The mathematics of juggling

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Abstract

Around 1985 three groups of jugglers discovered the same mathematical theory of juggling patterns, nowadays known as "siteswaps", and the same system was rediscovered many times since.

I'll explain this system, its naturality, and utility, with many examples.

Why a mathematical theory of juggling?

"Mathematical" means, above all, simple and precise.
With a mathematical description, we can

- record juggling patterns for posterity,
- find easier or harder "versions of" a given pattern, and
- discover fundamentally new patterns.

What's involved in mathematically modeling... anything?

1. *Restrict the description.*
   We shall **only** record the order in which balls are thrown and caught.

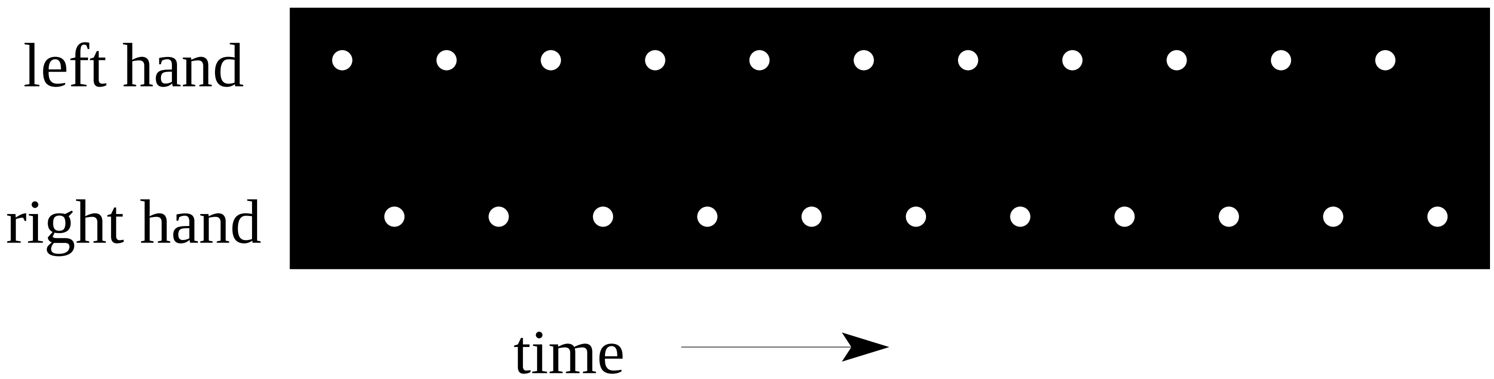
2. *Restrict the problem.*
   For now, we make two assumptions about the juggling:

   - The throws come one at a time from alternate hands left, right, left, right, . . .
   - Only one ball is thrown at any time: no "multiplexing".

   These assumptions rule out many interesting juggling patterns, so we'll weaken them later.

A first description: Space-time diagrams.

Imagine a camera is mounted in the ceiling, taking a long-exposure photograph of a walking juggler, who is juggling glowing balls in the dark.



