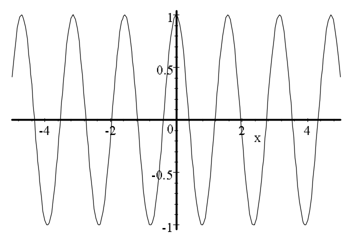
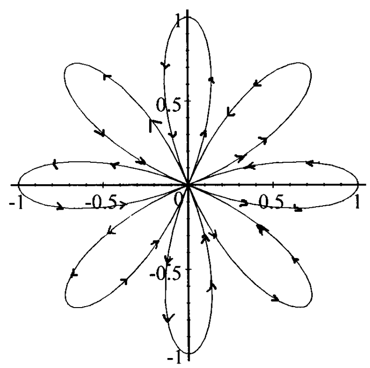
Solutions for the practice midterm Math 21C

1. We may look at this curve as a polar plot with $r = \cos(4\theta)$. The method we used to do this was to first plot the function $\cos(4x)$.



The part of interest to us is from 0 to 2π. From 0 to $\frac{\pi}{8}$ the graph goes from $(1,0)$ to $(0,0)$ above the x axis (on your paper you should indicate this with arrows) as we did in class. From $\frac{\pi}{8}$ to $\frac{\pi}{4}$ the value of r is negative Thus it traces part of the curve in the negative x, y quadrant. Here is the graph



(b) We have

$$\begin{aligned} \frac{df}{dt} &= -4\sin(4t)\cos(t) - \cos(4t)\sin(t) \\ \frac{dg}{dt} &= -4\sin(4t)\sin(t) + \cos(4t)\cos(t). \end{aligned}$$

This implies that

$$\begin{aligned} &\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 \\ = \; & 16\sin(4t)^2\cos(t)^2 + 8\sin(4t)\cos(t)\cos(4t)\sin(t) + \cos(4t)^2\sin(t)^2 + \\ & 16\sin(4t)^2\sin(t)^2 - 8\sin(4t)\sin(t)\cos(4t)\cos(t) + \cos(4t)^2\cos(t)^2. \end{aligned}$$

We note that the middle terms cancel and using the fact that $\cos(\theta)^2 + \sin(\theta)^2 = 1$ we have

$$\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 = 16\sin(4t)^2 + \cos(4t)^2 = 15\sin(4t)^2 + 1.$$

The arc length is therefore given by

$$\int_0^{2\pi} \sqrt{15\sin(4t)^2 + 1}dt.$$

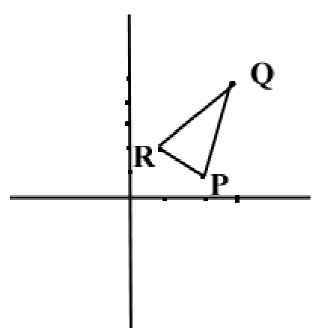
(c) From the picture if $t = \frac{\pi}{4}$ then $f(t) = \cos(\pi)\cos(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$ similarly we have $g(t) = -\frac{1}{\sqrt{2}}$. Now $\left(\frac{df}{dt}(\frac{\pi}{4}), \frac{dg}{dt}(\frac{\pi}{4})\right) = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Thus the line is given parametrically by

$$x = \frac{-1+t}{\sqrt{2}}, y = \frac{-1-t}{\sqrt{2}}.$$

2. a) $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\frac{3\pi}{4}) = 3 \times 7 \times \frac{-1}{\sqrt{2}} = -\frac{21}{\sqrt{2}}$.
$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\frac{3\pi}{4}) = 3 \times 7 \times \frac{1}{\sqrt{2}} = \frac{21}{\sqrt{2}}$.
b) $\mathbf{a} \cdot \mathbf{b} = 1 \cdot 2 + 3 \cdot (-1) + 1 \cdot 1 = 0$ and

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ 2 & -1 & 1 \end{vmatrix} = 4\mathbf{i} - (-1)\mathbf{j} - 6\mathbf{k} = 4\mathbf{i} + \mathbf{j} - 6\mathbf{k}.$$

3. A picture of the triangle is



The area of the parallelogram with adjacent sides $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $|\overrightarrow{PQ} \times \overrightarrow{PR}|$ after we think of $\langle x, y\rangle$ as $\langle x, y, 0\rangle$. Thus the area of the given triangle is 1/2 of that. We are thus looking at $\frac{1}{2}|\langle 2,3,0\rangle \times \langle 1,2,0\rangle| = \frac{1}{2}|\mathbf{k}| = \frac{1}{2}$.

4. The line is given by $\langle x, y, z \rangle = \langle 1, 1, -1 \rangle + t\langle 2, 0, 1 \rangle$. Thus $x = 1 + 2t, y = 1, z = -1 + t$ give parametric equations for the line. Every pair of vectors on this line have displacement vector a multiple of $\langle 2, 0, 1 \rangle$. The displacement vector $\overrightarrow{PQ} = \langle 2, -2, 3 \rangle$. Thus a normal vector for the plane is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}.$$

Thus an equation for the plane is $2x - 4y - 4z = 2$.

5. The curves $c = k$ are the circles $x^2 + y^2 = k + 2$. We note that $z \geq 2$. The vertical traces are parabolas. Thus the surface looks like: