2}$$

where $\hat{r}$ denotes the coefficient vector of $r \in \mathcal{P}_d$ relative to the basis for $\mathcal{P}_d$ consisting of the monomials in degree-lexicographic order. Thus, if L_y is positive, then $\langle M_d(y)\hat{p}, \hat{p} \rangle = L_y(p^2) \ge 0$. It is known that if L_y is positive and $M_d(y)$ is singular, then y need not have a representing measure; the simplest such example occurs with $n = 1$, $d = 2$ and $M_2(y)$ of the form

$$M_2(y) = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & b \end{bmatrix},$$

with $b > a > 0$ (cf. [10, Example 2.1]). Nevertheless, the following question, essentially asked in ([10, Question 2.9]), remains unsolved.

Question 1.2. Let $k = 2d$. If L_y is K-positive and $M_d(y)$ is positive definite, does y have a K-representing measure; equivalently, does L_y admit a K-positive extension $L_{\tilde{y}} : \mathcal{P}_{2d+2} \longrightarrow \mathbb{R}$?

In the sequel we say that K is a *determining set (of degree k)* if whenever $p \in \mathcal{P}_k$ and $p|K \equiv 0$, then $p \equiv 0$ (i.e., $p(x) = 0 \quad \forall x \in \mathbb{R}^n$); sets K with nonempty interior are clearly determining. It follows readily from (1.2) that if K is a determining set and L_y is strictly K-positive, then $M_d(y) \succ 0$. Our main tool in establishing K-representing measures is the following result, which complements Theorem 1.1 and partially answers Question 1.2.

Theorem 1.3. *Suppose K is a determining set of degree k and let y be a truncated moment sequence of degree k in n variables. If L_y is strictly K-positive, then y admits a K-representing measure.*

To discuss concrete applications of Theorem 1.3, we consider the following property:

> $(H_{n,d})$ Each $p \in \mathcal{P}_{2d}$ admits a sum-of-squares decomposition, $p = \sum p_i^2$, for certain polynomials $p_i \in \mathcal{P}_d$ (which depend on p).

If $(H_{n,d})$ holds and we set $k = 2d$, then positivity for L_y is equivalent to positivity of $M_d(y)$; indeed, in this case, if $M_d(y) \succeq 0$ and $p \in \mathcal{P}_{2d}$ is nonnegative on $\mathbb{R}^n$, then $L_y(p) = \sum L_y(p_i^2) = \sum \langle M_d(y)\hat{p}_i, \hat{p}_i\rangle \geq 0$. A well-known theorem of Hilbert (cf. [16, 17]) shows that $(H_{n,d})$ holds if and only if $n = 1$, $n = d = 2$, or $n > 1$ and $d = 1$. In these cases, whether or not y has a representing measure, Theorem 2.2 (cf. Section 2) implies that if $M_d(y) \succeq 0$, then y has a sequence of approximate representing measures. For $n = 1$, the truncated moment problem has been solved (cf. [4]): a multisequence y of degree $2d$ has a representing measure if and only if $M_d(y)$ is positive semidefinite and *recursively generated* (see below for terminology concerning moment matrices). In the sequel we address the truncated moment problem in the other cases covered by Hilbert's theorem.

Consider first the bivariate quartic moment problem ($n = d = 2$). For the case when $M_2(y)$ is singular, concrete necessary and sufficient conditions for representing measures are known (cf. [7, 9]): y has a representing measure if and only if

$$M_2(y) \succeq 0,\ M_2(y) \textit{ is recursively generated, and rank } M_2(y) \leq \textit{card } \mathcal{V}(M_2(y)), \tag{1.3}$$

where $\mathcal{V}(M_2(y))$ is the *algebraic variety* associated to $M_2(y)$ (see definition (1.5)) and *card* denotes the cardinality of a set. When 2 is replaced by d, the conditions of (1.3) apply more generally to any bivariate sequence y of degree $2d$ for which $M_2(y)$ is singular, i.e., the first 6 columns of $M_2(y)$ are dependent (cf. [9, Theorem 1.2]). Subsequent to [7], the case $M_2(y) \succ 0$ has been open (cf. [13]). In this case, it is easy to find a moment matrix extension $M_3(\tilde{y}) \succ 0$, but an example of [5] shows that for such $\tilde{y}$, $L_{\tilde{y}}$ need not be positive, so Theorem

1.1 cannot be applied to yield a representing measure for y. Instead, in Section 3 we will use Theorem 1.3, together with Hilbert's theorem, to establish that such y does indeed have a representing measure. This provides a positive answer to Question 1.2 for $n = d = 2$, with $K = \mathbb{R}^2$.

Consider next the case of the multivariate quadratic moment problem, where $n \geq 1$ and $d = 1$. For $n = 1,\ 2$, it was shown in [4] that if $M_1(y) \succeq 0$, then y has a *rank* $M_1(y)$-atomic representing measure, and in Section 4, Theorem 4.5, we prove the same result for $n \geq 1$. In the sequel, let $\mathcal{R}_{n,k}(K)$ denote the convex set of n-variable moment sequences of degree k which admit K-representing measures, and let $\overline{\mathcal{R}_{n,k}(K)}$ denote the closure of $\mathcal{R}_{n,k}(K)$ in $\mathbb{R}^\eta$, where $\eta = dim\ \mathcal{P}_k$. Now let q be a quadratic polynomial, and define the quadratic variety $E(q) = \{x \in \mathbb{R}^n : q(x) = 0\}$ and the quadratic semialgebraic set $S(q) := \{x \in \mathbb{R}^n : q(x) \geq 0\}$. We are interested in determining whether y has a representing measure supported in $E(q)$ or in $S(q)$. It is obvious that if y has a representing measure supported in $E(q)$ (resp., $S(q)$), then

$$M_1(y) \succeq 0, \quad L_y(q) = 0\ \ (resp., L_y(q) \geq 0). \tag{1.4}$$

For the case when $S(q)$ is compact, we will show in Theorem 4.7 that if y satisfies (1.4), then $y \in \mathcal{R}_{n,2}(E(q))$ (resp., $y \in \mathcal{R}_{n,2}(S(q))$). For the general case, we show in Theorem 4.8 that if (1.4) holds, then $y \in \overline{\mathcal{R}_{n,2}(E(q))}$ (resp., $y \in \overline{\mathcal{R}_{n,2}(S(q))}$). In Theorem 4.10, we further show that if $M_1(y) \succ 0$ and $L_y(q) = 0$ (resp., $L_y(q) > 0$), then $y \in \mathcal{R}_{n,2}(E(q))$ (resp., $y \in \mathcal{R}_{n,2}(S(q))$); this result implies an affirmative answer to Question 1.2 for $d = 1$ and $K = E(q)$ (resp., $K = S(q)$).

The preceding concrete results all concern the positive cases of Hilbert's theorem. In some cases where sums-of-squares are not available, it is still possible to use a sequence of approximate representing measures to establish positivity of a functional $L_y : \mathcal{P}_{2d} \longrightarrow \mathbb{R}$. In Example 2.5, for $n = 2$, $d = 3$, $k = 6$, we will use this approach to illustrate a multisequence y of degree 6 such that L_y is positive (whence $M_3(y) \succeq 0$), but y has no representing measure. We believe this is the first such example in a case where the positivity of L_y cannot be established by sums-of-squares, via positivity of $M_d(y)$.

We recall some additional terminology and results from [4, 8] concerning moment matrices and representing measures. Let $[x]_k$ denote the column vector of all n-variable monomials up to degree k in degree-lexicographic order, that is,

$$[x]_k^T = \begin{bmatrix} 1 & x_1 & \dots & x_n & x_1^2 & x_1x_2 & \dots & x_n^k \end{bmatrix}.$$

Throughout this paper, the superscript T denotes the transpose of a matrix or vector. Note that if $k = 2d$ and μ is a representing measure for y, then

$$M_d(y) = \int_{\mathbb{R}^2} [x]_d [x]_d^T d\mu(x),$$

which shows again that $M_d(y) \succeq 0$ is a necessary condition for representing measures. Moreover, in this case, *card supp* $\mu \geq$ *rank* $M_d(y)$ [4] (where *supp* μ denotes the closed support of μ). We denote the successive columns of $M_d(y)$ by

$$1,\ X_1, \dots,\ X_n,\ X_1^2,\ X_1X_2, \dots,\ X_n^2,\ \dots,\ X_n^d, \dots,\ X_n^d.$$

For $p = \sum_{\alpha \in \mathbb{Z}_+^n : |\alpha| \le d} p_\alpha x^\alpha \in \mathcal{P}_d$, we define an element $p(X)$ of the column space of $M_d(y)$ by

$$p(X) = \sum_{\alpha \in \mathbb{Z}_+^n : |\alpha| \le d} p_\alpha X^\alpha.$$

$M_d(y)$ is *recursively generated* if, whenever $p \in \mathcal{P}_d$ and $p(X) = 0$, then $(pq)(X) = 0$ for $q \in \mathcal{P}_d$ with $deg\ pq \le d$; recursiveness is a necessary condition for representing measures [4]. The *algebraic variety* associated to $M_d(y)$ is defined by

$$\mathcal{V}(M_d(y)) := \bigcap_{p \in \mathcal{P}_d,\ p(X)=0} \{x \in \mathbb{R}^n : p(x) = 0\}; \tag{1.5}$$

if y has a representing measure μ, then $supp\ \mu \subseteq \mathcal{V}(M_d(y))$ [4], whence

$$rank\ M_d(y) \le card\ \mathcal{V}(M_d(y)). \tag{1.6}$$

Recall that a measure ν is *p-atomic* if it is of the form $\nu = \sum_{i=1}^p \lambda_i \delta_{u_i}$, where $\lambda_i > 0$ and δ_{u_i} is the unit-mass measure supported at $u_i \in \mathbb{R}^n$. For $k = 2d$, a fundamental result of [4, 8] shows that y admits a $rank\ M_d(y)$-atomic representing measure if and only $M_d(y)$ is positive semi-definite and $M_d(y)$ admits a *flat* (i.e., rank-preserving) moment matrix extension $M_{d+1}(\tilde{y})$; in this case $\tilde{y}$ has a unique (and computable) representing measure, which is $rank\ M_d(y)$-atomic, with support precisely $\mathcal{V}(M_{d+1}(\tilde{y}))$. More generally, y admits a finitely atomic representing measure if and only if $M_d(y)$ admits a positive extension $M_{d+m}(\tilde{y})$ (for some $m \ge 0$), which in turn admits a flat extension M_{d+m+1} [8]. A remarkable result of Bayer and Teichmann [1] implies that a multisequence y of degree k admits a K-representing measure if and only if y admits a finitely atomic K-representing measure μ (with $card\ supp\ \mu \le dim\ \mathcal{P}_k$), so the preceding moment matrix criterion provides a complete characterization of the existence of representing measures when $k = 2d$. This characterization is more concrete than the criterion of Theorem 1.1, because it provides algebraic coordinates for constructing representing measures, although precise conditions for flat extensions are presently known only in special cases. For the case when K is a closed semialgebraic set, analogues of the preceding results appear in [8]. The papers [7, 9, 12] describe various concrete existence theorems for representing measures based on flat extensions. These results usually assume that $M_d(y)$ is positive semidefinite and *singular*, so that any representing measure is necessarily supported in the nontrivial algebraic variety $\mathcal{V}(M_d(y))$. By contrast, for the case when $M_d(y)$ is positive definite, very few results are known concerning the existence of representing measures. Our solutions to the positive definite cases of the bivariate quartic moment problem and the multivariate quadratic moment problem provide two such results. A notable feature of the proofs of these results is that they do not rely on flat extension techniques. For this reason, the results which depend on Theorem 1.3 (or Lemma 2.1) are purely existential and do not provide a procedure for explicitly computing representing measures (cf. Question 3.5 below).

