1 The decomposition of the space of "curvature operators".

Let M be an n-dimensional Riemannian manifold with metric tensor $\langle ..., ... \rangle$ and Levi-Cevita connection ∇. As usual, the curvature tensor is defined by

$$R_p(v,w)z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

at p for X, Y, Z vector fields with $X_p = v, Y_p = w, Z_p = z$. One has

$$\langle R_p(v,w)x, y \rangle_p$$

is skew-symmetric in v, w and in x, y and $\langle R_p(v,w)x, y \rangle_p = \langle R_p(x,y)z, w \rangle_p$. Thus the curvature tensor induces a linear map (also denoted R)

$$R_p : T_p(M) \bigwedge T_p(M) \to T_p(M) \bigwedge T_p(M).$$

Furthermore, if we use the inner product $\langle v \bigwedge w, x \bigwedge y \rangle_p = \langle v, x \rangle_p \langle w, y \rangle_p - \langle v, y \rangle_p \langle w, x \rangle_p$ then R_p is self adjoint. Thus if we choose an orthonormal basis of $T_p(M)$ we can consider R_p as an element of $S^2(\bigwedge^2 \mathbb{R}^n)$. If we choose a different orthonormal basis then R_p changes according to $S^2(\bigwedge^2 g)$ with $g \in O(n)$. in these notes we will look at how these tensors transform under these changes of bases. We will complexify and look at the decomposition of $S^2(\bigwedge^2 \mathbb{C}^n)$ as a representation of $SO(n, \mathbb{C})$ and then indicate what happens over $\mathbb{R}$. The decompositions are essentially the same for SO as for O. Since the theorem of the highest weight applies directly to the former case we have opted to emphasize it. As it turns out the decomposition of these curvature operators stabilizes in dimension 11. The decomposition of thos operators that satsify the Bianchi identity stabilizes in dimension 7. We will study the cases to that dimension case by case and when some new aspect of the decomposition appears we will consider it at that point. We will use the notation $\langle v, w \rangle = \sum v_i w_i$ and transpose will be relative to this nondegerate bilinear form.

1.1 $n = 2$.

In this case $\bigwedge^2 \mathbb{C}^2$ is one dimensional. Thus $S^2(\bigwedge^2 \mathbb{C}^2)$ is one dimensional and since the representation on $\bigwedge^2 \mathbb{C}^2$ is given by the determinant, $O(2, \mathbb{C})$ acts on the one dimensional space $S^2(\bigwedge^2 \mathbb{C}^2)$ by the trivial representation. This corresponds to the fact that the only curvature in 2 dimensions is the sectional curvature. In the general context in which we are working this case introduces the trivial representation whose highest weight is 0.Thus as a resepresentation of $SO(n, \mathbb{C})$ the representation $S^2(\bigwedge^2 \mathbb{C}^2)$ the only constituent has highest weight 0, F^0.

1.2 $n = 3$.

If $n = 3$ then as a representation of $SO(3, \mathbb{C})$ the representation $\bigwedge^2 \mathbb{C}^3$ is equivalent with the action of $SO(3, \mathbb{C})$ on $\mathbb{C}^3$. Thus $S^2(\bigwedge^2 \mathbb{C}^3)$ is equivalent with

$S^2(\mathbb{C}^3)$. We know that $S^2(\mathbb{C}^3)$ as a representation of $SO(3,\mathbb{C})$ is equivalent with the direct sum of the spherical harmonics of degree 2 and the trivial representation F^0. Now, $SO(3,\mathbb{C})$ has as a 2-fold cover $SL(2,\mathbb{C})$. For this group the irreducible representations are parametrized by the integers $0,1,2,...$ where, if the representation has parameter k then the dimension is $k+1$. The representation is also denoted by $\mathcal{H}^2$ for spherical harmonics of degree 2. We note that the decomposition into irreducibles $S^2(\bigwedge^2 \mathbb{C}^3) \equiv F^0 \oplus \mathcal{H}^2$ is also a decomposition for $O(3,\mathbb{C})$ and since the terms are invariant under complex conjugation it is also valid for $O(3)$. We will now give a general construction which is motivated by this case. If $A, B \in End(\mathbb{C}^n)$ we define

$$A \bigwedge B : \mathbb{C}^n \bigwedge \mathbb{C}^n \to \mathbb{C}^n \bigwedge \mathbb{C}^n$$

by

$$A \bigwedge B(v \bigwedge w) = Av \bigwedge Bw + Bv \bigwedge Aw.$$

We leave it to the reader to check that if A, B are self adjoint (relative to $\langle ..., ... \rangle$) then so is $A \bigwedge B$. We note that if $g \in GL(n,\mathbb{C})$ then

$$\bigwedge^2 g(A \bigwedge B) \bigwedge^2 g^{-1} = gAg^{-1} \bigwedge gBg^{-1}.$$