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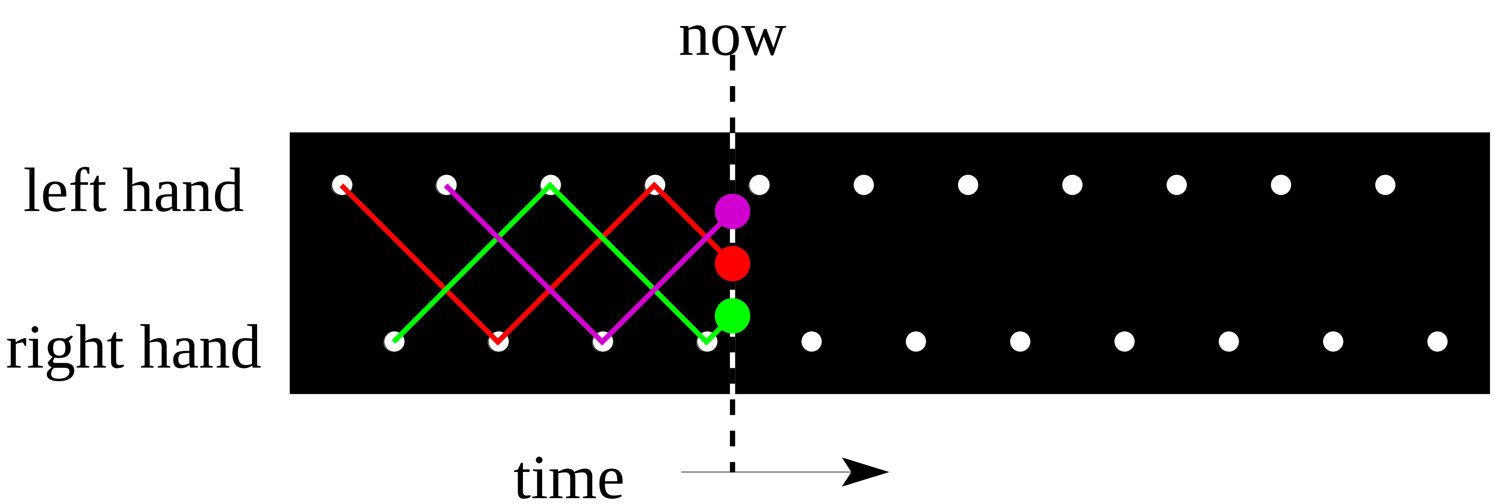

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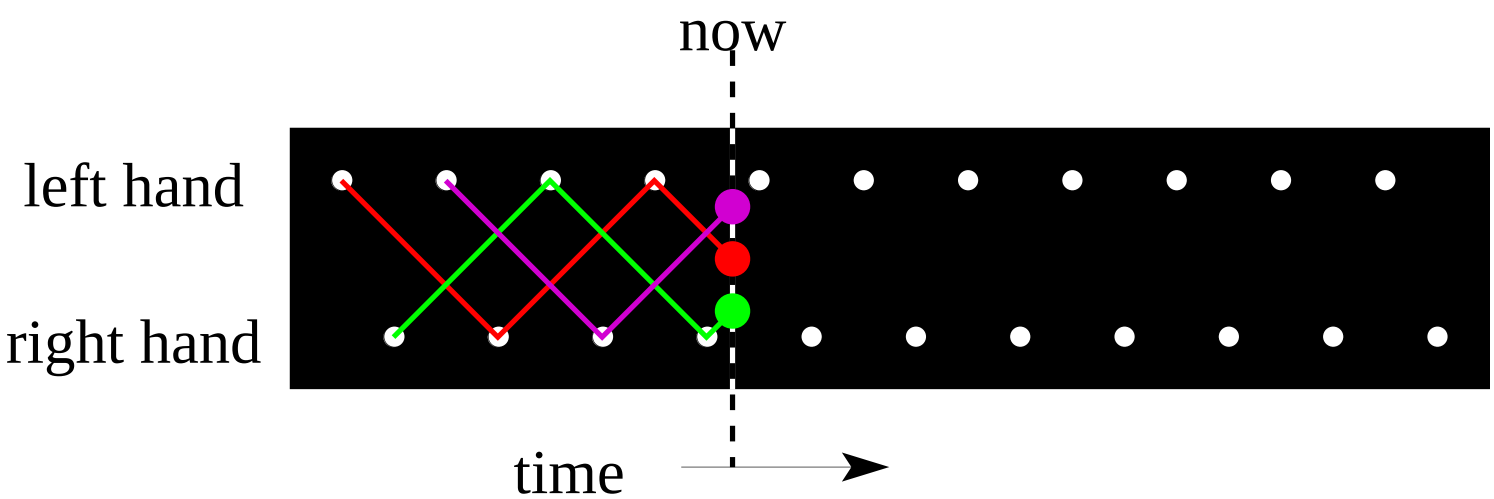

How does a trick look?

If a ball is thrown high, it crosses to the other hand slowly, giving a nearly horizontal line. Thrown low, it crosses quickly, giving a nearly vertical line. Or it may even go back to the same hand.