This paper is organized as follows. Section 2 contains an analysis of positivity of Riesz functionals, leading to a proof of Theorem 1.3. Section 3 shows that every bivariate quartic moment sequence y with $M_2(y) \succ 0$ admits a representing measure supported in $\mathbb{R}^2$. Section 4 gives a complete solution of quadratic K-moment problems when $K = \mathbb{R}^n$, or when $K \equiv S(q)$ or $K \equiv E(q)$ is defined by a quadratic multivariate polynomial $q(x)$.

2 Positivity, approximation, and representing measures.

In this section we will prove Theorem 1.3. Let

$$\mathcal{M}_{n,k} = \left\{ y \equiv (y_\alpha)_{\alpha \in \mathbb{Z}_+^n : |\alpha| \leq k} \right\},$$

the set of n-variable multisequences of degree k, and let

$$\mathcal{R}_{n,k}(K) = \left\{ y \in \mathcal{M}_{n,k} : y_\alpha = \int_K x^\alpha d\mu(x),\ \mu \geq 0,\ \ supp(\mu) \subseteq K \right\},$$

the multisequences with K-representing measures. When $K = \mathbb{R}^n$, we simply write $\mathcal{R}_{n,k}(\mathbb{R}^n) = \mathcal{R}_{n,k}$. Note that $\mathcal{R}_{n,k}(K)$ is a convex cone in $\mathcal{M}_{n,k}(K)$ and that $\mathcal{M}_{n,k}$ can be identified with the affine space $\mathbb{R}^\eta$, where $\eta \equiv dim\ \mathcal{P}_k = \binom{n+k}{k}$. $\mathbb{R}^\eta$ is equipped with the usual Euclidean norm $\|\cdot\|$, although we sometimes employ $\|\cdot\|_1$ as well. Note also that for $x \in K$, the truncated moment sequence $y \equiv [x]_k$ is an element of $\mathcal{R}_{n,k}(K)$, since δ_x is a K-representing measure. The truncated moment sequence y is said to be in the *interior* of $\mathcal{R}_{n,k}$ if there exists $\epsilon > 0$ such that for any truncated moment sequence y^* having the same degree as y, $y^* \in \mathcal{R}_{n,k}$ whenever $\|y^* - y\| < \epsilon$. Equivalently, the interior of $\mathcal{R}_{n,k}$ is defined in the standard way for a subset of the space $\mathbb{R}^\eta$.

Let us begin with a well-known fact about the interior and closure of convex sets.

Lemma 2.1. *If $\mathcal{C} \subset \mathbb{R}^N$ is a convex set, then $int(\mathcal{C}) = int(\overline{\mathcal{C}})$.*

The above lemma is a consequence of Theorem 25.20 (iii) of Berberian [2], which actually applies to convex sets in general topological vector spaces.

In the sequel, let $\mathcal{F}_{n,k}(K)$ denote the moment sequences $y \in \mathcal{M}_{n,k}$ having finitely atomic K-representing measures. $\mathcal{F}_{n,k}(K)$ is clearly a convex subset of $\mathcal{R}_{n,k}(K)$, and the Bayer-Teichmann theorem [1, Theorem 2] [14, Theorem 5.8] shows that $\mathcal{F}_{n,k}(K) = \mathcal{R}_{n,k}(K)$. The following result, which is implicit in the proof of [10, Theorem 2.4], is the basis for our approximation approach to K-positivity for Riesz functionals.

Theorem 2.2. *For $y \in \mathcal{M}_{n,k}$, the following are equivalent:*
i) L_y is K-positive;
ii) $y \in \overline{\mathcal{F}_{n,k}(K)}$.
iii) $y \in \overline{\mathcal{R}_{n,k}(K)}$.

Proof. We begin with $iii) \Longrightarrow i)$. If $y \in \mathcal{R}_{n,k}(K)$, with K-representing measure μ, then L_y is K-positive; indeed, if $p \in \mathcal{P}_k$ and $p|K \geq 0$, then $L_y(p) = \int_K p d\mu \geq 0$. Since the K-positive linear functionals form a closed positive cone in the dual space $\mathcal{P}_k^*$ (equipped with the usual norm topology), it follows that if $y \in \overline{\mathcal{R}_{n,k}(K)}$, then L_y is K-positive.

Since $ii) \Longrightarrow iii)$ is clear, it suffices to show $i) \Longrightarrow ii)$, which we prove by contradiction. Suppose L_y is K-positive, but $y \notin \overline{\mathcal{F}_{n,k}(K)}$. Since $\overline{\mathcal{F}_{n,k}(K)}$ is a closed convex cone in $\mathbb{R}^\eta$, it follows from the Minkowski separation theorem [2, (34.2)] that there exists a nonzero vector $p \in \mathbb{R}^\eta$ such that

$$p^T y < 0, \quad \text{and} \quad p^T w \geq 0, \forall\, w \in \overline{\mathcal{F}_{n,k}(K)}.$$

Now define the nonzero polynomial $\tilde{p}$ in $\mathcal{P}_k$ by

$$\tilde{p}(x) = p^T [x]_k.$$

Since, for each $x \in K$, the monomial vector $[x]_k$ is an element of $\mathcal{F}_{n,k}$ (with K-representing measure δ_x), then $\tilde{p}(x)$ is nonnegative on K. However, we have

$$L_y(\tilde{p}) = p^T y < 0,$$

which contradicts the K-positivity of L_y. Therefore, we must have $y \in \overline{\mathcal{F}_{n,k}(K)}$. $\square$

Lemma 2.3. *Let K be a determining set of degree k and let $y \in \mathcal{M}_{n,k}$. If the Riesz functional L_y is strictly K-positive, then there exists $\epsilon > 0$ such that $L_{\tilde{y}}$ is also strictly K-positive whenever $\|\tilde{y} - y\|_1 < \epsilon$.*

Proof. We equip $\mathcal{P}_k$ with the norm

$$\|p\| = \max_\alpha |p_\alpha| \quad (p \equiv \sum_{\alpha \in \mathbb{Z}_+^n : |\alpha| \le k} p_\alpha x^\alpha \in \mathcal{P}_k).$$

A sequence $\{p^{(i)}\}$ in $\mathcal{P}_k$ that is norm-convergent to $p \in \mathcal{P}_k$ is also pointwise convergent, so if $p^{(i)}|K \ge 0$ for each i, then $p|K \ge 0$. It follows that the set

$$\mathcal{T} := \{p \in \mathcal{P}_k : p|_K \ge 0, \|p\| = 1\}$$

is compact. Note that since K is a determining set, if $p \in \mathcal{T}$, then $p|K \not\equiv 0$. Thus, $L_{\tilde{y}}$ is strictly K-positive if and only if $L_{\tilde{y}}(p) > 0$ for every $p \in \mathcal{T}$. Since $\mathcal{T}$ is compact and $L_y : \mathcal{P}_k \longrightarrow \mathbb{R}$ is a norm-continuous functional on $\mathcal{T}$, there exists $\epsilon > 0$ such that

$$L_y(p) \ge 2\epsilon, \quad \forall\, p \in \mathcal{T}.$$

For any $p \in \mathcal{T}$, we have

$$|L_y(p) - L_{\tilde{y}}(p)| \le \|y - \tilde{y}\|_1.$$

So, if $\|y - \tilde{y}\|_1 < \epsilon$, then

$$L_{\tilde{y}}(p) \ge L_y(p) - \|y - \tilde{y}\|_1 \ge \epsilon > 0, \quad \forall\, p \in \mathcal{T},$$

whence $L_{\tilde{y}}$ is strictly positive. Thus, the lemma is proved. $\square$

We now prove Theorem 1.3, which we can restate as follows for convenience.

Theorem 2.4. *Let K be a determining set of degree k. If $y \in \mathcal{M}_{n,k}$ and L_y is strictly K-positive, then $y \in \mathcal{R}_{n,k}(K)$.*

Proof. By Theorem 2.2, we have $y \in \overline{\mathcal{R}_{n,k}(K)}$. Lemma 2.3 implies that y lies in the interior of $\overline{\mathcal{R}_{n,k}(K)}$. Lemma 2.1 tells us that $\overline{\mathcal{R}_{n,k}(K)}$ and $\mathcal{R}_{n,k}(K)$ have the same interior. Therefore we must have $y \in int(\mathcal{R}_{n,k}(K)) \subset \mathcal{R}_{n,k}(K)$. $\square$

Although we believe that the hypothesis that K is a determining set cannot be omitted from Theorem 2.4, at present we do not have an example illustrating this. We next present an example which shows how a sequence of approximate representing measures can be used to establish positivity of a functional $L_y : \mathcal{P}_{2d} \longrightarrow \mathbb{R}$ in a case where y has no representing measure and the positivity of L_y cannot be derived from the positivity of $M_d(y)$ via sums-of-squares arguments. Let $n = 2$ and consider the bivariate moment matrix $M_d(y)$. Denote the rows and columns by

$$1,\ X_1,\ X_2,\ X_1^2,\ X_1X_2,\ X_2^2,\ldots,\ X_1^d,\ X_1^{d-1}X_2,\ \ldots, X_1X_2^{d-1},\ X_2^d;$$

then y_{ij} is precisely the entry in row X_1^i, column X_2^j, the moment corresponding to the monomial $x_1^i x_2^j$.

Example 2.5. Let $n = 2$ and $d = 3$. We consider the general form of a moment matrix $M_3(y)$ with a column relation $X_2 = X_1^3$ (normalized with $y_{00} = 1$):

$$M \equiv M_3(y) = \begin{bmatrix} 1 & a & b & c & e & d & b & f & g & x \\ a & c & e & b & f & g & e & d & h & j \\ b & e & d & f & g & x & d & h & j & k \\ c & b & f & e & d & h & f & g & x & u \\ e & f & g & d & h & j & g & x & u & v \\ d & g & x & h & j & k & x & u & v & w \\ b & e & d & f & g & x & d & h & j & k \\ f & d & h & g & x & u & h & j & k & r \\ g & h & j & x & u & v & j & k & r & s \\ x & j & k & u & v & w & k & r & s & t \end{bmatrix}. \tag{2.1}$$

For suitable values of the moment data, M satisfies the following properties:

$$M \succeq 0, \quad X_2 = X_1^3, \quad rank\ M = 9; \tag{2.2}$$

this is the case, for example, with

$$\begin{gathered} a = b = f = g = u = v = w = 0,\ c = 1,\ e = 2,\ d = 5,\ h = 14, \\ j = 42,\ k = 132,\ r = 429,\ s = 1442,\ t = 4798,\ x = 0. \end{gathered} \tag{2.3}$$

In [12] we solved the truncated K-moment problem for $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1^3\}$. In particular, [12] provides a numerical test, that we next describe, for the existence of K-representing measures whenever M as in (2.1) satisfies (2.2). From [1] we know that if M admits a representing measure, then M admits a finitely atomic measure, and thus M admits a positive, recursively generated extension $M_4(\tilde{y})$. In any such extension, the moments must be consistent with the relation $x_2 = x_1^3$, so in particular, we must have $y_{44} = y_{15}(\equiv s)$. To insure positivity of $M_4(\tilde{y})$, we require a lower bound for the diagonal element y_{44}, which we may derive as in [12]. Let J denote the compression of M obtained by deleting row X_1^3 and column X_1^3; thus, $J \succ 0$. Let us write

$$J = \begin{bmatrix} N & U \\ U^T & \Delta \end{bmatrix},$$

where N is the compression of J to its first 8 rows and columns, U is a column vector, and $\Delta \equiv y_{06}(\equiv t) > 0$. Consider the corresponding block decomposition of J^{-1}, which is of the form

$$J^{-1} = \begin{bmatrix} P & V \\ V^T & \epsilon \end{bmatrix},$$

where $P \succ 0$ and $\epsilon > 0$. In extension $M_4(\tilde{y})$, we have $X_1^4 = X_1X_2$ and $X_1^3Y_2 = Y_2^2$, so by moment matrix structure, after deleting the element in row X_1^3, the first 8 remaining elements of column $X_1^2X_2^2$ must be $W \equiv (h,\ x,\ u,\ j,\ k,\ r,\ v,\ w)^T$. Let $\omega = \langle PW, W\rangle$ and define

$$\psi(y) := \frac{\omega\epsilon - \langle V, W\rangle^2}{\epsilon}. \tag{2.4}$$

In [12] we showed that in $M_4(\tilde{y})$ we must have $y_{44} \geq \psi(y)$, and [12, Theorem 2.4] implies that M has a representing measure if and only $y_{15} \equiv s > \psi(y)$.

