

Katherine Minakov

### René Descartes' *La Géométrie*

René Descartes was born on March 31, 1596 in Touraine, France. He enrolled at the Jesuit College of La Flèche in Anjou when he was only eight years old and continued studying there for four years until 1608. During these years as a student, Descartes studied subjects such as logic, traditional Aristotelian philosophy and the Classics. However, Descartes was able to find thorough satisfaction only in the subject of mathematics. His appreciation for the discipline and its use of concise logic found its way into his books such as *Discours de la méthode*, which bases all its philosophical axioms on mathematics rather than Aristotelian logic. Descartes was a Renaissance man in the sense that he was interested in many different fields of study that varied from philosophy to physics. His legacy includes a large number of volumes in these different areas whose contents have contained many innovative ideas. For example, as a philosopher, Descartes coined the famous phrase, “cogito, ergo sum” or “I think, therefore I am” and as a physicist he wrote the appendix *La dioptrique* which is the first publication of the law of refraction (Boyer 336). He was also the founder of modern biology and, in his spare time, a mathematician (Kline 304). In 1637 he published a very important book called *Discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences* (*Discourse on method to understand its reason well and to find truth in the sciences*.) The book contains three appendixes, namely: *La Géométrie*, *La Dioptrique*, and *Les Météores*. In *La Géométrie* Descartes makes his contributions in mathematics the greatest of which is his founding of analytic geometry.

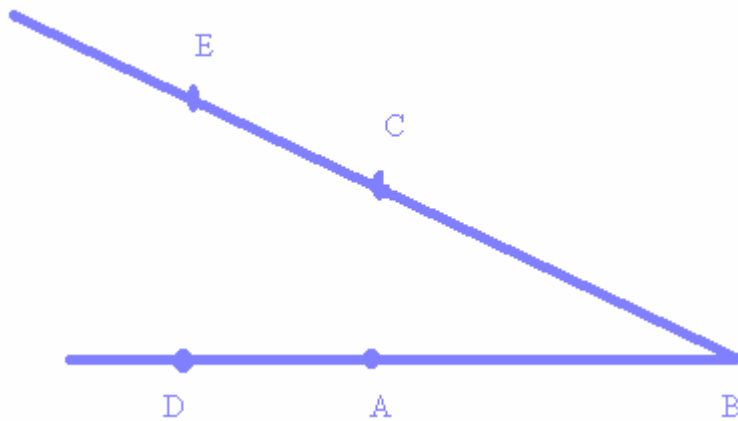
*La Géométrie* consists of three books. In a brief summary, Book I consists of Descartes' representation of numbers, which are explained through a combination of geometry and algebra. He also gives a geometric interpretation of the solutions of quadratic equations. The book concludes with a statement of the Pappus problem for four lines as well as an outline for its solution. In Book II, Descartes classifies curves by the highest power of  $x$ , which he refers to as the degree of the polynomial. He also solves the Pappus problem for four lines in great detail through use of what we call today, analytic geometry. Finally, in Book III Descartes constructs curves of degree three and higher and he also states what he believes to be a universal law for finding the roots of a polynomial. An important fact to note about Descartes' *La Géométrie* is that he establishes all the mathematical knowledge that the reader must have prior to giving a solution to a particular problem. Therefore, the reader must only have a basic knowledge of lines as a prerequisite for the book, since all other facts will be established by Descartes.

Descartes opens the work with the following phrase: "Tout les problèmes de Géométrie se peuvent facilement reduire a tells termes, qu'il n'est besoin par après que de connaitre la longuer de quelque lignes droites, pour les construire" (Descartes 297). In other words, all problems of geometry can easily be reduced to terms such that only a knowledge of the lengths of certain straight lines need be known for their construction. Thus Descartes proposes that all geometric problems can be simplified to forms that should be accessible to all people who know how to work with lines. Furthermore, he points out that working with these lines is not different from the basic procedures of arithmetic, namely addition, subtraction, multiplication, division and the extraction of

