Practice Final Examination Math 21C Fall 1999

1.(20 Points)Give parametric equations for the line of intersection of the plane $x+3y+z+1=0$ and the plane containing the points $(1,1,0),(1,1,1),(2,0,1)$.

Label the points P,Q,R respectively. Then $\overrightarrow{QP}=\langle 0,0,-1\rangle=-\mathbf{k}$ and $\overrightarrow{QR}=\langle 1,-1,0\rangle=\mathbf{i}-\mathbf{j}$. So $-\mathbf{k}\times(\mathbf{i}-\mathbf{j})=-\mathbf{j}-\mathbf{i}$. Thus a normal to the plane containing P,Q,R is $\langle 1,1,0\rangle$. $\overrightarrow{0P}\cdot\langle 1,1,0\rangle=2$. Thus the plane is givne by $x+y=2$. The intersection of the two planes is the set of points with

$$x+y=2$$

$$x+3y+z=-1$$

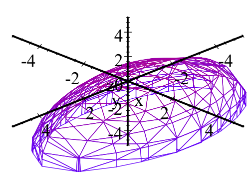
We can write this as $y=2-x$ and $z=-1-x-3y=-1-x-3(2-x)=-7+2x$. We therefore get a parametrization by taking $x=t,y=2-t,z=-7+2t$.

2. (20 Points)Sketch the surface defined by the equation $x^2+4y^2+z^2+2x+4y+9z+3=0$.

We can complete the squares getting $(x+1)^2+(2y+1)^2+(z+\frac{9}{2})^2-1-1-\frac{81}{4}+3=0$. That is

$$(x+1)^2+4(y+\frac{1}{2})^2+(z+\frac{9}{2})^2=\frac{77}{4}.$$

The surface is thus an ellipsoid centered at $(-1,-\frac{1}{2},-\frac{9}{2})$. The "top part" looks like:



3. (25 Points)Which of the following functions is continuous at $(0,0)$? (You must give reasons to get full credit.)

a) $f(x,y) = \frac{xy}{1+x^2y^2}$.

We note that $1 + x^2y^2$ is continuous at $(0,0)$ with value 1 and xy is coninuous at $(0,0)$ with value 0. Thus the limit is $\frac{0}{1} = 0$. This is the value so the function is continuous.

b) $f(x,y) = \frac{x^2+2y^2}{x^2+y^2}$ if $(x,y) \neq 0$, $f(0,0) = 0$.

If $y = 0$ then $f(x,0) = \frac{x^2}{x^2} = 1$ if $x \neq 0$. Thus 0 cannot be the limit as $(x,y) \to (0,0)$. Thus the function is not continuous

c) $f(x,y) = \frac{x^4+3y^3}{x^2+y^2}$ if $(x,y) \neq 0$, $f(0,0) = 0$.

If we test along lines $x = ta, y = tb$ with $(a,b) \neq 0$ then $f(ta,tb) = \frac{t^4a^4+3t^3b^3}{t^2a^2+t^2b^2} = t\left(\frac{ta^4+3b^3}{a^2+b^2}\right) \to 0$ as $t \to 0$. This indicates that the function might be continuous. We use $x^2 \leq x^2 + y^2$ and $|y| \leq \sqrt{x^2+y^2}$. Thus

$$
\begin{aligned}
0 &\leq |f(x,y)| = \frac{|x^4+3y^3|}{x^2+y^2} \leq \frac{x^4+3|y|^3}{x^2+y^2} \\
&\leq \frac{(x^2+y^2)^2 + 3(x^2+y^2)^{\frac{3}{2}}}{x^2+y^2} = (x^2+y^2) + 3(x^2+y^2)^{\frac{1}{2}}
\end{aligned}
$$

Since the expression at the far right goes to 0 as $(x,y) \to 0$ the limit is 0. Hence the function is continuous at 0. (A problem like this would have alot of partial credit, for example if you said that the numerator goes to 0 faster than the denominator you would get almost full credit.)

4. (20 Points)Give an equation for the tangent plane of the surface $z = x^2 - y^2 + 4$ at the point $(2,3,-1)$.

$f(x,y) = x^2 - y^2 + 4$. Thus $f_x = 2x$, $f_y = -2y$. Hence $f_x(2,3) = 4$, $f_y(2,3) = -6$. Thus an equation for the tangent plane is $z = -1 + 4(x - 2) - 6(y-3) = 4x - 6y + 9$.

5. (15 Points)A mountain climber is climbing a "mountain" that is given by the equation $z = x^2 + y^2$. Using his compass he knows that he is at the point $(1,1,2)$ in which direction should he head to ascend the fastest?

The direction of greatest increase is the direction of the gradient. If $f(x,y) = x^2 + y^2$ then $\nabla f(x,y) = \langle 2x, 2y \rangle$. Hence if $x = y = 1$ then $\nabla f(1,1) = \langle 2,2 \rangle$. The direction is therefore the unit vector $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. This is the direction (north-east).

6. (20 Points)Classify the critical points of the function $f(x,y) = x^4 - 2x^2 + y^4 - 2y^2$.