A calculation shows that for M as in (2.1) and satisfiying (2.2), with appropriate values of the moment data we can also have $\psi(y)$ *independent* of s and t. This is the case, for example, if we modify (2.3) so that $x = \frac{1}{10}$, $r = 600$, s is arbitrary and t is chosen sufficiently large so as to preserve positivity and the property *rank* $M_3(y) = 9$. More generally, this is the case if we modify (2.3) so that x, k, u, v, w, r, s, t are chosen, successively, to maintain positivity and the *rank* $M = 9$ property. (We conjecture that whenever $M_3(y)$ satisfies (2.2), then $\psi(y)$ is independent of s and t.) For any such M, with $\psi(y)$ independent of s and t, we now specify $s \equiv y_{1,5} = \psi(y)$ and we adjust t (if necessary) so that M continues to be positive with *rank* $M = 9$. (For a specific example, we may modify (2.3) so that $x = \frac{1}{10}$, $r = 600$, $s \equiv \psi(y) = \frac{526337068574699}{741609900} \approx 709722$, and $t \geq 11319100143$ (cf. [12, Example 3.2].)

We claim that L_y is K-positive for $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = x_1^3\}$, and thus positive. Since $y_{1,5} = \psi(y)$, positivity for L_y cannot be derived from the existence of a representing measure, since [12, Theorem 2.4] shows that y has no representing measure. Moreover, positivity for L_y cannot be established from the positivity of M via sums-of-squares arguments because, by Hilbert's theorem, there exist degree 6 bivariate polynomials that are everywhere nonnegative but are not sums of squares. To prove that L_y is K-positive, we employ a sequence of approximate representing measures. Since $J \succ 0$, $t \equiv \Delta > U^TN^{-1}U$. Thus, there exists $\delta > 0$ such that if we replace s $(= \psi(y))$ by $s + \frac{1}{m}$ (with $\frac{1}{m} < \delta$), then the resulting moment matrix, $M_3(y^{(m)})$, remains positive, with *rank* $M_3(y^{(m)}) = 9$ and $X_2 = X_1^3$. Since $\psi(y^{(m)})$ is independent of $y_{15}[y^{(m)}]$ and $y_{06}[y^{(m)}]$, we have $\psi(y^{(m)}) = \psi(y) = s < s + \frac{1}{m} = y_{15}[y^{(m)}]$. It now follows from [12, Theorem 2.4] that $y^{(m)}$ has a K-representing measure, whence $L_{y^{(m)}}$ is K-positive. Since $\|y^{(m)} - y\| = \frac{1}{m} \longrightarrow 0$, we conclude that L_y is K-positive, and thus positive. □

Remark 2.6. We have previously noted an example of [5, Section 4] (based on a construction of Schmüdgen [18]) which illustrates a case where, with $n = 2$, $M_3(y) \succ 0$ but L_y is not positive. Example 2.5 shows that if $M_3(y) \succeq 0$ and L_y is positive, y need not have a representing measure. Whether this can happen with $M_3(y) \succ 0$ is the content of Question 1.2.

Now we introduce a variety associated to L_y that provides a finer tool than $\mathcal{V}(M_d(y))$ for studying issues related to Question 1.2. For a moment sequence y of degree $2d$, we define the variety of L_y by

$$V(L_y) := \bigcap_{p \in \mathcal{P}_{2d},\ p|\mathcal{V}(M_d(y)) \geq 0,\ L_y(p) = 0} \mathcal{Z}(p).$$

Proposition 2.7. *If y has a representing measure μ, then supp $\mu \subseteq V(L_y)$.*

Proof. Suppose there exists $u \in supp\ \mu$ such that $u \notin V(L_y)$. Then there exists some $p \in \mathcal{P}_{2d}$, such that $p|\mathcal{V}(M_d(y)) \geq 0$ and $L_y(p) = 0$, but $p(u) \neq 0$. Since $supp\ \mu \subseteq \mathcal{V}(M_d(y))$, we have $p|supp\ \mu \geq 0$, and hence $p(u) > 0$. Thus, it follows that $L_y(p) = \int_{supp\ \mu} p(t)d\mu(t) > 0$, which contradicts $L_y(p) = 0$. $\square$

Proposition 2.8. *For each truncated moment sequence y, $V(L_y) \subseteq \mathcal{V}(M_d(y))$.*

Proof. Let p be an arbitrary polynomial such that $p \in \mathcal{P}_d$ and $p(X) = 0$ in the column space of $M_d(y)$. Then $L_y(p^2) = \langle M_d(y)\hat{p},\ \hat{p}\rangle = 0$. Since $p^2|\mathcal{V}(M_d(y)) \geq 0$, it follows that $V(L_y) \subseteq \mathcal{Z}(p^2) = \mathcal{Z}(p)$. By definition of $\mathcal{V}(M_d(y))$ in (1.5), the result is proved. $\square$

In view of Proposition 2.8, the following result refines the necessary condition $rank\ M_d(y) \leq card\ \mathcal{V}(M_d(y))$.

Corollary 2.9. *If y has a representing measure, then $rank\ M_d(y) \leq card\ V(L_y)$.*

Proof. Let μ be a representing measure for y. Then $rank\ M_d(y) \leq card\ supp\ \mu$ (see relation (1.6) in Section 1), and the result follows from Proposition 2.7. $\square$

We conclude this section with an example which shows that $V(L_y)$ may be a proper subset of $\mathcal{V}(M_d(y))$ (in a case where y has a representing measure).

Example 2.10. For $n = 2$, $d = 3$, consider the moment matrix

$$M_3(y) := \begin{bmatrix} 8 & 0 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 4 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 4 & 0 & 6 \\ 6 & 0 & 0 & 6 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 4 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 6 & 0 & 4 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 4 \\ 0 & 4 & 0 & 0 & 0 & 0 & 4 & 0 & 4 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 4 & 0 & 6 \end{bmatrix}.$$

A calculation shows that $M_3(y) \succeq 0$, with $rank\ M_3(y) = 8$. $\mathcal{V}(M_3(y))$ is determined by the column relations $X_1 = X_1^3$ and $X_2 = X_2^3$, and thus consists of the 9 points $u_1 = (0,0)$, $u_2 = (0,1)$, $u_3 = (0,-1)$, $u_4 = (-1,0)$, $u_5 = (-1,1)$, $u_6 = (-1,-1)$, $u_7 = (1,0)$, $u_8 = (1,1)$, $u_9 = (1,-1)$. Observe that y has the 8-atomic representing measure $\mu := \sum_{i=2}^{9} \delta_{u_i}$, and we will show that $V(L_y) = supp\ \mu$, so that $V(L_y)$ is a proper subset of $\mathcal{V}(M_3(y))$. To see this, we consider the dehomogenized Robinson polynomial,

$$r(x_1,x_2) = x_1^6 + x_2^6 - x_1^4x_2^2 - x_1^2x_2^4 - x_1^4 - x_2^4 - x_1^2 - x_2^2 + 3x_1^2x_2^2 + 1.$$

It is known that $r(x_1,x_2)$ is nonnegative on $\mathbb{R}^2$ and has exactly 8 zeros in the affine plane, namely the points in $supp\ \mu$ (cf. [17]). A calculation shows that $L_y(r) = 0$, so $V(L_y) \subseteq \mathcal{Z}(r) = supp\ \mu \subseteq V(L_y)$ (by Proposition 2.7), so $V(L_y) = supp\ \mu$ and thus $V(L_y)$ is a proper subset of $\mathcal{V}(M_3(y))$.

It is known that $r(x_1,x_2)$ is not a sum of squares (cf. [17]); to see this using variety methods, suppose to the contrary that $r = \sum_i r_i^2$, with each $r_i \in \mathcal{P}_3$. Then $supp\ \mu =$

$\mathcal{Z}(r) = \bigcap_i \mathcal{Z}(r_i)$, whence *supp* $\mu \subseteq \mathcal{Z}(r_i)$ for each i. It now follows from [4] that for each i, $r_i(X_1, X_2) = 0$ in the column space of $M_3(y)$. Thus, we have $\mathcal{V}(M_3(y)) \subseteq \bigcap_i \mathcal{Z}(r_i) = supp\ \mu$, a contradiction. This example also illustrates a moment sequence y with a *rank* $M_d(y)$-atomic representing measure and *rank* $M_d(y) <$ *card* $\mathcal{V}(M_3(y)) < +\infty$; the first such example appears in [11]. □

3 Solution of the bivariate quartic moment problem

Throughout this section, we consider bivariate quartic moment problems, that is, $n = 2$ and the degree $2d = 4$. Let $y \in \mathcal{M}_{2,4}$ be a truncated moment sequence of degree 4, which is associated with the second order moment matrix

$$M_2(y) := \begin{bmatrix} y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix}.$$

As noted in Introduction (cf. (1.3)), if $M_2(y)$ is singular, then y has a representing measure if and only if $M_2(y)$ is positive semidefinite, recursively generated, and *rank* $M_2(y) \leq$ *card* $\mathcal{V}(M_2(y))$.

Example 3.1. Consider

$$M_2(y) = \begin{bmatrix} 8 & 0 & 0 & 4 & 0 & 4 \\ 0 & 4 & 0 & 2 & 0 & -2 \\ 0 & 0 & 4 & 0 & -2 & 0 \\ 4 & 2 & 0 & 11 & 0 & a \\ 0 & 0 & -2 & 0 & a & 0 \\ 4 & -2 & 0 & a & 0 & b \end{bmatrix}.$$

With $a = 1$ and $b = 3$, $M_2(y)$ is positive and recursively generated, with column relations $X_1 = 1 - 2X_2^2$ and $X_2 = -2X_1X_2$, and *rank* $M_2(y) = 4$. A calculation shows that $x_1 = 1 - 2x_2^2$ and $x_2 = -2x_1x_2$ have only 3 common zeros, so $3 =$ *card* $\mathcal{V}(M_2(y)) <$ *rank* $M_2(y) = 4$, whence (1.3) implies that y has no representing measure. We will show below how to approximate y with truncated moment sequences having representing measures. □

For the case when $M_2(y) \succ 0$, it has been an open question as to whether y admits a representing measure. The aim of this section is to give an affirmative answer to this question. We begin, however, by showing that when $M_2(y)$ is merely positive semidefinite, then y admits approximate representing measures.

Theorem 3.2. *If $y \in \mathcal{M}_{2,4}$ and $M_2(y) \succeq 0$, then $y \in \overline{\mathcal{R}_{2,4}}$.*

Proof. Let $y \in \mathcal{M}_{2,4}$ be such that $M_2(y) \succeq 0$. To show $y \in \overline{\mathcal{R}_{2,4}}$, by Theorem 2.2, it suffices to show that the Riesz functional L_y is positive. If a polynomial $p(x) \in \mathcal{P}_4$ is nonnegative on the plane $\mathbb{R}^2$, then by Hilbert's theorem it must be a sum of squares, so there exist bivariate quadratic polynomials $q_1(x), \ldots, q_m(x)$, $deg\ q_i \le 2$ $(1 \le i \le m)$, such that

$$p(x) = q_1(x)^2 + \cdots + q_m(x)^2.$$

Hence, since $M_2(y) \succeq 0$, we have

$$L_y(p) = L_y(q_1^2) + \cdots + L_y(q_m^2) = \langle M_2(y)\hat{q_1}, \hat{q_1}\rangle + \cdots + \langle M_2(y)\hat{q_m}, \hat{q_m}\rangle \ge 0,$$

so L_y is positive. It now follows from Theorem 2.2 that $y \in \overline{\mathcal{R}_{2,4}}$. □

Note that if $M_2(y)$ is positive and singular, and y does not have a representing measure, then L_y is positive, but not strictly positive. Indeed, positivity follows from Theorem 3.2. Since $M_2(y)$ is singular, there exists $p \in \mathcal{P}_2$, $p \not\equiv 0$, such that $M_2(y)\hat{p} = 0$; then $p^2 \ge 0$ and $L_y(p^2) = \langle M_2(y)\hat{p}, \hat{p}\rangle = 0$, so L_y is not strictly positive.