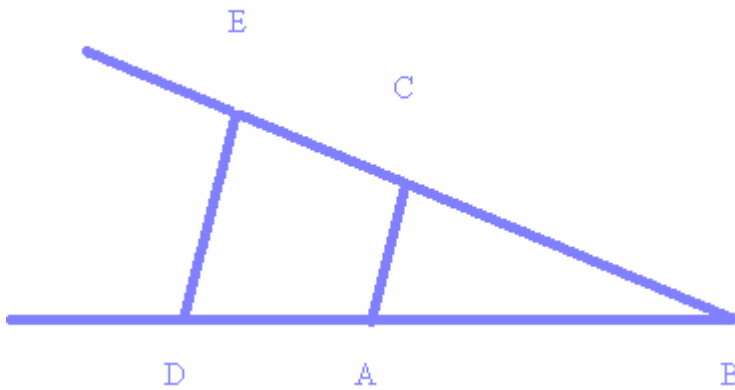
roots (Descartes 297). So the sum or difference of lines is simply obtained by adding a line to another or subtracting a line from another line:



However, the processes of multiplication, division and the extraction of roots are more complicated. In defining multiplication and division, Descartes introduces the concept of unity so that he may relate a line segment to a number so all lines can be thought of as some number 'a' times the unit. Therefore these lines can be thought of as quantities which is the way Euclid perceived them in his *Elements*. To derive the product or quotient, Descartes sets up ratios between the given lines and unity. This process is clearly illustrated by his diagram:



Descartes lets AB equal unity, and wishes to find the product  $BD \cdot BC$ . In order to do this, he extends a line from E to D and a line from C to A and these lines are parallel.

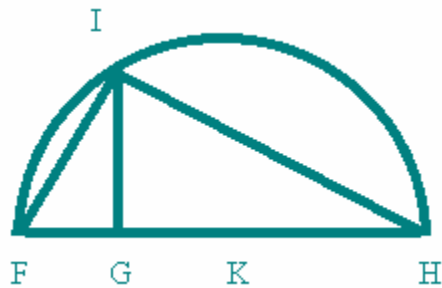


Since the triangles DBE and ABC are similar,  $EB:DB = CB:AB$  where  $a:b$  is  $a/b$ , or, Euclid's definition of a ratio. If we let  $AB=1$  then we get the result that  $EB*DB=1$ .

In order to extract roots, the following diagram is given by Descartes:



Descartes sets  $FG$  equal to unity, denotes  $K$  as the midpoint of  $FH$  and draws  $GI$  perpendicular to  $FH$ . One can now connect the points  $F$ ,  $I$ , and  $H$  to construct a triangle which is a right triangle since it is inscribed in a circle.



Since  $FIH$  is a right triangle the following identities hold:

$$(FI)^2 + (IH)^2 = (FH)^2$$

$$(FG)^2 + (GI)^2 = (FI)^2$$

$$(GH)^2 + (GI)^2 = (IH)^2$$

and so, we can make these calculations:

$$(FH)^2 = (FI)^2 + (IH)^2$$

$$\begin{aligned}
(1+(GH))^2 &= (FI)^2 + (HI)^2 \\
(GH)^2 + 2(GH) + 1 &= (FI)^2 + (HI)^2 = (FG)^2 + (GI)^2 + (GH)^2 + (GI)^2 \\
2(GH) + 1 &= (FG)^2 + 2(GI)^2 \\
2(GH) &= 2(GI)^2 \\
(GH) &= (GI)^2, \text{ hence it follows that } (GI) \text{ is the square root of } (GH).
\end{aligned}$$

So far, Descartes has shown a geometric interpretation of simple algebra. As mentioned previously, a line AB may be simply denoted as 'a' and so when two lines AB, CD are added, their sum can be denoted as a + b. Likewise, we may think of the subtraction, multiplication and division of lines as a-b, ab, a/b. Descartes' geometric interpretation of algebra is also of great value when considering quantities such as  $a^n$ . Before Descartes time, the quantities  $a^2$  and  $a^3$  were associated with the area and volume of an object of side length 'a,' so it really did not make much sense to talk of things of quantities such as  $a^4$  because there was no geometric representation for them. However, by introducing his multiplication, Descartes can think of  $a^2$  as a unit less quantity (since we divide by the unity when constructing the ratio) and can construct it to be a line rather than an area. So it is possible to write  $a^3 = a \cdot a^2$  where a and  $a^2$  are both lines which is very significant because it allows us to think of these quantities independent of areas and volumes. Based on this new concept,  $a^4$  becomes  $a \cdot a^3$  and makes sense as a number.