Also, if A, B are symmetric then so is $A \bigwedge B$. This implies that we have for all $n \geq 3$ the $O(n,\mathbb{C})$ intertwining operator $S^2(\mathbb{C}^n) \to S^2(\bigwedge^2 \mathbb{C}^n)$ given by $A \mapsto A \bigwedge I$.

We also note that we have an $O(n,\mathbb{C})$ intertwining operator

$$Ric : S^2(\bigwedge^2 \mathbb{C}^n) \to S^2(\mathbb{C}^n)$$

defined by

$$\langle Ric(R)v, w \rangle = \sum_{i=1}^{n} \langle R(e_i \bigwedge v), e_i \bigwedge w \rangle .$$

Exercise. Show that $A \mapsto Ric(A \bigwedge I)$ defines an invertible linear transformation of $S^2(\mathbb{C}^n)$.

This implies that in general $S^2(\bigwedge^2 \mathbb{C}^n) \cong F^0 \bigoplus \mathcal{H}^2 \bigoplus ?$ as a representation of $O(n,\mathbb{C})$ where $\mathcal{H}^2$ is the representation of $O(n,\mathbb{C})$ on the trace 0 symmetric matrices. We note that this is also true over $\mathbb{R}$. In the case of $n = 3$ this gives the full decomposition over either $\mathbb{R}$ or $\mathbb{C}$. Furthermore, it also implies that the Ricci tensor completely determines the curvature tensor in dimension 3.

1.3 $n = 4$.

This is a very special case. Here we note that over $\mathbb{C}$ we have a homomorphism of $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ onto $SO(4,\mathbb{C})$ that is defined by observing that on $\mathbb{C}^2 \bigotimes \mathbb{C}^2$ the tensor product of the $SL(2,\mathbb{C})$-invariant skew-symmetric form defines a non-degerate symmetric form. The map is thus $(g, h) \mapsto g \bigotimes h$ and the kernel is $\{(1,1), (-1,-1)\}$. This also corresponds to the fact that $Spin(3) \cong$

$SU(2)\times SU(2)$. We also note that as a representation of $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$, $\bigwedge^2\mathbb{C}^2\bigotimes\mathbb{C}^2$ is equivalent with the adjoint representation. That is equivalent with

$$S^2(\mathbb{C}^2)\bigotimes\mathbb{C}\bigoplus\mathbb{C}\bigotimes S^2(\mathbb{C}^2).$$

We now use the general fact that

$$S^2(V\bigoplus W)\cong S^2(V)\bigoplus V\bigotimes W\bigoplus S^2(W).$$

This implies that

$$S^2(\bigwedge^2(\mathbb{C}^2\bigotimes\mathbb{C}^2))\cong S^2(S^2(\mathbb{C}^2))\bigotimes\mathbb{C}\bigoplus S^2(\mathbb{C}^2)\bigotimes S^2(\mathbb{C}^2)\bigoplus\mathbb{C}\bigotimes S^2(S^2(\mathbb{C}^2)).$$

Here in the tensor products the first $SL(2,\mathbb{C})$ acts on the first factor and the second on the second. We have already observed that the adjoint representation of $SL(2,\mathbb{C})$ is equivalent with $S^2(\mathbb{C}^2)$. This leads to a second interpretation of $SO(4)$. The adjoint representation of $SO(3,\mathbb{C})$ (or $SO(3)$) is equivalent with its standard action on $\mathbb{C}^3$ (or $\mathbb{R}^3$). By the above the adjoint group of $SO(4,\mathbb{C})$ (or $SO(4,\mathbb{R})$) has image $SO(3,\mathbb{C})\times SO(3,\mathbb{C})$ $(SO(3,\mathbb{R})\times SO(3,\mathbb{R}))$. This and the above imply that

$$S^2(\bigwedge^2\mathbb{C}^4)\cong S^2(\mathbb{C}^3)\otimes\mathbb{C}\oplus\mathbb{C}^3\otimes\mathbb{C}^3\oplus\mathbb{C}\otimes S^2(\mathbb{C}^3)$$