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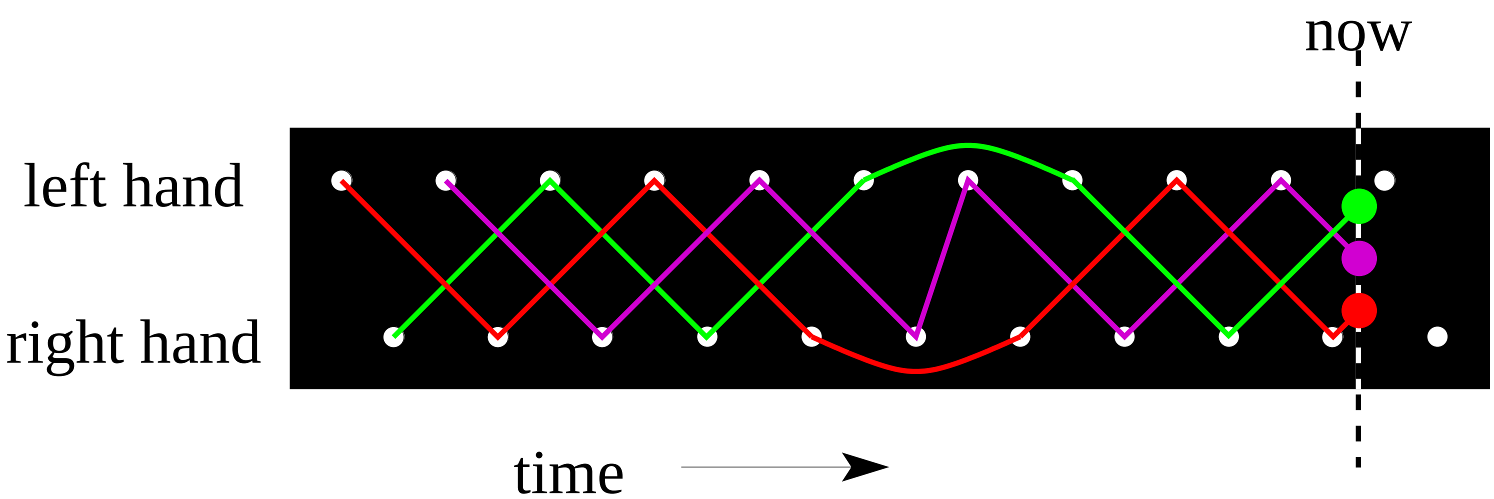

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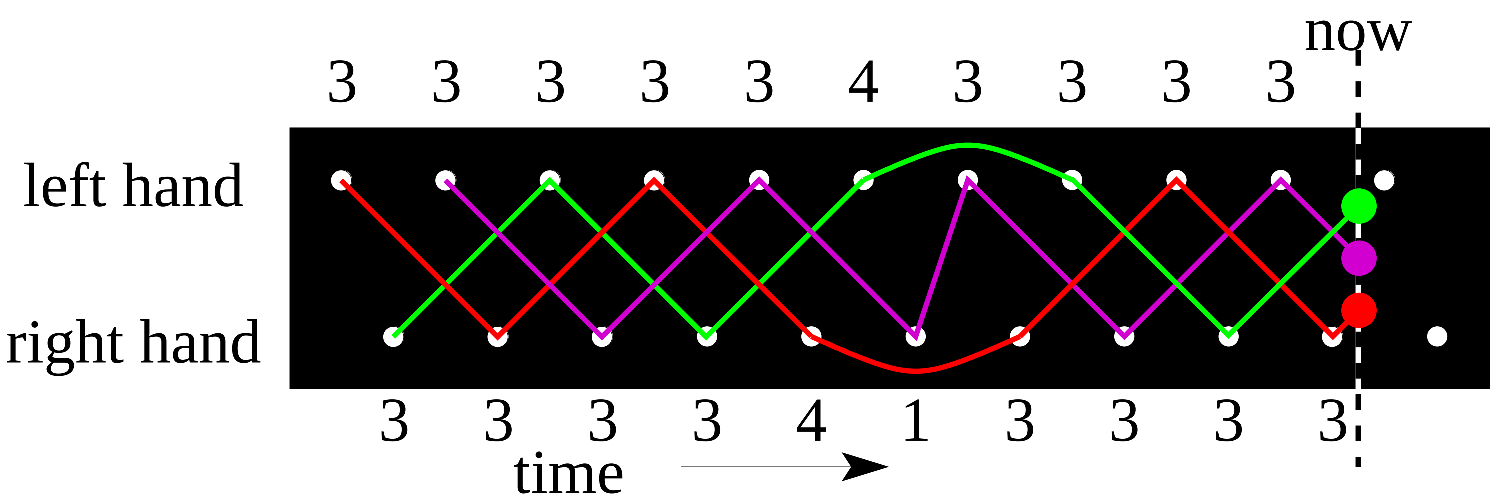

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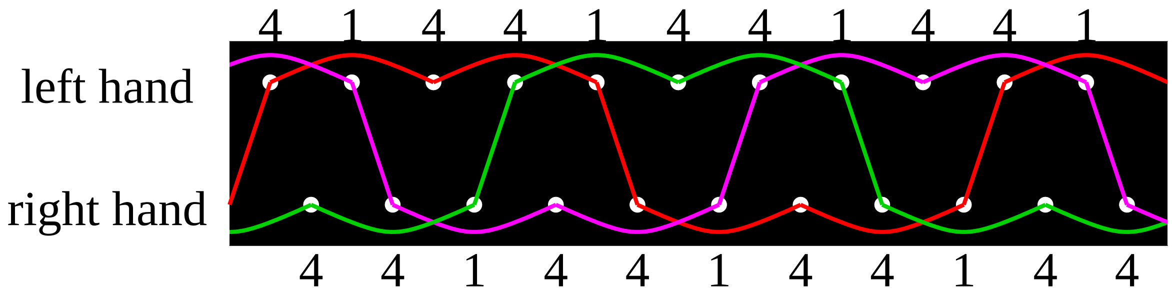
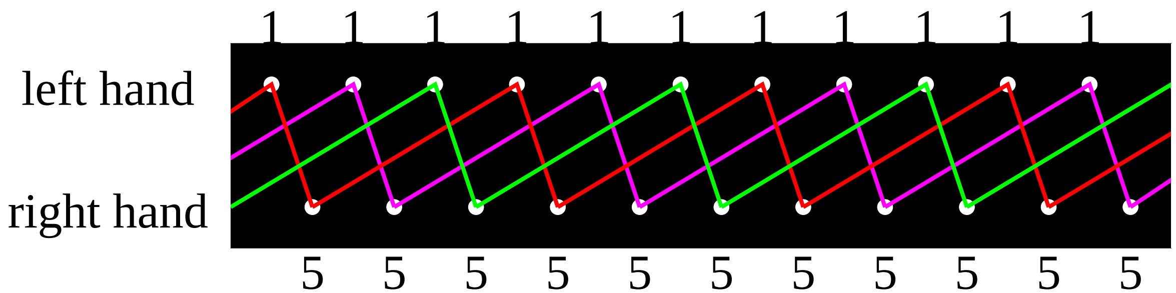
The idea: **label each throw by [the number of beats before this ball is thrown again].**

Examples of tricks and their space-time diagrams.

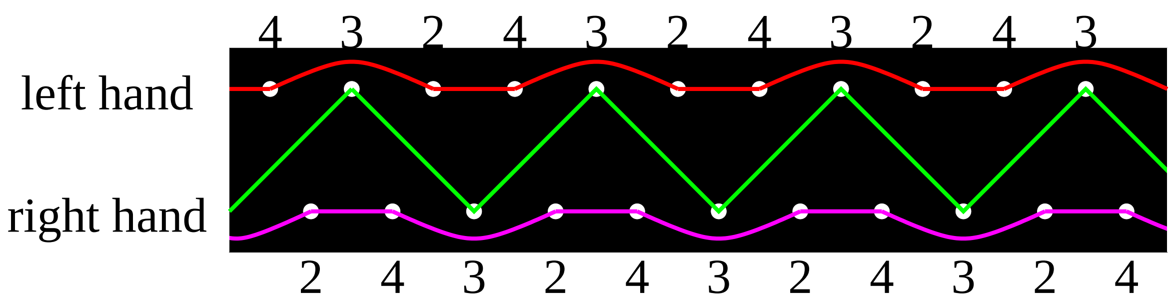
441:



51:



423:



Reconstructing the pattern from its "siteswap".

We can draw the space-time diagram given only the numbers, called the **siteswap**.
How do we read off properties of the pattern, straight from the numbers?

- If a throw number is even, the ball comes back to the same hand;
  if odd, then it crosses over.

- If the pattern is periodic (e.g. ...441441441441...), and the period is odd,
  then the pattern is left-right symmetric, e.g. 441 but not 4413.

- If the pattern is periodic, then the average of the numbers is the number of balls.
  (We'll prove this later.)
  In particular, the standard b-ball pattern is ...$bbbbb$...

Not every sequence of numbers (repeated) is a siteswap.

- **Obviously:** no two throws should come down at the same time,
  as in ...43... or ...5xy2...

- **Less obviously:** this is the *only* condition that one need check.
  In particular, under this condition the average is automatically a whole number.

Hereafter, we only write a pattern once, like "441" to indicate ...441441441...

Some strange throw numbers.

What is a 2? Recall the pattern

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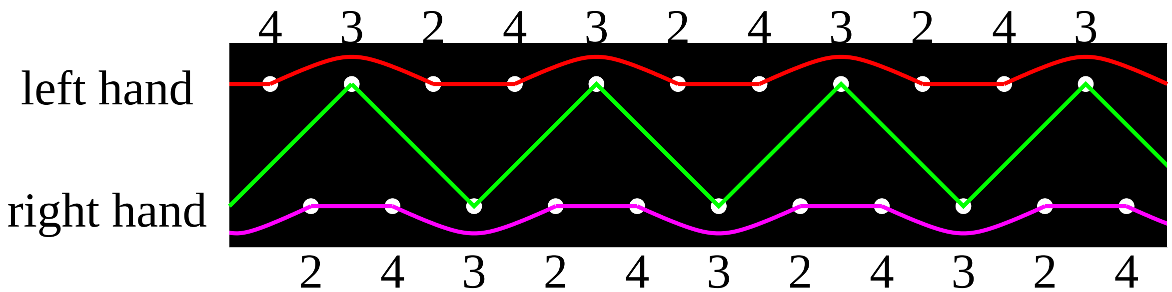

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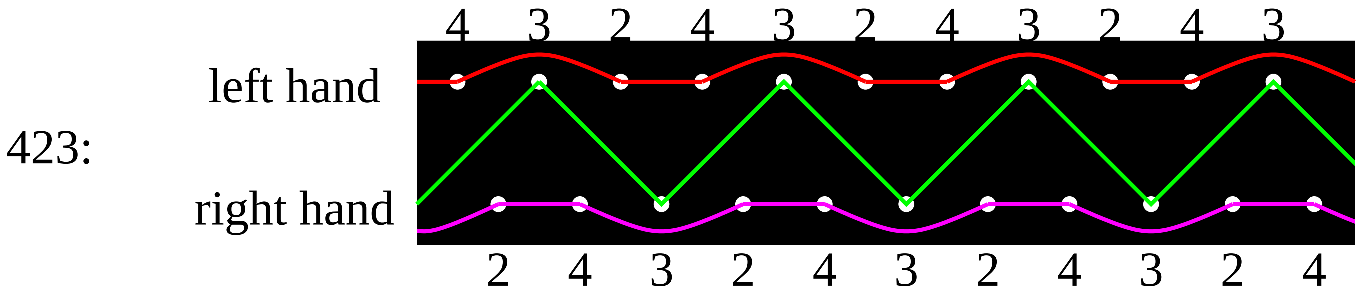

