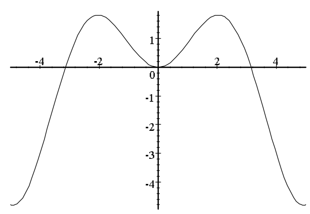
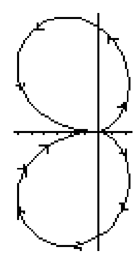
Solutions for the Midterm Fall 1999

1. (a) We note that the parametric curve $x = t\sin(t)\cos(t), y = t\sin(t)\sin(t)$ corresponds to considering in polar coordinates. We should thus graph r as a function of θ and get



The three x-intercepts are at $-\pi, 0, \pi$ the part that we need is from $-\pi$ to π. Tracing out the curve we have



(b) $\frac{dx}{dt} = \sin(t)\cos(t) + t\cos(t)^2 - t\sin(t)^2$, $\frac{dy}{dt} = \sin(t)^2 + 2t\cos(t)\sin(t)$ so

$$(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2$$

$$\begin{aligned} = \quad & \sin(t)^2\cos(t)^2 + 2t\sin(t)\cos(t)(\cos(t)^2 - \sin(t)^2) + t^2(\cos(t)^2 - \sin(t)^2)^2 \\ & \sin(t)^2\sin(t)^2 + 4t\cos(t)\sin(t)\sin(t)^2 + 4t^2\cos(t)^2\sin(t)^2. \end{aligned}$$

If we add the corresponding terms we get

$$\sin(t)^2 + 2t\cos(t)\sin(t) + t^2.$$

Thus the arc length is given by the integral

$$\int_{-\pi}^{\pi} \sqrt{\sin(t)^2 + 2t\cos(t)\sin(t) + t^2}dt.$$

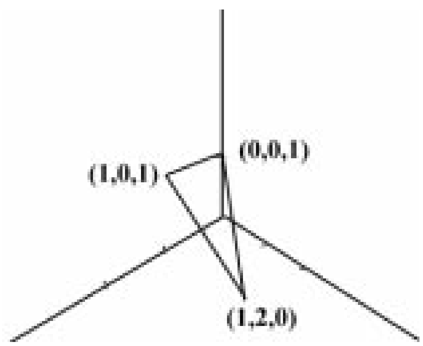
(c) $f(\frac{\pi}{4}) = (\frac{\pi}{4})(\frac{1}{\sqrt{2}})^2 = \frac{\pi}{8} = g(\frac{\pi}{4})$. $\frac{df}{dt}(\frac{\pi}{4}) = \frac{1}{2}$, $\frac{dg}{dt}(\frac{\pi}{4}) = \frac{1}{2} + \frac{\pi}{4}$. The tangent line is therefore given parametrically as $x = \frac{\pi}{8} + \frac{t}{2}$, $y = \frac{\pi}{8} + (\frac{\pi+2}{4})t$.

2. A solution is the unit vector in the direction of $\mathbf{a} \times \mathbf{b}$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 5\mathbf{i} - 5\mathbf{j} - 5\mathbf{k} = 5\langle 1, -1, -1 \rangle .$$

So the vector $\frac{\langle 1,-1,-1 \rangle}{\sqrt{3}}$ is a solution (the only other is the negative of this one).

3.The triangle can be pictured as follows:

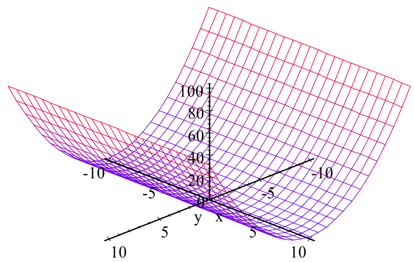


Now $\overrightarrow{PQ} = \langle 0, -2, 1 \rangle$ and $\overrightarrow{PR} = \langle -1, -2, 1 \rangle$ thus $\overrightarrow{PQ} \times \overrightarrow{PR} = -\mathbf{j} - 2\mathbf{k}$. The area is thus half the area, $|-\mathbf{j} - 2\mathbf{k}| = \sqrt{5}$ of the corresponding parallelogram. The answer is therefore $\frac{\sqrt{5}}{2}$.

4. The plane has as a normal vector $\overrightarrow{PQ} \times \overrightarrow{PR} = -\mathbf{j} - 2\mathbf{k}$ and contains the point $(1, 2, 0)$ thus an equation for the plane is $-y - 2z = -2$ that is $y + 2z - 2 = 0$. If $\mathbf{n} = \mathbf{j} + 2\mathbf{k}$ then the distance from the plane to the point (1,-1,1) is

$$\frac{|1 + 2 - 2|}{|\mathbf{n}|} = \frac{1}{\sqrt{4+1}} = \frac{1}{\sqrt{5}}.$$

5. The traces with y fixed are exactly the same and they are all the parabola in the x, z plane given by $z = x^2 + 2$. Thus the surface is gotten by sliding this parabola along the y-axis. Here is a picture


100
80
60
40
20
0
-10
-5
5
10
y
x
-10
-5
5
10