We now turn to the positive definite case. The following result provides an affirmative answer to Question 1.2 for the case $n = d = 2$, $K = \mathbb{R}^2$.

Theorem 3.3. *If $M_2(y) \succ 0$, then y has a representing measure.*

Proof. Clearly $\mathbb{R}^2$ is a determining set. By Theorem 2.4, it suffices to show that L_y is strictly positive. Proceeding as in the previous proof, if $p \in \mathcal{P}_4$ is nonnegative on $\mathbb{R}^2$ and not identically zero, then p is of the form $p(x) = q_1(x)^2 + \cdots + q_m(x)^2$, with $deg\ q_i \le 2$ $(1 \le i \le m)$ and every $q_i \not\equiv 0$. Since $M_2(y) \succ 0$, we have $L_y(p) = L_y(q_1^2) + \cdots + L_y(q_m^2) = \langle M_2(y)\hat{q_1}, \hat{q_1}\rangle + \cdots + \langle M_2(y)\hat{q_m}, \hat{q_m}\rangle > 0$, and the result follows. □

Remark 3.4. Theorem 3.3 shows that if $n = 2$ and $M_2(y) \succ 0$, then y has a representing measure, whence [1] implies that y has a representing measure μ with $card\ supp\ \mu \le dim\ \mathcal{P}_4 = 15$. We do not have a better upper bound for the size of the support, and it remains an open problem as to whether, in this case, $M_2(y)$ actually has a flat extension $M_3(\tilde{y})$, with a corresponding 6-atomic representing measure for y. In the case when $n = 2$ and $M_2(y)$ is positive semidefinite and singular, y has a representing measure if and only if the conditions of (1.3) hold, and in this case, the results of [9] show that either $M_2(d)$ has a flat extension $M_3(\tilde{y})$, or $M_2(y)$ admits a positive extension $M_3(\tilde{y})$ satisfying $rank\ M_3(\tilde{y}) = 1 + rank\ M_2(y)$, and $M_3(\tilde{y})$ has a flat extension. This leads to our next question (cf. [7, 13]).

Question 3.5. If $y \in \mathcal{M}_{2,4}$ and $M_2(y) \succ 0$, does $M_2(y)$ have a flat extension? Does y have an extension $\tilde{y} \in \mathcal{M}_{2,6}$ such that $M_3(\tilde{y})$ is positive and has a flat extension?

We next present two examples which illustrate Theorem 3.2 in cases where y has no representing measure.

Example 3.6. Consider the moment sequence $y \in \mathcal{M}_{2,4}$ such that

$$M_2(y) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 \end{bmatrix}.$$

Clearly, $M_2(y) \succeq 0$. Since $X_1 = 1$ but $X_1^2 \neq X_1$, $M_2(y)$ is not recursively generated, so y has no representing measure. However, by Theorem 3.2, y lies in the closure of moment sequences having representing measures. To see this explicitly, define the moment sequence $y(\epsilon)$ via the moment matrix $M_2(y(\epsilon)) :=$

$$\begin{bmatrix} 1 & 1+\epsilon^{3/4}-\epsilon & 1+\epsilon^{3/4}-\epsilon & 1+\epsilon^{1/2}-\epsilon & 1+\epsilon^{1/2}-\epsilon & 1+\epsilon^{1/2}-\epsilon \\ 1+\epsilon^{3/4}-\epsilon & 1+\epsilon^{1/2}-\epsilon & 1+\epsilon^{1/2}-\epsilon & 1+\epsilon^{1/4}-\epsilon & 1+\epsilon^{1/4}-\epsilon & 1+\epsilon^{1/4}-\epsilon \\ 1+\epsilon^{3/4}-\epsilon & 1+\epsilon^{1/2}-\epsilon & 1+\epsilon^{1/2}-\epsilon & 1+\epsilon^{1/4}-\epsilon & 1+\epsilon^{1/4}-\epsilon & 1+\epsilon^{1/4}-\epsilon \\ 1+\epsilon^{1/2}-\epsilon & 1+\epsilon^{1/4}-\epsilon & 1+\epsilon^{1/4}-\epsilon & 2-\epsilon & 2-\epsilon & 2-\epsilon \\ 1+\epsilon^{1/2}-\epsilon & 1+\epsilon^{1/4}-\epsilon & 1+\epsilon^{1/4}-\epsilon & 2-\epsilon & 2-\epsilon & 2-\epsilon \\ 1+\epsilon^{1/2}-\epsilon & 1+\epsilon^{1/4}-\epsilon & 1+\epsilon^{1/4}-\epsilon & 2-\epsilon & 2-\epsilon & 2-\epsilon \end{bmatrix}.$$

A calculation shows that $y(\epsilon)$ has the 2-atomic representing measure

$$(1-\epsilon)\delta_{(1,1)} + \epsilon\delta_{(\epsilon^{-1/4},\epsilon^{-1/4})},$$

and obviously $y(\epsilon) \to y$ as $\epsilon \to 0$. $\square$

Example 3.7. Let us return to Example 3.1. With $a = 1$ and $b = 3$, $M_2(y)$ is positive semidefinite, so although y has no representing measure, Theorem 3.2 implies that y can be approximated by moment sequences having measures. One way to do this is to replace $b = 3$ by $b = 3 + \frac{1}{m}$. The resulting moment sequence $y^{(m)}$ satisfies $M_2(y^{(m)}) \succeq 0$ and $M_2(y^{(m)})$ is recusrsively generated. Further, $\mathcal{V}(M_2(y^{(m)})) = \{(x_1, x_2) : x_2 = -2x_1x_2\}$, and since the variety is infinite, (1.3) implies that $y^{(m)}$ has a representing measure. Following [9, Proposition 3.6], a calculation shows that although $M_2(y^{(m)})$ admits no flat extension $M_3(\widetilde{y^{(m)}})$ (so $y^{(m)}$ has no 5-atomic representing measure), $M_2(y^{(m)})$ does admit a positive extension $M_3(\widetilde{y^{(m)}})$, with *rank* $M_3(\widetilde{y^{(m)}}) = 6$, such that $M_3(\widetilde{y^{(m)}})$ has a flat extension $M_4(\widetilde{\widetilde{y^{(m)}}})$. Thus, $y^{(m)}$ has a 6-atomic representing measure.

Another approach is to replace $a = 1$ by $a = 1 + \frac{1}{m}$ and $b = 3$ by $b = 3 + \frac{1}{4m^2}$. Then the resulting moment sequence $y^{(m)}$ has $M_2(y^{(m)}) \succ 0$, so $y^{(m)}$ has a representing measure by Theorem 3.3. Indeed, a *Mathematica* calculation shows that with $\widetilde{y^{(m)}}_{4,1} = \widetilde{y^{(m)}}_{2,3} = \widetilde{y^{(m)}}_{1,4} = \widetilde{y^{(m)}}_{0,5} = 0$, $M_2(y^{(m)})$ admits two distinct flat extensions $M_3(\widetilde{y^{(m)}})$ (and corresponding 6-atomic representing measures for $y^{(m)}$). $\square$

We conclude this section with an application of Theorem 3.3 to a solution to the bivariate cubic moment problem, with y of the form

$$y = \{y_{00},\ y_{10},\ y_{01},\ y_{20},\ y_{11},\ y_{02},\ y_{30},\ y_{21},\ y_{12},\ y_{03}\},$$

with $y_{00} > 0$. To such a sequence we may associate $M_1(y)$ and the block

$$B(2) := \begin{bmatrix} y_{20} & y_{11} & y_{02} \\ y_{30} & y_{21} & y_{12} \\ y_{21} & y_{12} & y_{03} \end{bmatrix}.$$

Theorem 3.8. *Suppose $y \in \mathcal{M}_{2,3}$. If y has a representing measure, then $M_1(y) \succeq 0$. Conversely, suppose $M_1(y) \succeq 0$.*
i) If $M_1(y) \succ 0$, then y has a representing measure.
ii) If rank $M_1(y) = 2$, then y has a representing measure if and only if Ran $B(2) \subseteq$ Ran $M_1(y)$ and $\begin{bmatrix} M_1(y) & B(2) \end{bmatrix}$ is recursively generated.
iii) If rank $M_1(y) = 1$, then y has a representing measure if and only if Ran $B(2) \subseteq$ Ran $M_1(y)$.

Proof. Since a representing measure for y is, in particular, a representing measure for $M_1(y)$, the necessity of the condition $M_1(y) \succeq 0$ is clear. Conversely, suppose $M_1(y) \succeq 0$. For i), if $M_1(y) \succ 0$, then it is not difficult to see that $M_1(y)$ admits a positive definite moment matrix extension M_2, of the form

$$M_2 \equiv \begin{bmatrix} M_1(y) & B(2) \\ B(2)^T & C(2) \end{bmatrix},$$

where

$$C(2) = \begin{bmatrix} y_{40} & y_{31} & y_{22} \\ y_{31} & y_{22} & y_{13} \\ y_{22} & y_{13} & y_{04} \end{bmatrix}.$$

Indeed, by choosing y_{40}, y_{22}, and y_{04} successively, and sufficiently large, we can insure that $C(2) \succ B(2)^T M_1(y)^{-1} B(2)$. By Theorem 3.3, M_2 then has a representing measure, which is obviously a representing measure for y.

Suppose next that y has a representing measure. It follows from [1] that y has a finitely atomic representing measure μ, and thus $M_2[\mu]$ is a positive semidefinite and recursively generated extension of $M_1(y)$. In particular, we must have *Ran* $B(2) \subseteq$ *Ran* $M_1(y)$ and $\begin{bmatrix} M_1(y) & B(2) \end{bmatrix}$ must be recursively generated. Now suppose that these conditions hold and that *rank* $M_1(y) = 2$. Since $y_{00} > 0$, we may assume without loss of generality that there exist scalars α and β so that in the column space of $M_1(y)$ we have a column dependence relation

$$X_2 = \alpha 1 + \beta X_1. \tag{3.1}$$

Since $\begin{bmatrix} M_1(y) & B(2) \end{bmatrix}$ is recursively generated, we then have the column relations

$$X_1 X_2 = \alpha X_1 + \beta X_1^2, \tag{3.2}$$

$$X_2^2 = \alpha X_2 + \beta X_1 X_2. \tag{3.3}$$

Since *Ran* $B(2) \subseteq$ *Ran* $M_1(y)$, there is a matrix W such that $B(2) = M_1(y)W$, and we may thus define a positive, rank-preserving extension M of $M_1(y)$ by

$$M := \begin{bmatrix} M_1(y) & B(2) \\ B(2)^T & C \end{bmatrix},$$

where $C := B(2)^T W$ $(= W^T M_1(y) W)$. It is straightforward to check that the columns of M satisfy (3.1)-(3.3), from which it also follows that M has the form of a moment matrix M_2. Thus M is a flat, positive moment matrix extension of $M_1(y)$, whence [8] implies the existence of a representing measure for M, and thus for y.