with the same interpretation of the tensor factors. We note that $S^2(\mathbb{C}^4)$ is equivalent with $\mathbb{C}^3\otimes\mathbb{C}^3\oplus\mathbb{C}$ (with the same decomposition over $\mathbb{R}$). Also (as in the case $n=3$) $S^2(\mathbb{C}^3)\cong\mathcal{H}_3^2\oplus F^0$ (here $\mathcal{H}_n^k$ denotes the space of spherical harmonics in n-variables of degree k) and Thus we have

$$S^2(\bigwedge^2\mathbb{C}^4)\cong F^0\oplus\mathcal{H}_4^2\oplus\left(\mathcal{H}_3^2\otimes\mathbb{C}\oplus\mathbb{C}\otimes\mathcal{H}_3^2\right)\oplus F^0.$$

We can interpret this in terms of highest weights as follows. We take the roots to be $\varepsilon_1-\varepsilon_2,\varepsilon_1+\varepsilon_2$ thus the basic highest weights are $\varpi_1=\frac{1}{2}(\varepsilon_1-\varepsilon_2),\varpi_2=\frac{1}{2}(\varepsilon_1+\varepsilon_2)$. One checks that the highest weight of $\mathcal{H}_4^2$ is $2(\varpi_1+\varpi_2)$ the highest weight of $\mathcal{H}_3^2\otimes\mathbb{C}$ is $2\varpi_1$ and that of $\mathbb{C}\otimes\mathcal{H}_3^2$ is $2\varpi_2$. We think of the first F^0 as contained in $S^2(\mathbb{C}^4)$ as discussed in the case $n=3$. We will now interpret the second F^0 term.

We note that using the Grassmann algebra multiplication we have a map

$$\bigwedge^2\mathbb{C}^4\oplus\bigwedge^2\mathbb{C}^4\to\bigwedge^4\mathbb{C}^4.$$

The even Grassmann algebra is commutative. Hence, this induces an intertwining operator

$$S^2(\bigwedge^2\mathbb{C}^4)\to\bigwedge^4\mathbb{C}^4.$$

This is the other F^0. On can show that the kernel of this map is exactly the space of curvature operators satisfying the Bianchi identities. We also note that this is true for all $n\geq 4$. That is, Grassmann multiplication induces a map

$$S^2(\bigwedge^2\mathbb{C}^n)\to\bigwedge^4\mathbb{C}^n$$

and the kernel of this map is the space of curvature operators satisfying the Bianchi identities. We therefore have

$$S^2(\bigwedge^2 \mathbb{C}^n) = S^2(\bigwedge^2 \mathbb{C}^n)_B \oplus \bigwedge^4 \mathbb{C}^n$$

where the sub-B indicates Bianchi. Lastly we look at $\left(\mathcal{H}_3^2 \otimes \mathbb{C} \oplus \mathbb{C} \otimes \mathcal{H}_3^2\right)$. The moment we will call this the Weyl part of the decomposition. Thus if we denote by the image of $S^2(\mathbb{C}^n)$ in $S^2(\bigwedge^2 \mathbb{C}^n)$ under the map $A \mapsto A \wedge I$ by $S^2(\bigwedge^2 \mathbb{C}^n)_{Ric}$ and the orthogonal complement in $S^2(\bigwedge^2 \mathbb{C}^n)_B$ of $S^2(\bigwedge^2 \mathbb{C}^n)_{Ric}$ by $S^2(\bigwedge^2 \mathbb{C}^n)_W$ then we have for all n the decomposition

$$S^2(\bigwedge^2 \mathbb{C}^n)_B = S^2(\bigwedge^2 \mathbb{C}^n)_{Ric} \oplus S^2(\bigwedge^2 \mathbb{C}^n)_W$$

and $S^2(\bigwedge^2 \mathbb{C}^n)_{Ric} \cong F^0 \oplus \mathcal{H}_n^2$.

1.4 $n = 5$.

