Problems for Math 204 Fall 1999

1. Let B be a ring and let A be a subring such that B is integral over A. Show that if $\mathfrak{m} \subset A$ is a maximal ideal in A and if $\mathfrak{p} \subset B$ is prime and such that $\mathfrak{p} \supset \mathfrak{m}$ then $\mathfrak{p}$ is maximal in B.

2. Show that if K is a field and if $|...|$ is a norm (valuation in the terminology of the book) on K then $|...|^a$ is a norm for $0 < a \leq 1$. Show that if $\|...\|$ is a norm on $\mathbb{Q}$ equivalent with $|x| = \sqrt{x^2}$ then $\|...\| = |...|^a$ with $0 < a \leq 1$.

3. Let K be a number field (i.e. finite algebraic extension of $\mathbb{Q}$) and and let R be the integral closure of $\mathbb{Z}$ in K. Show that if $\mathfrak{p}$ is a prime in R and if $x \in \mathfrak{p}$ is such the $norm_{K/Q}(x)\mathbb{Z} = norm(\mathfrak{p})$ then $\mathfrak{p} = xR$.

4. Exercise 1 page 73 in the text.

5. Let $K = \mathbb{C}(x)$, the rational functions in one variable with coefficients in $\mathbb{C}$. Show that any norm on K is trivial when restricted to $\mathbb{C}$.

6. Let p be a prime and let $\theta \neq 1$ be a p-th root of unity. If n is not congruent to 0 mod p then $\left(\frac{n}{p}\right)$ denotes the Legendre symbol (value 1 if $n \equiv m^2 \bmod p$, -1 if not). Set $\alpha = \sum_{1 \leq n \leq p-1} \left(\frac{n}{p}\right) \theta^n$. Show that $\alpha^2 = \left(\frac{-1}{p}\right) p$. (Hint: $\sum_{1 \leq n \leq p-1} \left(\frac{n}{p}\right) = 0$.)

7. One knows that if p is a prime less than 23 and if θ is as in problem 6. then $\mathbb{Z}[\theta]$ is a PID. However, in this problem you are asked to show using only what we know from this course that if $p = 13$ then the order of the ideal class group is at most 6.