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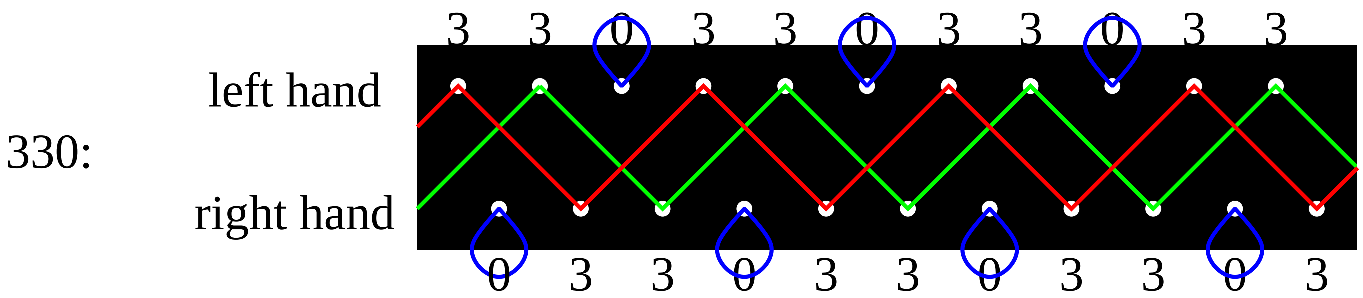
What is a 0? These give odd little loops on the space-time diagram:



0s are usually interpreted as empty hand "throws", dropping the loops altogether.

While both 2s and 0s are pauses, they are very different, viz. 42 vs. 4440, 522 vs. 55500.

Some small examples.

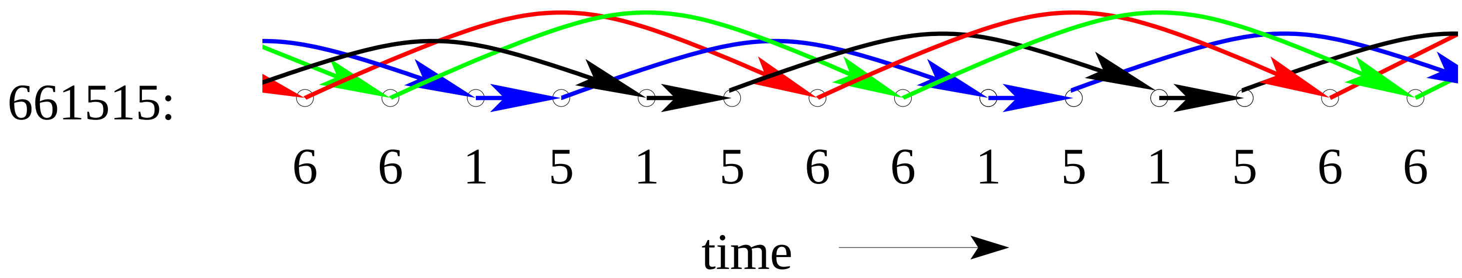
3	the standard 3-ball "cascade"
51	the standard 3-ball "shower"
4	the standard 4-ball "(asynchronous) fountain"
71	the 4-ball shower
5313	an asymmetric trick in which one ball is never thrown high
531	a symmetric version in which one ball is only cascaded
501	a 2-ball version, with the cascaded ball removed
552	four balls out of five, with a hold (i.e. lazily)
5551	four balls out of five, with a handoff one direction
55550	four balls out of five, with an occasionally empty hand (i.e. actively)
612	the "shower-box"
4413	the poor man's shower-box
52512	the "baby-juggling pattern"

Many have no good description other than the numbers:

441, 51414, 561, 7162, ...

Orbits.