In this case we consider the choice of simple roots $\varepsilon_1 - \varepsilon_2, \varepsilon_2$. The fundamental highest weights for $Spin(5, \mathbb{C})$ $(= B_2)$ are $\varpi_1 = \varepsilon_1$ and $\varpi_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)$. The highest weight of $\mathcal{H}_5^2$ is $2\varpi_1$ and $\wedge^4\mathbb{C}^5$ is equivalent with $\mathbb{C}^5$ as an $SO(5, \mathbb{C})$ representation. There is one additional property for all n that appears in this dimension. If F^Λ is an irreducible representation with highest weight Λ then if $v \neq 0$ is in the highest weight space then $v \otimes v$ has weight 2Λ. The cyclic space of $v \otimes v$ in $S^2(F^\Lambda)$ is an irreducible representation with highest weight 2Λ and this determines the full occurrence of $F^{2\Lambda}$ in $S^2(F^{2\Lambda})$. Thus since $\varepsilon_1 + \varepsilon_2$ is the highest weight of $\bigwedge^2 \mathbb{C}^n$ which is irreducible for $n \geq 5$ we have $F^{2(\varepsilon_1+\varepsilon_2)}$ occurs in $S^2(\bigwedge^2 \mathbb{C}^n)$. In the case of $n = 5$ we have $S^2(\bigwedge^2 \mathbb{C}^n)_{Ric} \cong F^0 \oplus F^{2\varpi_1}$ and $\wedge^4\mathbb{C}^5 \cong F^{\varpi_1}$ thus $F^{2(\varepsilon_1+\varepsilon_2)}$ must occur in $S^2(\bigwedge^2 \mathbb{C}^n)_W$. If we count dimensions (using the Weyl dimension formula for $F^{2(\varepsilon_1+\varepsilon_2)}$) we have

$$\dim(F^0 \oplus F^{2\varpi_1} \oplus F^{2(\varepsilon_1+\varepsilon_2)} \oplus F^{\varpi_1}) = 1 + 14 + 35 + 5 = 55$$

since $\dim S^2(\bigwedge^2 \mathbb{C}^n) = 55$ we see that (using $2(\varepsilon_1 + \varepsilon_2) = 4\varpi_2$) that

$$S^2(\bigwedge^2 \mathbb{C}^5)_{Ric} \cong F^0 \oplus F^{2\varpi_1},$$

$$S^2(\bigwedge^2 \mathbb{C}^5)_W \cong F^{2\varpi_2}$$

and

$$\wedge^4\mathbb{C}^5 \cong F^{\varpi_1}.$$

We note that all of these decompositions are defined over $\mathbb{R}$. Furthermore, these representations all extend trivially to $O(5) = \{\pm I\}SO(5)$ except for the last where we note that $\wedge^4\mathbb{C}^5 \cong \det \otimes \mathbb{C}^5$. In general the first fundamental weight for $SO(n, \mathbb{C})$ for $n \geq 5$ is always $\varpi_1 = \varepsilon_1$ and $S^2(\bigwedge^2 \mathbb{C}^n)_{Ric} \cong F^0 \oplus F^{2\varpi_1}$ for all $n \geq 5$ and the decomposition is invariant under conjugation.

1.5 $n = 6$.

In this case we take the simple roots to be $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_2 + \varepsilon_3$ then we have $\varpi_1 = \varepsilon_1$ (as pointed out in the case of $n = 5$), $\varpi_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3)$ and $\varpi_3 = \frac{1}{2}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$. Thus $2(\varepsilon_1+\varepsilon_2) = 2\varpi_2+2\varpi_3$. Also $\wedge^4\mathbb{C}^6 \cong \det \otimes \wedge^2\mathbb{C}^6$ as a representation of $O(2,\mathbb{C})$. We have from the above $S^2(\bigwedge^2 \mathbb{C}^6)_{Ric} \cong F^0 \oplus F^{2\varpi_1}$, $S^2(\bigwedge^2 \mathbb{C}^6)_W \cong F^{2\varpi_2+2\varpi_3} \oplus ?, \wedge^4\mathbb{C}^6 \cong F^{\varpi_2+\varpi_3}$ for $SO(6)$. As above we calculate dimensions of all of the known constituents as $1 + 20 + 84 + 15 = 120$. Since this is the dimension of $S^2(\bigwedge^2 \mathbb{C}^6)$ we see that the question mark is 0.

$$S^2(\textstyle\bigwedge^2 \mathbb{C}^6)_{Ric} \cong F^0 \oplus F^{2\varpi_1},$$

$$S^2(\textstyle\bigwedge^2 \mathbb{C}^6)_W \cong F^{2\varpi_2+2\varpi_3}$$

and

$$\wedge^4\mathbb{C}^6 \cong F^{\varpi_2+\varpi_3}.$$

Further, except for the last term the result is the same over $\mathbb{R}$ and for $O(n)$ (the last term must be tensored with det).

1.6 $n = 7$.