The proof of iii) is similar to the proof of ii), but simpler. It is straightforward to check that if *rank* $M_1(y) = 1$ and *Ran* $B(2) \subseteq$ *Ran* $M_1(y)$, then the dependence relations in the columns of $M_1(y)$ propagate recursively so as to define a rank one (flat, positive) moment matrix extension $M_2(y)$ of $M_1(y)$. The result follows as above. □

4 Quadratic moment problems

Let $y = (y_\alpha)_{\alpha \in \mathbb{Z}_+^n : |\alpha| \le 2}$ be a quadratic moment sequence such that $M_1(y) \succeq 0$. Does y have a representing measure? For this question, we may assume without loss of generality that $y_0 = 1$ and we may write $M_1(y)$ as

$$M_1(y) = \begin{bmatrix} 1 & v_1^T \\ v_1 & U \end{bmatrix},$$

where $v_1 \in \mathbb{R}^n$. Since $M_1(y) \succeq 0$, then $U - v_1 v_1^T \succeq 0$, so the Spectral Theorem implies that there exist vectors $v_2, \ldots, v_r$ in $\mathbb{R}^n$ such that

$$U = v_1 v_1^T + v_2 v_2^T + \ldots + v_r v_r^T.$$

A calculation now shows that we have

$$M_1(y) = \frac{1}{r-1} \sum_{i=2}^{r} \left(\begin{bmatrix} 1 \\ v_1 \end{bmatrix} \begin{bmatrix} 1 \\ v_1 \end{bmatrix}^T + (r-1) \begin{bmatrix} 0 \\ v_i \end{bmatrix} \begin{bmatrix} 0 \\ v_i \end{bmatrix}^T \right).$$

For $i = 2, \ldots, r$, let $u_i^+ = v_1 + \sqrt{r-1} v_i$, $u_i^- = v_1 - \sqrt{r-1} v_i$. Then we have the representation

$$M_1(y) = \frac{1}{2(r-1)} \sum_{i=2}^{r} \left(\begin{bmatrix} 1 \\ u_i^+ \end{bmatrix} \begin{bmatrix} 1 \\ u_i^+ \end{bmatrix}^T + \begin{bmatrix} 1 \\ u_i^- \end{bmatrix} \begin{bmatrix} 1 \\ u_i^- \end{bmatrix}^T \right),$$

and hence we know y has a $(2r-2)$-atomic representing measure

$$\mu = \sum_{i=2}^{r} \frac{1}{2(r-1)} (\delta_{u_i^+} + \delta_{u_i^-}).$$

In the sequel, we will show that y actually has a *rank* $M_1(y)$-atomic representing measure (equivalently, $M_1(y)$ admits a flat extension $M_2(\tilde{y})$).

Now we turn to the quadratic truncated moment problem on an algebraic set $E(q) := \{x \in \mathbb{R}^n : q(x) = 0\}$ or a semialgebraic set $S(q) := \{x \in \mathbb{R}^n : q(x) \ge 0\}$, where $q(x)$ is a quadratic polynomial in x. If $y \in \mathcal{M}_{n,2}$ has a representing measure supported in $E(q)$, it is necessary that

$$M_1(y) \succeq 0, \quad L_y(q) = 0.$$

Is the above also sufficient for y to have a representing measure supported in $E(q)$? If $y \in \mathcal{M}_{n,2}$ has a representing measure supported in $S(q)$, it is necessary that

$$M_1(y) \succeq 0, \quad L_y(q) \ge 0.$$

Is the above also sufficient for y to have a representing measure supported in $S(q)$? These questions will be answered affirmatively under certain suitable conditions.

Throughout this section, we will employ a well-known connection between nonnegative polynomials and positive semidefinite real symmetric matrices (cf. [14]), which we apply in the case of quadratic polynomials. Let $p(x) = \sum_{\alpha \in \mathbb{Z}_+^n : |\alpha| \le 2} p_\alpha x^\alpha$. Let y_p denote the degree 2 moment sequence whose moment corresponding to a monomial of degree 1, or to a monomial of the form $x^\alpha = x_i x_j$ $(i \neq j)$, is $p_\alpha/2$, and whose moment corresponding to a monomial of degree 0, or of the form $x^\alpha = x_i^2$, is p_α. A calculation shows that

$$p(x) = [x]_1^T M_1(y_p)[x]_1. \tag{4.4}$$

From this it follows immediately that $p(x)$ is nonnegative on $\mathbb{R}^n$ if and only if there exists a matrix P such that $P = P^T$, $P \succeq 0$, and

$$p(x) = [x]_1^T P[x]_1 \ (x \in \mathbb{R}^n). \tag{4.5}$$

In the case when $p(x)$ is a homogeneous quadratic, by compressing $M_1(y_p)$ to the rows and columns indexed by the variables x_i, and similarly for $[x]_1$, we see that $p(x)$ admits a representation of the form

$$p(x) = x^T P x \ (x \in \mathbb{R}^n), \tag{4.6}$$

where $P = P^T$; further, $p(x)$ is nonnegative on $\mathbb{R}^n$ if and only if $P \succeq 0$.

In the sequel, for $m \times m$ real matrices $R \equiv (r_{ij})$ and $S \equiv (s_{ij})$, we denote by $R \bullet S$ the Frobenius inner product, defined by $R \bullet S = Trace(RS^T) = \sum_{1 \leq i,j \leq m} r_{ij} s_{ij}$. A calculation shows that if p has a representation as in (4.5) and y is a quadratic moment sequence, then

$$L_y(p) = P \bullet M_1(y). \tag{4.7}$$

If $R = R^T \succeq 0$ and $S = S^T \succeq 0$, then $R = LL^T$ and $S = MM^T$, and thus

$$\begin{aligned} R \bullet S &= Trace(LL^T MM^T) = Trace(M^T LL^T M) \\ &= Trace(M^T L(M^T L)^T) = (M^T L) \bullet (M^T L) \geq 0. \end{aligned}$$

It now follows that

$$if\ R = R^T \succeq 0 \ and\ S = S^T \succeq 0, \ then\ R \bullet S \geq 0. \tag{4.8}$$

4.1. Quadratic polynomials nonnegative on quadratic sets

A useful tool in quadratic moment theory, which we will employ repeatedly, is the following matrix decomposition developed by Sturm and Zhang [19]. In the sequel, let $\mathbb{M}_m$ denotes the space of real $m \times m$ matrices, endowed with the norm induced by the Frobenius inner product.

Proposition 4.1 (Corollary 4 [19]). *Let $Q \in \mathbb{M}_m$ be a symmetric matrix. If $X \in \mathbb{M}_m$ is symmetric positive semidefinite and has rank r, then there exist nonzero vectors $u_1, \ldots, u_r \in \mathbb{R}^m$ such that*

$$X = u_1 u_1^T + \cdots + u_r u_r^T, \quad u_1^T Q u_1 = \cdots = u_r^T Q u_r = \frac{Q \bullet X}{r}.$$

We will also utilize the following representation of quadratic polynomials that are nonnegative on $S(q)$.

Proposition 4.2 (S-Lemma, Yakubovich (1971), [21])**.** *Let $f(x), q(x)$ be two quadratic polynomials in x. Suppose there exists $\xi \in \mathbb{R}^n$ such that $q(\xi) > 0$. Then $f(x) \geq 0$ for all $x \in S(q)$ if and only if there exists $t \geq 0$ such that*

$$f(x) - tq(x) \geq 0, \quad \forall\, x \in \mathbb{R}^n.$$

When $f(x)$ and $g(x)$ are homogeneous and quadratic, if $f(x)$ is nonnegative on the algebraic set $E(q) = \{x \in \mathbb{R}^n : q(x) = 0\}$, then a certificate like that provided by S-Lemma holds, but without requiring $t \geq 0$, as pointed out by Luo, Sturm and Zhang [15]. However, we are not able to find a complete proof from [15] and the references therein. Moreover, this result can also be generalized to the case when $f(x)$ and $g(x)$ are non-homogeneous. So here we summarize these results and include a proof for completeness.

Proposition 4.3. *Let $f(x), q(x)$ be two quadratic polynomials in x, and assume $E(q) \neq \emptyset$. Suppose $f(x) \geq 0$ for all $x \in E(q)$, and suppose there exist $\xi, \zeta \in \mathbb{R}^n$ such that $q(\xi) > 0 > q(\zeta)$. Then there exists $t \in \mathbb{R}$ such that*

$$f(x) - tq(x) \geq 0, \quad \forall\, x \in \mathbb{R}^n.$$

Proof. **Step 1** Consider first the case when both f and g are homogeneous quadratics. From (4.6), we may write $f(x) = x^T F x$ and $q(x) = x^T Q x$ for symmetric matrices $F, Q \in \mathbb{M}_n$. In the sequel we view $\mathbb{M}_n$ as a locally convex normed real vector space, with norm induced by the Frobenius inner product. By finite dimensionality, each linear functional on $\mathbb{M}_n$ is of the form $F \longrightarrow F \bullet X$ for some $X \in \mathbb{M}_n$. Let $\mathcal{E} = \{S + tQ : S^T = S \succeq 0, t \in \mathbb{R}\}$. Obviously $\mathcal{E}$ is a convex set, and we claim that $\mathcal{E}$ is also closed. To see this, let $\{A_k \equiv S_k + t_k Q\} \subset \mathcal{E}$ be sequence such that $A_k \to A$. Note that every $S_k \succeq 0$ and thus

$$\xi^T A_k \xi = \xi^T S_k \xi + t_k \xi^T Q \xi \geq t_k \xi^T Q \xi, \quad \zeta^T A_k \zeta = \zeta^T S_k \zeta + t_k \zeta^T Q \zeta \geq t_k \zeta^T Q \zeta.$$

From this, and the hypothesis $\xi^T Q \xi = q(\xi) > 0 > q(\zeta) = \zeta^T Q \zeta$, it follows that

$$\frac{\zeta^T A_k \zeta}{q(\zeta)} \leq t_k \leq \frac{\xi^T A_k \xi}{q(\xi)}.$$

Since $\{A_k\}$ is bounded, $\{\zeta^T A_k \zeta\}$ and $\{\xi^T A_k \xi\}$ are also bounded, whence the sequence $\{t_k\}$ is bounded too. Thus $\{S_k\}$ is also bounded, so we may assume $S_k \to S_* \succeq 0$ and $t_k \to t_*$, whence $A_k \to A = S_* + t_* Q \in \mathcal{E}$.

Now we show that F belongs to the closed convex set $\mathcal{E}$. Suppose to the contrary that $F \notin \mathcal{E}$. It follows from a version of the Hahn-Banach Theorem [3, Proposition 14.15] that there exist a nonzero symmetric matrix X and a scalar η such that

$$F \bullet X < \eta, \quad (S + tQ) \bullet X \geq \eta, \forall\, S^T = S \succeq 0,\ t \in \mathbb{R}.$$

By choosing $S = 0$, we see that $Q \bullet X = 0$. Thus $S \bullet X \geq \eta \ \ \forall\, S^T = S \succeq 0$, whence $X \succeq 0$. The preceding implies that

$$Q \bullet X = 0, \quad X \succeq 0, \quad \eta \leq 0.$$

Then, by Proposition 4.1, there exist vectors $u_1, \ldots, u_r$ such that

$$X = u_1 u_1^T + \cdots + u_r u_r^T, \quad u_i^T Q u_i = 0 \quad (1 \leq i \leq r).$$

From $\sum_{i=1}^{r} u_i^T F u_i = F \bullet X < \eta \leq 0$, we see that at least one u_i satisfies

$$u_i^T F u_i < 0, \quad u_i^T Q u_i = 0.$$

Thus, $q(u_i) = 0$, but $f(u_i) < 0$, which is a contradiction. So F must belong to $\mathcal{E}$. With $F = S + tQ$, for some $S^T = S \succeq 0$ and $t \in \mathbb{R}$, we have $f(x) = s(x) + tq(x)$ for some nonnegative quadratic $s(x)$ corresponding to S via (4.6), so the result follows in this case.

Step 2 We next consider the case when at least one of q and f is non-homogeneous, and without loss of generality in the following argument we may assume both are non-homogeneous. Since $E(q) \neq \emptyset$, we may further assume that $q(0) = 0$ (for if $q(a) = 0$, we may replace q and f by $q(x+a)$ and $f(x+a)$). Let $\tilde{q}(x_0, x) = x_0^2 q(x/x_0)$ (resp. $\tilde{f}(x_0, x) = x_0^2 f(x/x_0)$) be the homogenization of $q(x)$ (resp. $f(x)$). Denote $\tilde{x}^T := [x_0 \quad x^T]^T$, and note that

$$\tilde{f}(\tilde{x}) = x_0^2 f_0 + x_0 f_1(x) + f_2(x), \quad \tilde{q}(\tilde{x}) = x_0 q_1(x) + q_2(x),$$

where every f_i and q_i are homogeneous of degree i.

Now we claim that

$$\tilde{f}(\tilde{x}) \geq 0, \quad \forall\, \tilde{x} : \tilde{q}(\tilde{x}) = 0. \tag{4.9}$$

From the hypothesis that $f(x) \geq 0$ whenvever $q(x) = 0$, (4.6) follows easily from the homogenization formulas when $x_0 \neq 0$. For the case when $x_0 = 0$, we need to prove

$$f_2(x) \geq 0, \quad \forall\, x : q_2(x) = 0.$$

Let u be an arbitrary point such that $q_2(u) = 0$. Consider the equation

$$\tilde{q}(\epsilon, x) = \epsilon q_1(x) + q_2(x) = 0.$$

If $q_1(u) = 0$, then $q(\alpha u) = 0$ for all real α. Thus $\alpha u \in E(q)$ and $f(\alpha u) \geq 0$ for all α, which implies $f_2(u) \geq 0$. If $q_1(u) \neq 0$, then the rational function

$$\epsilon(x) = -\frac{q_2(x)}{q_1(x)}$$

is continuous in a neighborhood $\mathcal{O}_u$ of u. Choose a sequence $\{u^{(i)}\} \subset \mathcal{O}_u$ such that $q_2(u^{(i)}) \neq 0$ and $u^{(i)} \to u$. Then $\epsilon(u^{(i)}) \neq 0$ and $\epsilon(u^{(i)}) \to 0$. Since $\tilde{q}(\epsilon(u^{(i)}), u^{(i)}) = \epsilon(u^{(i)}) q_1(u^{(i)}) + q_2(u^{(i)}) = 0$, it follows that $q(\frac{u^{(i)}}{\epsilon(u^{(i)})}) = 0$. The hypothesis now implies that $f(\frac{u^{(i)}}{\epsilon(u^{(i)})}) \geq 0$, whence $\tilde{f}(\epsilon(u^{(i)}), u^{(i)}) \geq 0$. Letting $i \to \infty$, we get $f_2(u) = \tilde{f}(0, u) \geq 0$. Therefore, claim (4.9) is proved. The existence of $\xi, \zeta \in \mathbb{R}^n$ such that $q(\xi) > 0 > q(\zeta)$ implies that $\tilde{q}(1, \xi) > 0 > \tilde{q}(1, \zeta)$. Now the homogeneous case can be applied to yield $t \in \mathbb{R}$ such that $\tilde{f}(x_0, x) - t\tilde{q}(x_0, x) \geq 0 \quad \forall\, (x_0, x) \in \mathbb{R}^{n+1}$, and the result follows by setting $x_0 = 1$. $\square$

In Proposition 4.3, if there do not exist $\xi, \zeta \in \mathbb{R}^n$ such that $q(\xi) > 0 > q(\zeta)$, then the conclusion might fail. For instance, for polynomials $f(x) = x_1 x_2$ and $q(x) = -x_1^2$, the summation $f(x) - tq(x)$ is never globally nonnegative for any scalar t. However, Proposition 4.3 can be weakened as follows.

Proposition 4.4. *Let $f(x), q(x)$ be two quadratic polynomials.*
(a) If $S(q) \neq \emptyset$ and $f(x) \geq 0$ for all $x \in S(q)$, then for any $\epsilon > 0$ there exists $t \geq 0$ such that

$$f(x) + \epsilon(1 + \|x\|_2^2) - tq(x) \geq 0, \quad \forall\, x \in \mathbb{R}^n.$$

(b) If $E(q) \neq \emptyset$ and $f(x) \geq 0$ for all $x \in E(q)$, then for any $\epsilon > 0$ there exists $t \in \mathbb{R}$ such that

$$f(x) + \epsilon(1 + \|x\|_2^2) - tq(x) \geq 0, \quad \forall\, x \in \mathbb{R}^n.$$

Proof. As in (4.4), write $f(x)$ and $q(x)$ as

$$f(x) = \begin{bmatrix} 1 \\ x \end{bmatrix}^T \underbrace{\begin{bmatrix} f_0 & f_1^T \\ f_1 & F_2 \end{bmatrix}}_{F} \begin{bmatrix} 1 \\ x \end{bmatrix}, \quad q(x) = \begin{bmatrix} 1 \\ x \end{bmatrix}^T \underbrace{\begin{bmatrix} q_0 & q_1^T \\ q_1 & Q_2 \end{bmatrix}}_{Q} \begin{bmatrix} 1 \\ x \end{bmatrix}.$$

(a) If there exists $\xi \in \mathbb{R}^n$ such that $q(\xi) > 0$, then we are done by Proposition 4.2. So we need only consider the case when $q(x) \leq 0$ for every $x \in \mathbb{R}^n$. Since $S(q) \neq \emptyset$, without loss of generality we may assume that the origin belongs to $S(q)$, which implies that $q_0 = 0$. Let

$$\mathcal{E} = \{S + tQ : S^T = S \succeq 0, t \geq 0\}.$$

Note that $\mathcal{E}$ is a convex set (but not necessarily closed). We claim that for each $\epsilon > 0$,

$$F(\epsilon) := F + \epsilon I_{n+1} = \begin{bmatrix} f_0 + \epsilon & f_1^T \\ f_1 & F_2 + \epsilon I_n \end{bmatrix} \in \mathcal{E}.$$

Suppose to the contrary that $F(\epsilon) \notin \mathcal{E}$ for some $\epsilon > 0$. Then as in the proof of Proposition 4.3, there exist a nonzero symmetric matrix X and a scalar η such that

$$F(\epsilon) \bullet X \leq \eta, \quad (S + tQ) \bullet X \geq \eta, \forall\, S \succeq 0, \forall\, t \geq 0.$$

The above implies that

$$Q \bullet X \geq 0, \quad X \succeq 0, \quad \eta \leq 0.$$

Then, by Proposition 4.1, there exist nonzero vectors $u_1, \ldots, u_r$ such that

$$X = u_1 u_1^T + \cdots + u_r u_r^T, \quad u_i^T Q u_i = \frac{Q \bullet X}{r} \geq 0 \quad (1 \leq i \leq n).$$

Write every u_i as

$$u_i = \begin{bmatrix} \tau_i \\ v_i \end{bmatrix}, \quad \tau_i \in \mathbb{R}, \quad v_i \in \mathbb{R}^n.$$

Order u_i such that $\tau_i \neq 0$ $(1 \leq i \leq k)$, and $\tau_{k+1} = \cdots = \tau_r = 0$ (the nonzero terms may be absent, or the zero terms may be absent). For every $i = 1, \ldots, k$ if $k > 0$, we have

$$\tau_i^2 q(v_i/\tau_i) = u_i^T Q u_i \geq 0.$$

Thus every $v_i/\tau_i \in S(q)$ and hence $f(v_i/\tau_i) \geq 0$. For every $i = k+1, \ldots, r$, we have

$$v_i^T Q_2 v_i = u_i^T Q u_i \geq 0.$$

Then we must have $v_i^T Q_2 v_i = 0$, because otherwise $q(\alpha v_i) > 0$ for $\alpha > 0$ big enough contradicts the assumption that $q(x) \leq 0$ for all $x \in \mathbb{R}^n$ at the beginning. So

$$q(\alpha v_i) = 2\alpha q_1^T v_i.$$

Replacing u_i by $-u_i$ if necessary, we may assume that $q_1^T v_i \geq 0$. So $q(\alpha v_i) \geq 0$ and $\alpha v_i \in S(q)$ for all $\alpha > 0$. Then $f(\alpha v_i) \geq 0$ for all $\alpha > 0$, and hence $v_i^T F_2 v_i \geq 0$. So we have

$$\begin{aligned} F(\epsilon) \bullet X &= \sum_{i=1}^{r} u_i F(\epsilon) u_i \\ &= \sum_{i=1}^{k} \tau_i^2 (f(v_i/\tau_i) + \epsilon(1 + \|v_i/\tau_i\|_2^2)) + \sum_{i=k+1}^{r} (v_i^T F_2 v_i + \epsilon \|v_i\|_2^2) \\ &\geq \sum_{i=1}^{k} \tau_i^2 \epsilon + \sum_{i=k+1}^{r} \epsilon \|v_i\|_2^2. \end{aligned}$$

Since every u_i is nonzero, we have either $\tau_i > 0$ or $v_i \neq 0$. Thus we must have

$$F(\epsilon) \bullet X > 0,$$

which contradicts that $F(\epsilon) \bullet X \leq \eta \leq 0$. So $F(\epsilon)$ must belong to $\mathcal{E}$, and the result follows.

(b) If there exist ξ, ζ such that $q(\xi) > 0 > q(\zeta)$, then we are done by applying Proposition 4.3. Replacing q by $-q$ if necessary, we may thus assume that $q(x) \geq 0$ for all $x \in \mathbb{R}^n$. Let us recall the decomposition $q(x) = q_0 + 2q_1^T x + x^T Q_2 x$ given just before the proof of (a). Since $E(q) \neq \emptyset$, we may assume that the origin belongs to $E(q)$, i.e., $q_0 = q(0) = 0$. Since we assumed $q(x) \geq 0$ for all $x \in \mathbb{R}^n$, the origin is a minimizer of $q(x)$, whence $\nabla q(0) = 0$. Thus it follows that

$$q_1 = \frac{1}{2} \nabla q(0) = 0.$$

We now proceed to derive a contradiction similar to that used in (a), but now we define $\mathcal{E}$ as

$$\mathcal{E} = \{S + tQ : \, S^T = S \succeq 0, t \in \mathbb{R}\}.$$

As in part (a), if $F(\epsilon) \notin \mathcal{E}$, then there exist a nonzero symmetric matrix X and a scalar η such that

$$F(\epsilon) \bullet X \leq \eta, \quad (S + tQ) \bullet X \geq \eta, \, \forall \, S^T = S \succeq 0, \, \forall \, t \in \mathbb{R},$$

which implies

$$Q \bullet X = 0, \quad X \succeq 0, \quad \eta \leq 0.$$

Again, applying Proposition 4.1, we get nonzero vectors $u_1, \ldots, u_r$ such that

$$X = u_1 u_1^T + \cdots + u_r u_r^T, \quad u_1^T Q u_1 = \cdots = u_r^T Q u_r = \frac{Q \bullet X}{r} = 0.$$

As before, write u_i as

$$u_i = \begin{bmatrix} \tau_i \\ v_i \end{bmatrix},$$

and reorder the u_i so that $\tau_i \neq 0$ $(1 \leq i \leq k)$, and $\tau_{k+1} = \cdots = \tau_r = 0$. For $i = 1, \ldots, k$, we have

$$\tau_i^2 q(v_i/\tau_i) = u_i^T Q u_i = 0,$$

so $v_i/\tau_i \in E(q)$, and hence $f(v_i/\tau_i) \geq 0$. For every $i = k+1, \ldots, r$, we have that for all $\alpha \in \mathbb{R}$,

$$0 = \alpha^2 u_i^T Q u_i = \alpha^2 v_i^T Q_2 v_i = q(\alpha v_i).$$

Thus we get

$$0 \leq f(\alpha v_i) = f_0 + 2\alpha f_1^T v_i + \alpha^2 v_i^T F_2 v_i, \quad \forall \;\; \alpha \in \mathbb{R},$$

whence $v_i^T F_2 v_i \geq 0$ for $i = k+1, \ldots, r$. As in part (a), we have

$$\begin{aligned} F(\epsilon) \bullet X &= \sum_{i=1}^{k} \tau_i^2 (f(v_i/\tau_i) + \epsilon(1 + \|v_i/\tau_i\|_2^2)) + \sum_{i=k+1}^{r} (v_i^T F_2 v_i + \epsilon \|v_i\|_2^2) \\ &\geq \sum_{i=1}^{k} \tau_i^2 \epsilon + \sum_{i=k+1}^{r} \epsilon \|v_i\|_2^2 > 0, \end{aligned}$$

which contradicts $F(\epsilon) \bullet X \leq 0$. So $F(\epsilon)$ must belong to $\mathcal{E}$, and the result follows. $\square$

4.2. Quadratic moment problems

We now apply the preceding results to quadratic moment problems. Recall from [4] that for $n = 1,\ 2$, if $M_1(y) \succeq 0$, then $M_1(y)$ has a flat extension, and thus y has admits a *rank* $M_1(y)$-atomic representing measure. We begin by generalizing the latter result to $n \geq 1$.