Imagine drawing a space-time diagram on a line (losing "space-", really),



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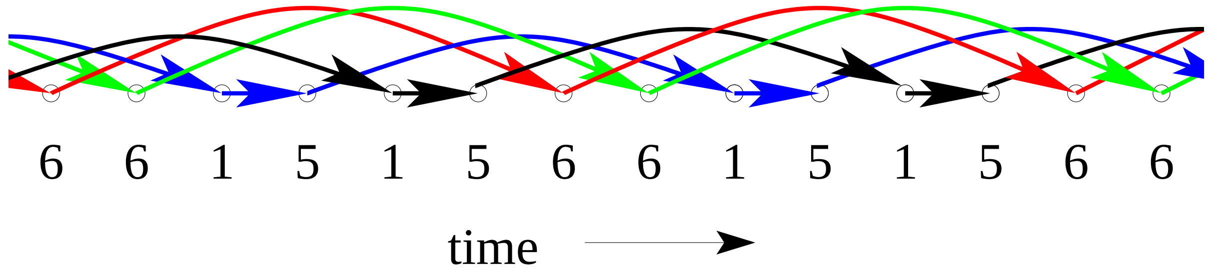
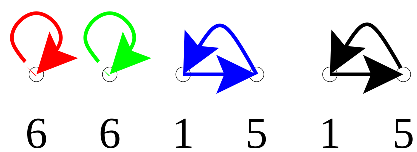
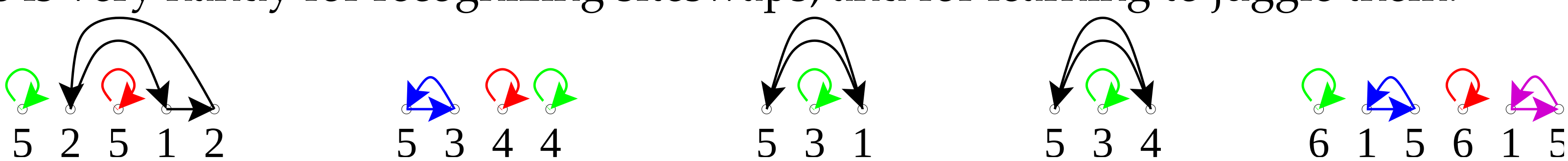


then draw only one period, essentially wrapping the line around a circle,



and the siteswap breaks up into **orbits** from which no ball escapes.

This is very handy for recognizing siteswaps, and for learning to juggle them:

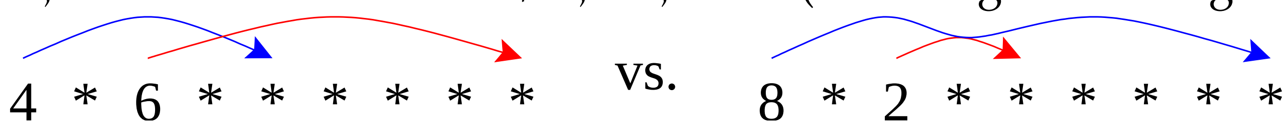


Making new siteswaps from old.

In a siteswap of length N, one can add or subtract N from any number, without changing the orbit structure, e.g. 501, 531, 534, 561, 261, 234 are all related this way.

Usually N is too big for this to be very useful.

A subtler change is to replace two successive numbers A, B with $B+1, A-1$. This is sometimes called an "elementary site swap". More generally one can replace two throws $A, \ldots, B$ at distance d with $B+d, \ldots, A-d$ (watching out for negative throws!).



If $A > B$, this operation "smooths out" the pattern, without changing the average or the number of balls, e.g. (smoothing the underlined pair in each case)

$$\underline{51}414 \longrightarrow 24\underline{41}4 \longrightarrow 2\underline{42}34 \longrightarrow 23334 \sim 333\underline{42} \longrightarrow 33333$$