Here we take the simple roots to be $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3$. We have $\varpi_1 = \varepsilon_1, \varpi_2 = \varepsilon_1 + \varepsilon_2$ and $\varpi_3 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$. Here we get a new stabilization we have for all $n \geq 7$, $\bigwedge^2 \mathbb{C}^n \cong F^{\varpi_2}$. This also implies that $S^2(\bigwedge^2 \mathbb{C}^n)_W \cong F^{2\varpi_2} \oplus ?$. We now prove that the question mark is always 0 for $n \geq 7$. We take the simple roots as follows for $n = 2k$

$$\varepsilon_1 - \varepsilon_2, ..., \varepsilon_{k-1} - \varepsilon_k, \varepsilon_{k-1} + \varepsilon_k$$

and for $n = 2k + 1$

$$\varepsilon_1 - \varepsilon_2, ..., \varepsilon_{k-1} - \varepsilon_k, \varepsilon_k.$$

We first note that

$$\dim S^2(\textstyle\bigwedge^2 \mathbb{C}^n)_B = \dim S^2(\bigwedge^2 \mathbb{C}^n) - \dim \bigwedge^4 \mathbb{C}^n = \dfrac{n^4 - n^2}{12}.$$

One can check that $\dim F^{2\varpi_2} = \frac{(n+2)(n+1)n(n-3)}{12}$ for $n \geq 7$ (here one can use the Weyl dimension formula and do a direct calculation although on needs to look at the cases n odd and even separately) and $\dim(F^0 \oplus F^{2\varpi_1}) = \dim S^2(\mathbb{C}^n) = \frac{n(n+1)}{2}$. This clearly implies that the ? above is 0. We thus have in all cases $n \geq 7$

$$S^2(\textstyle\bigwedge^2 \mathbb{C}^n)_{Ric} \cong F^0 \oplus F^{2\varpi_1},$$

$$S^2(\textstyle\bigwedge^2 \mathbb{C}^n)_W \cong F^{2\varpi_2}.$$

The only part that is special to $n = 7$ is $\wedge^4\mathbb{C}^7$. Ignoring the det factor for the orthogonal groups we see that this representation is equivalent with $\wedge^3\mathbb{C}^7$ so has highest weight $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 2\omega_3$. So

$$\wedge^4\mathbb{C}^7 \cong F^{2\varpi_3}.$$

1.7 $n = 8, 9, 10.$

The Bianchi part is stable and is indicated in the discussion of the case $n = 7$. For $n \geq 8$ we have $\varpi_1 = \varepsilon_1, \varpi_2 = \varepsilon_1 + \varepsilon_2$. For $n = 8$ we have $\varpi_3 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)$ and $\varpi_4 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$. We note that the representations $F^{2\varpi_3}$ and $F^{2\varpi_4}$ are dual and since $F^{2\varpi_4}$ occurs in $\wedge^4\mathbb{C}^8$ (since $2\varpi_4$ corresponds to the weight of $\wedge^4 W$ of a maximal isotropic subspace of $\mathbb{C}^8$) so must $F^{2\varpi_3}$. One can then prove that

$$\wedge^4\mathbb{C}^8 \cong F^{2\varpi_3} \oplus F^{2\varpi_4}$$

Indeed, this observation is a special case of

$$\wedge^n\mathbb{C}^{2n} \cong F^{2\varpi_{n-1}} \oplus F^{2\varpi_n}$$

here we take the simple roots as in the discussion for $n = 7$.

For $n = 9$. The only term that hasn't stabilized is $\wedge^4\mathbb{C}^9$ in this case the highest weight is $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$ and this is $2\varpi_4$.

Finally, for $n = 10$ highest weight of $\wedge^4\mathbb{C}^{10}$ is $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 = \varpi_4 + \varpi_5$.

1.8 $n \geq 11.$

For $n \geq 11$ we have

$$S^2(\bigwedge^2 \mathbb{C}^n) \cong F^0 \oplus F^{2\varpi_1} \oplus F^{2\varpi_2} \oplus F^{\varpi_4}$$

this is because ϖ_4 is (finally) equal to $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4$. Each of these representations is the complexification of an irreducible representation of $SO(n)$. So the same decomposition is true over $\mathbb{R}$. The types are given as follows

$$S^2(\bigwedge^2 \mathbb{R}^n)_{Ric} \cong F^0 \oplus F^{2\varpi_1},$$

$$S^2(\bigwedge^2 \mathbb{R}^n)_W \cong F^{2\varpi_2},$$

$$\wedge^4\mathbb{R}^n \cong F^{\varpi_4}.$$