Similarly, Descartes denotes the square root of a quantity  $a^2 + b^2$  as  $(a^2 + b^2)^{1/2}$  where all components of a line must be of dimension 2 if the square root is to be extracted.

Furthermore, Descartes legitimizes taking the cube root of a quantity such as  $a^2 b^2 - b$  by stating that we can think of  $a^2 b^2$  as divided once by unity and b as multiplied twice by unity so that each component will in fact be of degree 3. Thus the flexibility of unity allows one to "substitute homogeneity in thought for homogeneity in form" (Boyer 338). It is important to note that the way that Descartes interprets square roots and quantities of

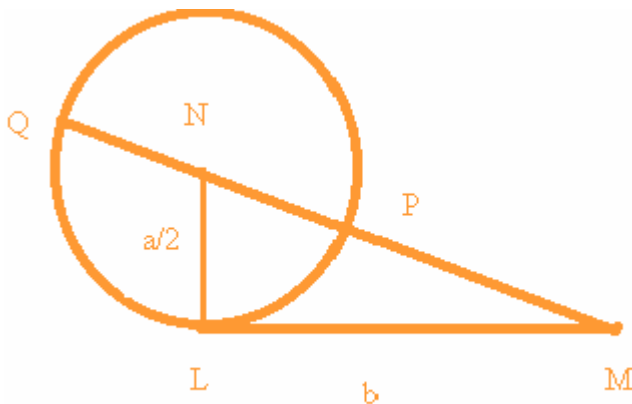
the form  $a^n$  is exactly the way we think of them today. Also, Descartes' notation is the notation that is used nowadays because he is the founder of modern mathematical notation.

After having established these principles, Descartes explains his method for working on geometric problems. One must assume that the conclusion is given and then name all lines that are necessary for the construction of the solution. The relationships between the lines need be established in the form of equations. If there are not as many equations as there are lines, then there is an infinity of solutions to the problem. Otherwise, by solving simultaneous equations it should be possible to derive the answer (Descartes 11).

Furthermore, Descartes states that if a problem can be solved by simple geometry so that only circles and lines are used to solve it, then the solution can be derived from an equation of at most degree 2. (Because a line will intersect with a circle in at most two points.) He then considers the three possible cases for the quadratic equation:

1.  $z^2 = az + b^2$
2.  $z^2 = -az + b^2$
3.  $z^2 = az - b^2$ .

He constructs the solution for the first equation by using the following diagram:

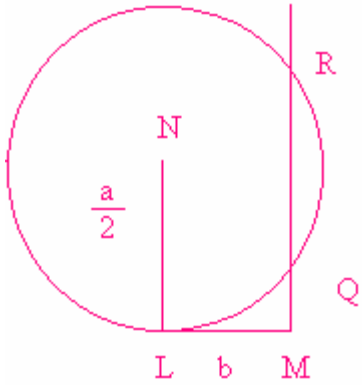


The triangle NLM is a right triangle where  $NM = (a^2/4 + b^2)^{1/2}$  and  $QM = a/2 + (a^2/4 + b^2)^{1/2}$

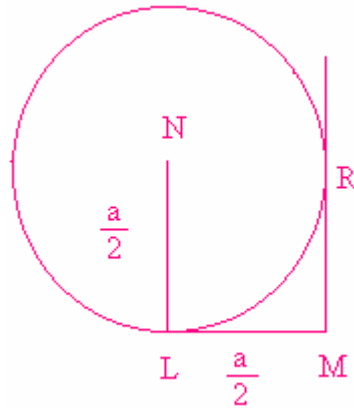
then  $QM^2 = a(QM) + b^2$  so this shows that  $QM = z$  is the solution to the quadratic equation.

Similarly, it can be shown that if one is trying to solve the second quadratic equation, then  $z = PM$  is a solution. Finally, to solve the last quadratic equation, three cases need to be considered:

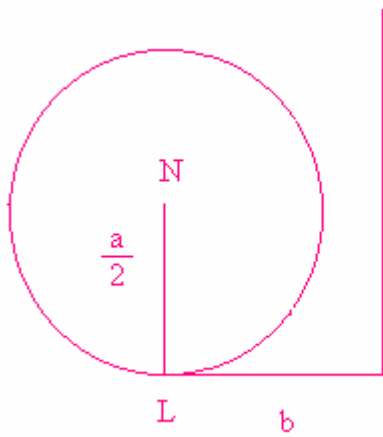
1.