$f_x(x,y) = 4x^3 - 4x, f_y(x,y) = 4y^3 - 4y$. Thus the critical points are the pairs (x,y) with $x^3 = x, y^3 = y$. The solutions to $v^3 = v$ are $v = 0$ and $v = \pm 1$.Thus the critcal points are $(0,0), (0,\pm 1), (\pm 1, 0), (1, \pm 1), (\pm 1, 1)$. We now apply the second derivative test to find $f_{xx}(x,y) = 12x^2 - 4, f_{xy}(x,y) = 0, f_{yy}(x,y) = 12y^2 - 4$. Thus

$$f_{xx}f_{yy} - f_{xy}^2 = (12x^2 - 4)(12y^2 - 4)$$

The values at the critical points are 16 for $(0,0)$, -32 for each of $(\pm 1, 0), (0, \pm 1)$, and 64 for $(\pm 1, \pm 1)$. Thus $(0,\pm 1)$ and $(\pm 1, 0)$ are saddle points. $f_{xx}(0,0) = -4, f_{xx}(\pm 1, \pm 1) = 8$. Thus $(0,0)$ is a local maximum and $(\pm 1, \pm 1)$ are local minima.

7. (20 Points)Find the maximum and minimum value of the function $f(x,y)$ of the previous problem in the set $x^2 + y^2 \leq 16$.

The minimum and maximum must occur at critical points if they occur in the set $x^2 + y^2 < 16$. If either occurs in the set $x^2 + y^2 = 16$ then we should try to find it using Lagrange multipliers. We first do the Lagrange multipliers. Set $g(x,y) = x^2 + y^2$. Then we are looking for $\nabla f = \lambda \nabla g$. That is $4x^3 - 4x = 2\lambda x, 4y^3 - 4y = 2\lambda y$. Thus we have

$$\begin{aligned} 2x^3 &= (2+\lambda)x \\ 2y^3 &= (2+\lambda)y \end{aligned}$$

If $x = 0$ then $y^2 = 16$ so $y = \pm 4$. This gives the points $(0,4)$ and $(0,-4)$ (so far). If $y = 0$ then we get $(4,0)$ and $(-4,0)$. If neither x nor y is 0 then we can divide $2x$ from both sides of the first equation and $2y$ from both sides of the second getting

$$\begin{aligned} x^2 &= \frac{2+\lambda}{2} \\ y^2 &= \frac{2+\lambda}{2} \end{aligned}$$

Thus $x^2 = y^2$. So $2x^2 = 16$ or $x = \pm 2\sqrt{2}$. This gives four more points

$$(2\sqrt{2}, 2\sqrt{2}), (2\sqrt{2}, -2\sqrt{2}), (-2\sqrt{2}, 2\sqrt{2}), (-2\sqrt{2}, -2\sqrt{2}).$$

We thus see that the extreme values of f in the set $x^2+y^2=16$ are in the set $f(0,4), f(4,0), f(2\sqrt{2},2\sqrt{2})$ taking into account that changing the sign of x or y doesn't change the value of $f(x,y)$. The numbers are 48, 96 and 224. In the set $x^2+y^2<16$ there are 4 critical points 2 of which are saddle points. The other 2 are the local maximum $(0,0)$ giving the value 0 and the local minimum $(1,1)$ giving the value -2. We therefore have the maximum is 96 and the minimum is -2.

8. (25 Points)Calculate the following double integrals.

a) The integral of $f(x,y)=\cos(x)\cos(y)-\sin(x)\sin(y)$ over the rectangle $1\le x\le 3,\ 0\le y\le 1$.

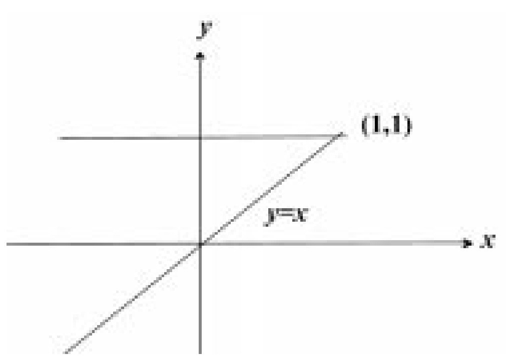
b) The integral of $f(x,y)=xy$ over the triangle with vertices $(0,0),(0,1),(1,1)$.

c) The integral of $f(x,y)=x+y$ over the region between the curves $y=x^2+1, y=2-x^2$.

a) $\int_0^1\int_1^3(\cos(x)\cos(y)-\sin(x)\sin(y))dxdy =$
$\int_0^1\int_1^3(\cos(x)\cos(y)dxdy - \int_0^1\int_1^3(\sin(x)\sin(y)dxdy =$
$\sin(x)|_1^3\sin(y)|_0^1 - \cos(x)|_1^3\cos(y)|_0^1 = (\sin(3)-\sin(1))\sin(1)+(\cos(3)-\cos(1))(\cos(1)-1) =$
$\sin(3)\sin(1)-\sin(1)\sin(1)+\cos(3)\cos(1)-\cos(3)-\cos(1)\cos(1)+\cos(1) =$
-$\cos(4)+\cos(2)+\cos(3)-\cos(1)$.