Theorem 4.5. *If $y \in \mathcal{M}_{n,2}$ and $M_1(y) \succeq 0$, then y has a rank $M_1(y)$-atomic representing measure.*

Proof. Without loss of generality, we may normalize y so that $y_0 = 1$. Write the moment matrix $M_1(y)$ as follows:

$$M_1(y) = \begin{bmatrix} 1 & z^T \\ z & W \end{bmatrix},$$

where $z \in \mathbb{R}^n$. Since $y_0 = 1$, we can choose a number $\alpha > 0$ small enough such that the matrix

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha I_n \end{bmatrix}$$

satisfies $Q \bullet M_1(y) \geq 0$. Then, by Proposition 4.1, there exist nonzero (column) vectors $u_1, \ldots, u_r \in \mathbb{R}^{n+1}$ ($r = rank\ M_1(y)$) such that

$$M_1(y) = u_1 u_1^T + \cdots + u_r u_r^T, \quad u_1^T Q u_1 = \cdots = u_r^T Q u_r = \frac{Q \bullet M_1(y)}{r} \geq 0.$$

Write the vectors u_i as

$$u_i = \begin{bmatrix} \tau_i \\ w_i \end{bmatrix}, \tau_i \in \mathbb{R}, w_i \in \mathbb{R}^n.$$

Then $u_i^T Q u_i \geq 0$ implies that $\tau_i^2 \geq \alpha \|w_i\|_2^2$. So, if $\tau_i = 0$, then $w_i = 0$. Note that $\|u\|_i^2 = \tau_i^2 + \|w_i\|_2^2$. Since all u_i are nonzero, every $\tau_i \neq 0$, and hence we can write u_i as

$$u_i = \tau_i \begin{bmatrix} 1 \\ v_i \end{bmatrix}, v_i \in \mathbb{R}^n.$$

Thus, we have

$$M_1(y) = \tau_1^2 \begin{bmatrix} 1 \\ v_1 \end{bmatrix} \begin{bmatrix} 1 \\ v_1 \end{bmatrix}^T + \cdots + \tau_r^2 \begin{bmatrix} 1 \\ v_r \end{bmatrix} \begin{bmatrix} 1 \\ v_r \end{bmatrix}^T. \tag{4.10}$$

The above gives a r-atomic representing measure for y. $\square$

We pause to give an application of Theorem 4.5 to the multivariable degree one moment problem.

Corollary 4.6. *A degree one multisequence y has a representing measure if and only if $y_0 > 0$.*

Proof. Note that if v denotes the vector of moments in y, in degree-lexicographic order, then $v^T v$ has the form of a positive moment matrix M_1, so the existence of a representing measure follows from Theorem 4.5 . $\square$

We next turn to the quadratic K-moment problem where q is a quadratic polynomial and $K = E(q)$ or $K = S(q)$. For the case when $n = 2$ and $q(x) = 1 - \|x\|_2^2$, it is known that the conditions $M_1(y) \succeq 0$ and $L_y(q) = 0$ (resp., $L_y \geq 0$) imply the existence of representing measures supported in $E(q)$ [6, Theorem 3.1] (resp., $S(q)$ [6, Theorem 1.8]). This can be generalized to $n \geq 1$ and $S(q)$ compact.

Theorem 4.7. *Suppose $q(x)$ is quadratic and $S(q)$ is compact and nonempty.*
(a) $y \in \mathcal{M}_{n,2}$ has a representing measure supported in $E(q)$ if and only if

$$M_1(y) \succeq 0, \quad L_y(q) = 0.$$

(b) $y \in \mathcal{M}_{n,2}$ has a representing measure supported in $S(q)$ if and only if

$$M_1(y) \succeq 0, \quad L_y(q) \geq 0.$$

Proof. We write $q(x)$ as

$$q(x) = q_0 + 2q_1^T x + x^T Q_2 x = \begin{bmatrix} 1 \\ x \end{bmatrix}^T \underbrace{\begin{bmatrix} q_0 & q_1^T \\ q_1 & Q_2 \end{bmatrix}}_{Q} \begin{bmatrix} 1 \\ x \end{bmatrix}. \tag{4.11}$$

Since $S(q)$ is nonempty, we can assume $0 \in S(q)$, i.e., $q_0 \geq 0$, without loss of generality. From the compactness of $S(q)$, we know $q(x)$ must be strictly concave, that is, Q_2 must be negative definite ($Q_2 \prec 0$). To see this, suppose otherwise, i.e., that Q_2 is not negative definite. Then there exists a nonzero $u \in \mathbb{R}^n$ such that $u^T Q_2 u \geq 0$. We can also further choose u so that $q_1^T u \geq 0$ (otherwise replace u by $-u$). Thus, for any $t > 0$, we have $q(tu) \geq 0$, which implies $S(q)$ is unbounded. However, this contradicts the compactness of $S(q)$. Therefore, Q_2 must be negative definite.

(a) We need only prove the sufficiency direction. Suppose $y \in \mathcal{M}_{n,2}$ and let $X = M_1(y)$. Then we have

$$X \succeq 0, \quad Q \bullet X = L_y(q) = 0.$$

By Proposition 4.1, there exist nonzero vectors $u_1, \dots, u_r \in \mathbb{R}^{n+1}$ such that

$$X = \sum_{i=1}^{r} u_i u_i^T, \quad u_1^T Q u_1 = \cdots = u_r^T Q u_r = \frac{Q \bullet X}{r} = 0.$$

Write $u_i = \begin{bmatrix} \tau_i & w_i^T \end{bmatrix}^T$ for some scalar τ_i and some vector $w_i \in \mathbb{R}^n$. Then $u_i^T Q u_i = 0$ implies that

$$q_0 \tau_i^2 + 2\tau_i q_1^T w_i + w_i^T Q_2 w_i = 0. \tag{4.12}$$

If $\tau_i = 0$ for some i, then $w_i^T Q_2 w_i = 0$, and hence $w_i = 0$ because of negative definiteness of Q_2. Since u_i is nonzero, it follows that every $\tau_i \neq 0$, and we can write $u_i = \tau_i [1\ v_i^T]^T$. (4.11) and (4.12) now imply that $q(v_i) = 0$, so $v_i \in E(q)$. Therefore, we have

$$M_1(y) = \tau_1^2 \begin{bmatrix} 1 \\ v_1 \end{bmatrix} \begin{bmatrix} 1 \\ v_1 \end{bmatrix}^T + \cdots + \tau_r^2 \begin{bmatrix} 1 \\ v_r \end{bmatrix} \begin{bmatrix} 1 \\ v_r \end{bmatrix}^T,$$

and it follows that $\mu \equiv \sum_{i=1}^r \tau_i^2 \delta_{v_i}$ is a representing measure for y supported in $E(q)$.

(b)The proof is very similar to part (a). Suppose $y \in \mathcal{M}_{n,2}$ and let $X = M_1(y)$. Then

$$X \succeq 0, \quad Q \bullet X = L_y(q) \geq 0.$$

By Proposition 4.1, there exist nonzero vectors $u_1, \dots, u_r \in \mathbb{R}^{n+1}$ such that

$$X = \sum_{i=1}^{r} u_i u_i^T, \quad u_1^T Q u_1 = \cdots = u_r^T Q u_r = \frac{Q \bullet X}{r} \geq 0.$$

Write $u_i = \begin{bmatrix} \tau_i & w_i^T \end{bmatrix}^T$ for some $w_i \in \mathbb{R}^n$. Then $u_i^T Q u_i \geq 0$ implies that

$$q_0 \tau_i^2 + 2\tau_i q_1^T w_i + w_i^T Q_2 w_i \geq 0. \tag{4.13}$$

If $\tau_i = 0$ for some i, then $w_i^T Q_2 w_i \geq 0$ and hence $w_i = 0$ because of negative definiteness of Q_2. But this is also impossible, since otherwise $u_i = [\tau_i\ w_i^T]^T$ is a zero vector. Thus, every $\tau_i \neq 0$. So we can further write $u_i = \tau_i \begin{bmatrix} 1 & v_i^T \end{bmatrix}^T$. Then (4.11) and (4.13) imply that $q(v_i) \geq 0$, and so $v_i \in S(q)$. Hence we get

$$M_1(y) = \tau_1^2 \begin{bmatrix} 1 \\ v_1 \end{bmatrix} \begin{bmatrix} 1 \\ v_1 \end{bmatrix}^T + \cdots + \tau_r^2 \begin{bmatrix} 1 \\ v_r \end{bmatrix} \begin{bmatrix} 1 \\ v_r \end{bmatrix}^T,$$

and it follows as above that y has a representing measure supported in $S(q)$. □

When $E(q)$ or $S(q)$ is not compact, the conclusions of Theorem 4.7 might fail. However, we can get a sightly weakened version.

Theorem 4.8. *Let $y \in \mathcal{M}_{n,2}$ and let $q(x)$ be a quadratic polynomial.*
(i) Suppose $E(q) \neq \emptyset$. Then $M_1(y) \succeq 0$ and $L_y(q) = 0$ if and only if $y \in \overline{\mathcal{R}_{n,2}(E(q))}$.
(ii) Suppose $S(q) \neq \emptyset$. Then $M_1(y) \succeq 0$ and $L_y(q) \geq 0$ if and only if $y \in \overline{\mathcal{R}_{n,2}(S(q))}$.

Proof. (i) The sufficiency direction is obvious, so we only need prove necessity. Suppose to the contrary that $M_1(y) \succeq 0$ and $L_y(q) = 0$, but $y \notin \overline{\mathcal{R}_{n,2}(E(q))}$. Since $\overline{\mathcal{R}_{n,2}(E(q))}$ is a closed convex cone, Minkowski's separation theorem implies that there exists a nonzero polynomial $p \in \mathcal{P}_2$ such that

$$L_y(p) \equiv \hat{p}^T y < 0, \quad \text{and} \quad \hat{p}^T w \geq 0, \forall\, w \in \overline{\mathcal{R}_{n,2}(E(q))}.$$

For $1 \leq i \leq n$, let y_{2e_i} denote the element of y corresponding to the monomial x_i^2. Choose $\epsilon > 0$ small enough so that

$$\hat{p}^T y + \epsilon(1 + \sum_{i=1}^n y_{2e_i}) L_y(1) < 0, \tag{4.14}$$

and define the nonzero polynomial

$$\tilde{p}(x) = \hat{p}^T [x]_2 + \epsilon(1 + \|x\|_2^2). \tag{4.15}$$

Since, for each $x \in E(q)$, the monomial vector $[x]_2$ belongs to $\mathcal{R}_{n,2}(E(q))$ (with $E(q)$-representing measure δ_x), the polynomial $\hat{p}^T[x]_2$ is nonnegative on $E(q)$. By Proposition 4.4-(b), there exists $t \in \mathbb{R}$ such that

$$\hat{p}^T [x]_2 + \epsilon(1 + \|x\|_2^2) - tq(x) \geq 0 \quad \forall x \in \mathbb{R}^n.$$

It follows from (4.5) that there exists a matrix P, with $P = P^T \succeq 0$, such that

$$\hat{p}^T [x]_2 + \epsilon(1 + \|x\|_2^2) - tq(x) = [x]_1^T P [x]_1 \quad \forall x \in \mathbb{R}^n,$$

whence

$$\tilde{p}(x) = [x]_1^T P [x]_1 + tq(x).$$

Since $M_1(y) \succeq 0$ and $L_y(q) = 0$, applying L_y on both sides of the above (see equation (4.7)) implies that

$$L_y(\tilde{p}) = P \bullet M_1(y) + tL_y(q) = P \bullet M_1(y) \geq 0.$$

However, from (4.14)-(4.15) we have

$$L_y(\tilde{p}) = \hat{p}^T y + \epsilon(1 + \sum_{i=1}^n y_{2e_i}) L_y(1) < 0,$$

which is a contradiction. So we must have $y \in \overline{\mathcal{R}_{n,2}(E(q))}$.