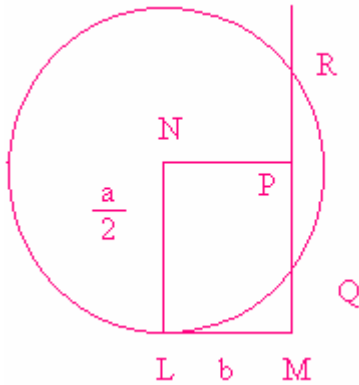
2.



3.



In the first case,  $b < a/2$ . Descartes claims that either  $MQ$  or  $MR$  may equal  $z$ . This can be shown by considering  $RM = RP + PM$ :



Since  $NP=b$  and  $NR=a/2$ ,  $PR$  can be found by using the Pythagorean theorem so  $PR^2=NR^2-NP^2$  therefore  $PR= ((a^2/4)-b^2)^{1/2}$ . Similarly,  $PQ$  has the same length as  $PR$ . Therefore  $MR=(a/2)+((a^2/4)-b^2)^{1/2}$  and since  $MQ=MR-RQ$ ,  $MQ= (a/2)+((a^2/4)-b^2)^{1/2} - 2((a^2/4)-b^2)^{1/2} = (a/2) - ((a^2/4)-b^2)^{1/2}$ . Now, it is easily verified that both  $MQ$  and  $MR$  satisfy the given quadratic equation. In case 2,  $b=(a/2)$  so the solution is actually a double root. In this case  $z = (a/2)$ . In the last case,  $b>(a/2)$  so the line does not intersect the circle in any place therefore the solutions are imaginary. Thus far, Descartes has shown how the solutions of algebraic problems can be interpreted geometrically (Boyer 338).

He now decides to Pappus' problem of four lines. The problem is as follows:

If three straight lines are given in position, and if straight lines be drawn from one and the same point, making given angles with three given lines so drawn to the square of the other, the point lies on a solid locus given in position, namely, one of the three conic sections. Again, if the lines be drawn making given angles with four straight lines given in position, and if the rectangle of two of the lines so drawn bears a given ratio to the rectangle of the other two; then in like manner, the point lies on a conic section given in position. (qtd. Descartes 21)

So, in essence we are looking for a figure such that any point on the figure satisfies a given relationship with the given lines. Before Descartes, Euclid worked on this problem in the case of two lines, and Apollonius, in the case of four lines but their solutions were not simple and did not provide a method of extending the problem to a higher number of lines. Let us now consider Zeuthen's reconstruction of Apollonius' proof for the locus

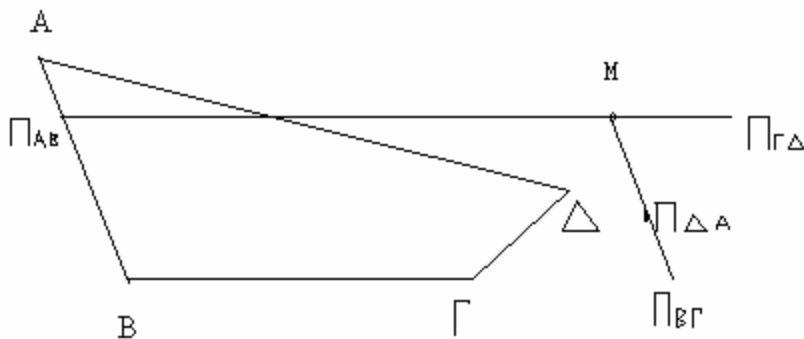
problem on four lines.