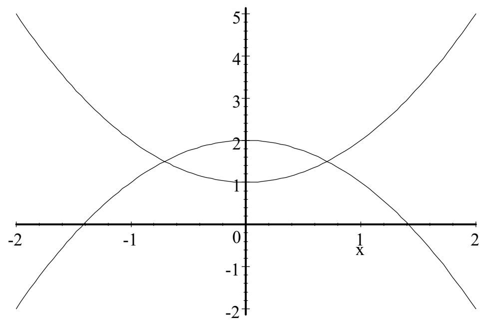
Alternatively, $\cos(x)\cos(y)-\sin(x)\sin(y)=\cos(x+y)$. So the integral is
$\int_0^1\int_1^3(\cos(x+y)dxdy = \int_0^1(\sin(x+y)|_{x=1}^{x=3})dy = \int_0^1(\sin(3+y)-\sin(1+y))dy = -\cos(3+1)+\cos(3+0)+\cos(2)-\cos(1)$.

b) The region can be graphed as follows:



This is most easily described as a domain of type II. The set of all (x, y) with $0 \le y \le 1$ and $0 \le x \le y$. Thus we are required to integrate
$\int_0^1 \int_0^y xydxdy = \int_0^1 \frac{x^2}{2}|_0^y ydy = \int_0^1 \frac{y^3}{2}dy = \frac{1}{8}$.

c) We are looking at the region between the two curves.



The curves intersect at $(\frac{1}{\sqrt{2}}, \frac{3}{2})$ and $(-\frac{1}{\sqrt{2}}, \frac{3}{2})$. If $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$ then $1 + x^2 \le 2 - x^2$. Thus we have a domain of type I. The integral is given as
$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{1+x^2}^{2-x^2} (x + y)dydx = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (x((2 - x^2) - (1 + x^2)) + \frac{(2-x^2)^2}{2} - \frac{(1+x^2)^2}{2})dx$
$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (x - 2x^3 + \frac{3}{2} - 3x^2)dx = \sqrt{2}$.

9. (15 Points)Calculate the integral of $f(x, y) = e^{x^2+y^2}$ over the set $x^2 + y^2 < 16$. (Hint: Try polar coordinates.)

In polar coordinates $f(r\cos\theta, r\sin\theta) = e^{r^2}$. In the domain $0 \le r \le 4$ and $0 \le \theta \le 2\pi$. Thus we are looking at

$$\int_0^{2\pi} \int_0^4 e^{r^2} rdrd\theta.$$

If we substitute $u = r^2$ then $du = 2rdr$ and u runs from 0 to 16 thus the inner inegral is

$$\frac{1}{2}\int_0^{16} e^u du = \frac{e^{16} - 1}{2}.$$

Now integrating in θ we get $2\pi\frac{e^{16}-1}{2} = \pi\,(e^{16} - 1)$.

10. (20 Points)Calculate the following triple integrals.

a) The integral of $f(x,y,z) = 2x - 3y + z$ over the box $0 \leq x \leq 1, -1 \leq y \leq 1, 1 \leq z \leq 3$.

b) The integral of $f(x,y,z) = xyz$ over the three dimensional domain given by $x^2 + y^2 < 4$, $x \geq 0, y \geq 0, 0 \leq z \leq 1$.

a) Is the iterated integral

$$\int_1^3 \int_{-1}^1 \int_0^1 (2x - 3y + z)dxdydz = \int_1^3 \int_{-1}^1 (1 - 3y + z)dydz = \int_1^3 3zdz = 12.$$

b) The domain, D, in (x,y) space given by $x^2 + y^2 = 4$, $x \geq 0, y \geq 0$ is given in polar coordinates as $x = r\cos\theta$, $y = r\sin\theta$ with $0 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$. The domain in (x,y,z) space is type I with the upper function 1 and the lower 0. Thus the integrals is (the last formula involves conversion to polar coordiantes

$$\int\int_D (\int_0^1 xyzdz)dA = \int\int_D \frac{xy}{2}dA = \int_0^{\frac{\pi}{2}} \int_0^2 \frac{r^2\cos\theta\sin\theta}{2} rdrd\theta.$$

Now $\sin(2\theta) = 2\cos\theta\sin\theta$. Thus the integral that we are calculating is

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \int_0^2 r^3 \sin(2\theta)drd\theta.$$

Now the integral of $\sin(2\theta)$ is $\frac{-\cos(2\theta)}{2}$ so we have

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \int_0^2 r^3 \sin(2\theta)drd\theta = \frac{1}{8}\int_0^{\frac{\pi}{2}} r^4|_0^2 \sin(2\theta)d\theta = 2\int_0^{\frac{\pi}{2}} \sin(2\theta)d\theta = -\cos 2\theta|_0^{\frac{\pi}{2}} = 1.$$