(ii) Sufficiency is again obvious, so we focus on necessity. The proof is very similar to the argument of (i), but we replace $E(q)$ by $S(q)$. Thus, the polynomial $\hat{p}^T[x]_2$ is now nonnegative on $S(q)$. Using Proposition 4.4-(a), it follows as above that there exists $t \geq 0$ and a matrix P with $P = P^T \succeq 0$, such that $\tilde{p}(x) = [x]_1^T P[x]_1 + tq(x)$. Since $t \geq 0$ and $L_y(q) \geq 0$, it follows as before that $L_y(\tilde{p}) \geq 0$, which leads to the same contradiction as in (i). □

Theorem 4.8 implies that if q is a quadratic polynomial and if $M_1(y) \succeq 0$ and $L_y(q) = 0$ (resp. $L_y(q) \geq 0$), then y is in the closure of the quadratic moment sequences which admit representing measures supported in $E(q)$ (resp. $S(q)$). But this does not necessarily imply that y admits a representing measure supported in $E(q)$ or $S(q)$, as the following example shows.

Example 4.9. Let $n = 2$ and let $y \in \mathcal{M}_{2,2}$ be the quadratic moment sequence such that

$$M_1(y) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Let 1, X_1, X_2 denote the columns of $M_1(y)$. Obviously, $M_1(y)$ is positive semidefinite with *rank* $M_1(y) = 2$, so y admits 2-atomic representing measures by Theorem 4.5. Since $1 = X_1$, Proposition 3.1 of [4] implies that any representing measure μ must be supported in the variety $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 1\}$.

Let $q(x) = x_2 - x_1^2$. Then $S(q)$ is convex but noncompact, and $E(q)$ is nonconvex and noncompact. Note that $L_y(q) = y_{01} - y_{20} = 0$, so of course $L_y(q) \geq 0$. But y does not have a representing measure μ supported in either $E(q)$ or $S(q)$. Indeed, suppose a representing measure μ with *supp* $\mu \subseteq S(q)$ exists. For any $x = (x_1, x_2) \in$ *supp* $\mu \subseteq S(q)$, we must have $x_1 = 1$ and $x_2 \geq 1$. Then the relation

$$\int_{\mathbb{R}^2} x_2 d\mu(x) = y_{01} = 1,$$

together with $y_{00} = 1$, implies that $x_2 = 1$ on the support of μ. So μ is supported at the single point $(1, 1)$, which is obviously false. Therefore, y does not have a representing measure μ supported in $S(q)$ or $E(q)$.

In keeping with Theorem 4.7, we next show that an arbitrarily small perturbation can be applied to make the perturbed y have a representing measure supported in $E(q)(\subset S(q))$. For $1 > \epsilon > 0$, let the moment sequence $\bar{y}(\epsilon)$ be defined by

$$M_1(\bar{y}(\epsilon)) = (1-\epsilon) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T + \epsilon \begin{bmatrix} 1 \\ \epsilon^{-1/4} \\ \epsilon^{-1/2} \end{bmatrix} \begin{bmatrix} 1 \\ \epsilon^{-1/4} \\ \epsilon^{-1/2} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & 1-\epsilon+\epsilon^{3/4} & 1+\epsilon^{1/2}-\epsilon \\ 1-\epsilon+\epsilon^{3/4} & 1+\epsilon^{1/2}-\epsilon & 1+\epsilon^{1/4}-\epsilon \\ 1+\epsilon^{1/2}-\epsilon & 1+\epsilon^{1/4}-\epsilon & 2-\epsilon \end{bmatrix}.$$

We see that $\bar{y}(\epsilon) \to y$ as $\epsilon \to 0$, and $\bar{y}(\epsilon)$ has the 2-atomic $E(q)$-representing measure

$$(1-\epsilon)\delta_{(1,1)} + \epsilon\delta_{(\epsilon^{-\frac{1}{4}}, \epsilon^{-\frac{1}{2}})}.$$

$\square$

Despite the preceding example, if, in Theorem 4.8, the quadratic moment sequence y is such that $M_1(y) \succ 0$ and $L_y(q) = 0$ (resp. $L_y(q) > 0$), then y does have a representing measure supported in $E(q)$ (resp. $S(q)$). The following result thus provides some affirmative evidence for Question 1.2.

Theorem 4.10. *Let $y \in \mathcal{M}_{n,2}$ and let $q(x)$ be a quadratic polynomial.*
(i) If $E(q) \neq \emptyset$, $M_1(y) \succ 0$ and $L_y(q) = 0$, then $y \in \mathcal{R}_{n,2}(E(q))$.
(ii) If $S(q) \neq \emptyset$, $M_1(y) \succ 0$ and $L_y(q) > 0$, then $y \in \mathcal{R}_{n,2}(S(q))$.

Proof. (i) Define the affine subspace $\mathcal{N}(q)$ and set $\mathcal{F}_E$ as follows:

$$\mathcal{N}(q) = \{y \in \mathcal{M}_{n,2} : L_y(q) = 0\}, \quad \mathcal{F}_E = \{y \in \mathcal{N}(q) : M_1(y) \succeq 0\}.$$

Note that $\mathcal{R}_{n,2}(E(q))$ and $\mathcal{F}_E$ are both convex sets contained in the space $\mathcal{N}(q)$. Theorem 4.8 says that $\mathcal{F}_E = \overline{\mathcal{R}_{n,2}(E(q))}$. If $M_1(y) \succ 0$, then y lies in the interior of $\mathcal{F}_E$. By Lemma 2.1, we know $y \in \mathcal{R}_{n,2}(E(q))$.

(ii) Let $\mathcal{F}_S$ be the following convex set

$$\mathcal{F}_S = \{y \in \mathcal{M}_{n,2} : M_1(y) \succeq 0, L_y(q) \geq 0\}.$$

Theorem 4.8 says that $\mathcal{F}_S = \overline{\mathcal{R}_{n,2}(S(q))}$. If $M_1(y) \succ 0$ and $L_y(q) > 0$, then y lies in the interior of $\mathcal{F}_S$. Hence Lemma 2.1 implies $y \in \mathcal{R}_{n,2}(S(q))$. $\square$

Using Theorem 4.10, we can now show that Question 1.2 has an affirmative answer when $d = 1$ and $K = E(q)$ or $K = S(q)$ for a quadratic polynomial $q(x)$.

Corollary 4.11. *Let $y \in \mathcal{M}_{n,2}$ and let $q(x)$ be a quadratic polynomial.*
(i) Suppose $E(q) \neq \emptyset$. If $M_1(y) \succ 0$ and L_y is $E(q)$-positive, then y has an $E(q)$-representing measure.
(ii) Suppose $S(q) \neq \emptyset$. If $M_1(y) \succ 0$ and L_y is $S(q)$-positive, then y has an $S(q)$-representing measure.

Proof. (i) From Theorem 4.10 (i), it suffices to show that $L_y(q) = 0$. Since L_y is $E(q)$-positive, we have $L_y(q) \geq 0$ and $L_y(-q) \geq 0$, so $L_y(q) = 0$.

(ii) Suppose first that $E(q) \neq \emptyset$. Since L_y is $S(q)$-positive, $L_y(q) \geq 0$. If $L_y(q) = 0$, Theorem 4.10 (i) implies that y has a representing measure supported in $E(q) \subseteq S(q)$. If $L_y(q) > 0$, then Theorem 4.10 (ii) shows that y has a representing measure supported in $S(q)$. Suppose next that $E(q) = \emptyset$. Since $S(q) \neq \emptyset$, then $S(q) = \mathbb{R}^n$, so in this case the result follows from Theorem 4.5. $\square$

References

[1] C. Bayer and J. Teichmann. The proof of Tchakaloff's Theorem, *Proc. Amer. Math. Soc.*, 134(2006), 3035-3040.

[2] S. Berberian. Lectures in Functional Analysis and Operator Theory, Springer-Verlag, 1973.

[3] A. Brown and C. Pearcy. Introduction to Operator Theory I, Elements of Functional Analysis, Springer-Verlag, 1977.

[4] R. Curto and L. Fialkow. Solution of the truncated complex moment problem for flat data. *Memoirs of the American Mathematical Society*, 119(1996), No. 568, Amer. Math. Soc. Providence, RI, 1996.

[5] R. Curto and L. Fialkow. Flat extensions of positive moment matrices: Relations in analytic or conjugate terms, *Operator Th.: Adv. Appl.* 104(1998), 59-82.

[6] R. Curto and L. Fialkow. The quadratic moment problem for the unit circle and unit disk. *Integr. equ. oper. theory*, 38(2000), 377-409.

[7] R. Curto and L. Fialkow. Solution of the singular quartic moment problem. *Journal of Operator Theory*, 48(2002), pp. 315-354.

[8] R. Curto and L. Fialkow. Truncated K-moment problems in several variables. *Journal of Operator Theory*, 54(2005), pp. 189-226.

[9] R. Curto and L. Fialkow. Solution of the truncated hyperbolic moment problem, *Integral Equations and Operator Theory*, 52(2005), 181-219.

[10] R. Curto and L. Fialkow. An analogue of the Riesz-Haviland Theorem for the truncated moment problem, *J. Functional Analysis*, 225(2008), 2709-2731.

[11] L. Fialkow. Truncated multivariable moment problems with finite variety, J. Operator Theory 60(2008), 343-377.

[12] L. Fialkow. Solution of the truncated moment problem with variety $y = x^3$, preprint 2008.

[13] C. Li and S.H. Lee The quartic moment problem, *J. Korean Math. Soc.* 42(2005), 723-747.

[14] M. Laurent. Sums of squares, moment matrices and optimization over polynomials, Emerging Applications of Algebraic Geometry, Vol. 149 of IMA Volumes in Mathematics and its Applications, M. Putinar and S. Sullivant (eds), Springer, pages 157-270, 2009.

[15] Z. Luo, J. Sturm and S. Zhang. Multivariate nonnegative quadratic mappings. *SIAM J. Optim.* 14 (2004), no. 4, 1140–1162

[16] B. Reznick. Some concrete aspects of Hilbert's 17^{th} problem. In *Contemp. Math.*, volume 253, pages 251-272. American Mathematical Society, 2000.

[17] B. Reznick. On Hilbert's construction of positive plynomials. *Preprint*, 2007.

[18] K. Schmüdgen. An example of a positive polynomial which is not a sum of squares of polynomials. A positive, but not strongly positive functional. *Math. Nachr.* 88(1979), 385-390.

[19] J. Sturm and S. Zhang. On Cones of Nonnegative Quadratic Functions. *Mathematics of Operations Research*, Vol. 28, No.2, pp. 246 - 267, 2003.

[20] V. Tchakaloff. Formules de cubatures mécanique à coefficients non négatifs, *Bull. Sci. Math.* (2)82(1957), 123-134.

[21] V.A. Yakabovich. The $\mathcal{S}$-procedure in non-linear control theory. *Vestnik Leningrad Univ. Math.*, 4:73-93, 1977. In Russian, 1971.