The four lines  $AB$ ,  $B\Gamma$ ,  $\Gamma\Delta$ ,  $\Delta A$  are given such that no line is parallel to another. The ratio  $a:b$  is also given. Then by choosing a point  $M$ , we construct the following segments:

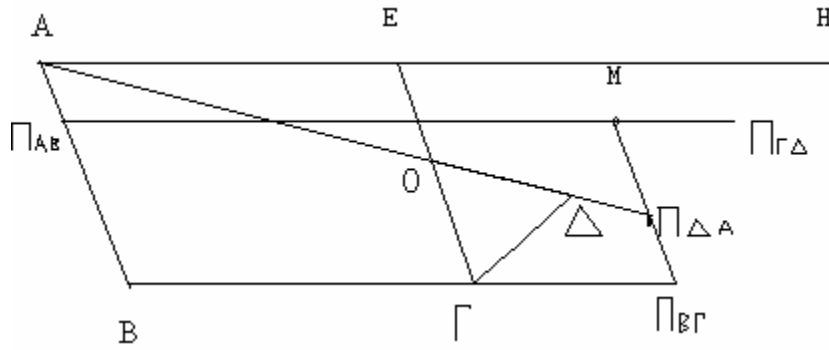
$M\Pi_{AB}$ ,  $M\Pi_{B\Gamma}$ ,  $M\Pi_{\Gamma\Delta}$ ,  $M\Pi_{\Delta A}$  which are drawn in specified relations to the given lines.

Furthermore, when the equality of the ratios  $^1(M\Pi_{AB} * M\Pi_{\Gamma\Delta}) : (M\Pi_{B\Gamma} * M\Pi_{\Delta A}) = a:b$  holds, then it can be shown that  $M$  is a point on a given conic section.

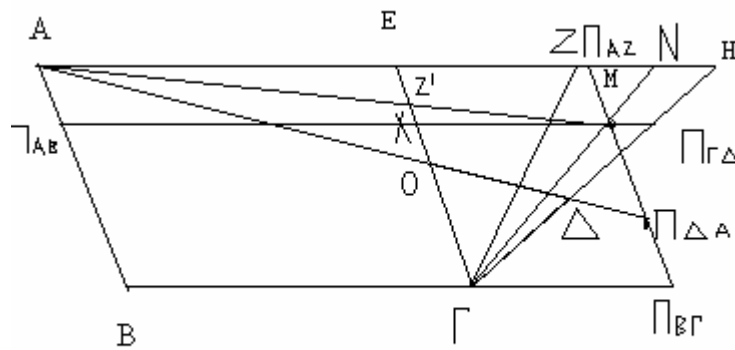
In Book 3 of his *Conics*, Apollonius constructs a solid locus (i.e. one of the conic sections) through various geometric arguments. However, in its construction he requires two of the four lines to be parallel so when working with four nonparallel lines, it is necessary to reduce the problem to a locus on four lines where two are parallel. From the initial given data, the figure that we have so far is the following:



Without loss of generality, we may assume that we can draw lines from the locus to the given lines such that these new lines are parallel to the given lines since there is no restriction on the direction in which the new lines should be drawn in (as long as the new lines are extended to the given lines). So let  $AEH$  be parallel to  $B\Gamma$  while  $\Gamma OE$  is parallel to  $AB$ :



By construction, we see that  $\Pi_{AB}\Pi_{\Gamma\Delta}$  is parallel to  $B\Pi_{B\Gamma}$  and  $\Pi_{\Delta A}\Pi_{B\Gamma}$  is parallel to  $AB$ . Now, if we draw the segment  $MA$  and call  $Z'$  the intersection of  $MA$  with  $\Gamma OE$  and if we extend the segment  $\Pi_{\Delta A}\Pi_{B\Gamma}$  to the line  $AH$  and call the intersection of the two,  $\Pi_{AZ}$ , then we can state that triangle  $AOZ'$  is similar to triangle  $A\Pi_{\Delta A}M$ . Furthermore, these are also similar triangles:  $AEZ' \sim A\Pi_{AZ}M$  and  $AEO \sim A\Pi_{AZ}\Pi_{\Delta A}$ . Since the figure  $A\Pi_{AZ}M\Pi_{AB}$  is a parallelogram, the length of the side  $\Pi_{AB}M$  equals that of  $A\Pi_{AZ}$ . Then  $AE:(EZ'+Z'O) = \Pi_{AB}M:(\Pi_{AZ}M + M\Pi_{\Delta A}) = c:d$ . Because the triangles  $AEZ'$  and  $A\Pi_{AZ}M$  are similar, this implies that triangles  $AZ'O$  and  $AM\Pi_{\Delta A}$  are similar. So the following ratio holds:  ${}^2M\Pi_{AB}:M\Pi_{\Delta A} = AE:OZ'$ .



Also,  $\Pi_{\Gamma\Delta}X\Gamma \sim HE\Gamma$  and  $MX\Gamma \sim NE\Gamma$ . Let  $EN=b$ ,  $NH=c$ ,  $XM=e$ ,  $M\Pi_{\Gamma\Delta}=d$ ,  $X\Gamma=a$  and  $\Gamma E=f$ . Then  $(e+d)/a = (b+c)/f$  and because of the fact that we are dealing with similar triangles, we can assert that  ${}^3M\Pi_{\Gamma\Delta}:M_{B\Gamma} = HN:\Gamma E$ . So from having established  ${}^2$  and  ${}^3$  the

following is derived:  $M\Pi_{AB}=AE:OZ' * M\Pi_{\Delta A}$  and  $M\Pi_{\Gamma\Delta}=HN:GE * M\Pi_{B\Gamma}$ . By plugging these into <sup>1</sup>, it turns out that  $HN:OZ'=a:b*(\Gamma E:AE)$ . Now instead of N on the line segment AH, let Z be the point such that  $HZ:OE=a:b*(\Gamma E:AE)$  because we want the locus to lie on four lines, two of which are parallel. Since the new ratio  $HZ:OE$  satisfies  $a:b*(\Gamma E:AE)$ , we can use similar triangles once again, to derive the ratio  $ZN:Z'E=a:b*(\Gamma E:AE)$ . By the fact that AE is parallel to  $\Pi_{AB}M$ , and by using techniques used to demonstrate that  $M\Pi_{AB}:M\Pi_{DA}=AE:OZ'$  it is possible to show that  $M\Pi_{AB}:M\Pi_{AZ}=AE:Z'E$ . Since  $\Gamma EN$  and  $GXM$  are similar triangles, the ratio  $M\Pi_{\Gamma Z}:M\Pi_{B\Gamma}=ZN:GE$  holds. But now,

$$(M\Pi_{AB} * M\Pi_{\Gamma Z}) : (M\Pi_{AZ} * M\Pi_{B\Gamma}) = (AE * ZN) : (Z'E * \Gamma E), \text{ and}$$

$$(M\Pi_{AZ} * M\Pi_{B\Gamma}) : (M\Pi_{AB} * M\Pi_{\Gamma Z}) = (\Gamma E : AE) * (Z'E : ZN) \text{ which is equivalent to } (ZN : Z'E) = (\Gamma E : AE) * ((M\Pi_{AB} * M\Pi_{\Gamma Z}) : (M\Pi_{B\Gamma} * M\Pi_{AZ}))$$

but thus implies that the ratio in italics equals a:b which means that the point M now lies on a different four line locus where the lines are AB,  $\Gamma Z$ ,  $B\Gamma$  and AZ and ZA is parallel to  $B\Gamma$  (Jones 588). After this point one must refer to Apollonius' Book 3 and after "more than fifty carefully worded propositions, all proved by synthetic methods" (Boyer, 151) the reader will see that the solution to the locus problem on four lines is in fact a conic section. But the lengthy demonstration implies that more complex geometric techniques are required to continue solving this problem. In contrast, Descartes' way of showing that the conic sections satisfy the problem only requires the knowledge of similar triangles and ratios.

Given the segments AB, AD, EF, and GH it is required to find a point C such that the lines CB, CD, CF and CH make the angles CBA, CDA, CFE and CHG with the given segments and so that the ratio of some of the segments equals the ratio of the rest of



triangle ABR are known. AB:VR is arbitrarily set to equal z:b but because AB=x, this means that BR:xb:z. So RC=y+(xb/z). Since all the angles of the triangle DRC are known, the ratios between the sides are known, and CR:CD is set equal to z:c. Since CR=RC=y+(xb/z) this implies that

$$\frac{CR}{CD} = \frac{yz+xb}{z} = \frac{z}{c} \quad \frac{c(yz+xb)}{z^2} = CD$$

Now, by what information we are given, we know the following distances: EA, AB, BG, AD, AR, FE, FS, GT and GH. It remains to find BT, BS, CF, and CH. Since the angles of ESB are all known now, it is possible to set up a ratio so that BE:BS= z:d. This process of using known angles and the sine formula to find the lengths of the sides of a triangle is continued until finally the following is the result (where AE=k and AG=l are given by the fact that these segments lie on the lines that we start with initially):

$$\begin{aligned} AB &= x \\ RB &= bx/z \\ CR &= y+(bx/z) \\ CD &= (cy/z)+bcx/(z^2) \\ EB &= k+x \\ BS &= (dk+dx)/z \\ CS &= (zy+dk+dx)/z \\ CF &= (ezy+dek+dex)/(z^2) \\ BG &= l-x \\ BT &= (fl-fx)/z \\ CT &= (zy+fl-fx)/z \\ CH &= (gzy+fgl-fgx)/(z^2). \end{aligned}$$

Every segment consists of a sum of known and unknown quantities, where

b,c,d,e,f,g,k,and z are known while x and y are not. But as you can see, it is possible to express each segment in terms of the unknowns x and y. Now, the problem of the locus

states that C is a point such that the rectangle of two segments is equal to the rectangle of two other segments where “rectangle” refers to the product of two of the following: CB, CD, CF, CH. In this particular example, the locus that corresponds to the ratio  $CB \cdot CF = CD : CH$  is to be found. By plugging in the corresponding formulas for the lines and using elementary algebra, it is possible to derive the following equation:

$$y^2 = \frac{(cflz - dckz^2)y - (dez^2 + cfgz - bcz)xy + bcflx - bcfgz^2}{ez^3 - cz^2}$$

Let  $(cflz - dckz^2)/(ez^3 - cz^2) = A$ ,  $(dez^2 + cfgz - bcz)/(ez^3 - cz^2) = B$ ,  $(bcfl)/(ez^3 - cz^2) = C$  and  $(bcfg)/(ez^3 - cz^2) = D$ , then the above equation may be written as  $Y^2 = Ay - Bxy + Cx - Dx^2$ . So as can be seen, this general form can yield all the conic sections and the specific conic section is determined by the signs in front of A, B, C, D. To summarize, in his solution Descartes randomly chooses two lines that have a certain angle between them and in this particular case, these two lines were AB and CD. Furthermore, he denotes these lines by x and y, which denote some unknown quantities. Then by establishing geometric relations between the x and y he is able to set up relations between all the lines in terms of x and y. So essentially Descartes is using the oblique coordinate system in dealing with this problem. The coordinate axes that is used today is actually a specific case of the oblique coordinate system used by Descartes because the coordinate axes is the case when the angle between two given lines is 90 degrees. Furthermore, Descartes' method of expressing curves algebraically by expressing relations between lines in terms of the coordinate axes and then using a given ratio to set up simultaneous equations has led to a great way to classify curves. As it was stated

earlier, it took Apollonius no less than fifty propositions to construct the conic sections, which is indicative of the complex nature of the constructions. However, by his combination of algebra and geometry, Descartes is easily able to find the algebraic formulas for these curves and can represent them geometrically on the coordinate axes. Furthermore, since this method of is so efficient, Descartes is able to demonstrate the solution to Pappus' problem for a higher number of lines (i.e. 5, 6, 7 and 8).

In conclusion, Descartes' innovative approach to representing curves in the plane by combining geometry and algebra has turned the construction of curves into a relatively easy matter. Mathematics has benefited greatly through the development of the coordinate axes and analytic geometry. For example, had Descartes not made these discoveries, the calculus would have been a very difficult subject to study. Ironically, although he was a brilliant mathematician, Descartes wasn't interested in the beauty of pure mathematics as many of his contemporaries. Instead, he considered it a useful science (Kline 301) that he wished to apply to problems outside of mathematics.

## Bibliography:

Boyer, Carl B. A History of Mathematics. 2<sup>nd</sup> ed.

New York: John Wiley & Sons Inc., 1991

Descartes René, La Géométrie. La Salle: The Open Court  
Publishing Company, 1952

Jones, Alexander ed. Pappus of Alexandria: Book 7 of the  
Collection. Part 2. Berlin: Springer-Verlag 1986.

Kline, Morris. Mathematical Thought from Ancient to  
Modern Times. New York: Oxford University Press,  
1972