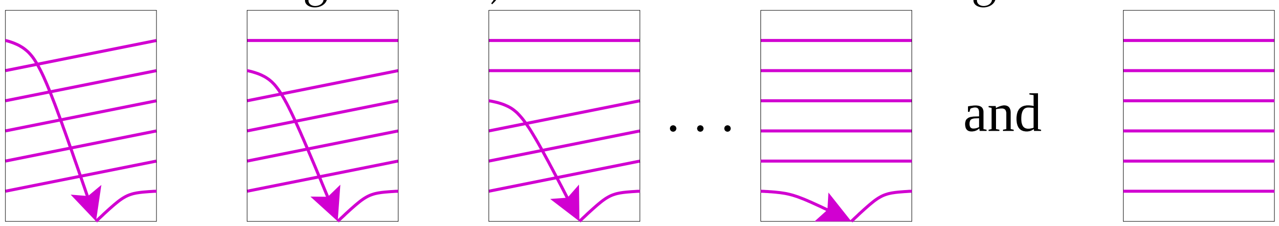
Theorem. The average of a siteswap is the number of balls.
Proof. This smoothing operation lets one reduce to the case that the siteswap is constant. (There is a momentary worry about the case $A = B + 1$, where the smoothing operation does nothing.) **QED.**

There is a better version: the number of balls in an orbit is the sum over that orbit, divided by the length of the whole pattern.

Counting siteswaps.

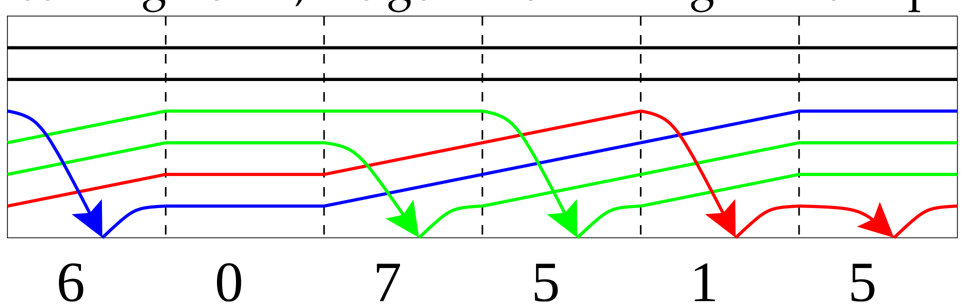
Theorem [Buhler-Eisenbud-Graham-Wright 1994].
The number of siteswaps of length n, with at most b balls, is $(b+1)^n$.

Proof (simpler than the original one). Consider the following deck of cards:



Each card connects b points on the left edge to b on the right, where one point on the left is pulled to the bottom edge and everyone below rises to take the space left over – or possibly no one is pulled down. So there are $b+1$ such cards.

If we lay out n such cards in a row (allowing repeats), and ignore any lines that float across the top never coming down, we get an orbit diagram for a pattern with $\leq b$ balls.



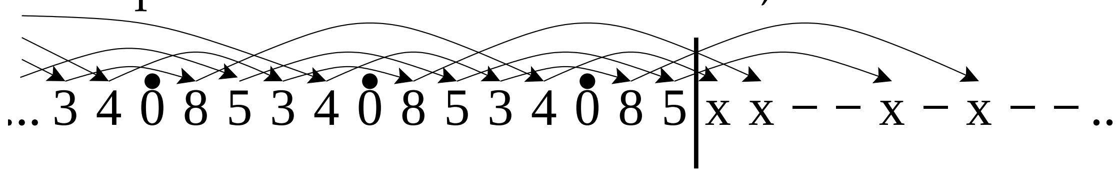
It is just as easy to go backwards, from a siteswap to its list of cards. **QED.**

The states of a juggling pattern.

At any moment, if a juggler stopped juggling and dropped everything, the b balls would land at b different times in the future (possibly including now). Call that set of landing times the **state** of the juggler.

It is sometimes drawn as a word in -, x with "x" indicating a ball coming down and "-" not, stopping at the last landing time, e.g. $\{0, 1, 3, 6\}$ is denoted `xx-x--x`.

For example, if we stop ...340853408534085... after a 5,



we get the state `xx--x-x`.

A siteswap of length n gives a list of n juggling states, and each throw tells how to change from the previous state to the next. For example 85340 has the states

`xxxx` $\xrightarrow{8}$ `xxx----x` $\xrightarrow{5}$ `xx--x-x` $\xrightarrow{3}$ `x-xx-x` $\xrightarrow{4}$ `-xxxx` $\xrightarrow{0}$ `xxxx`

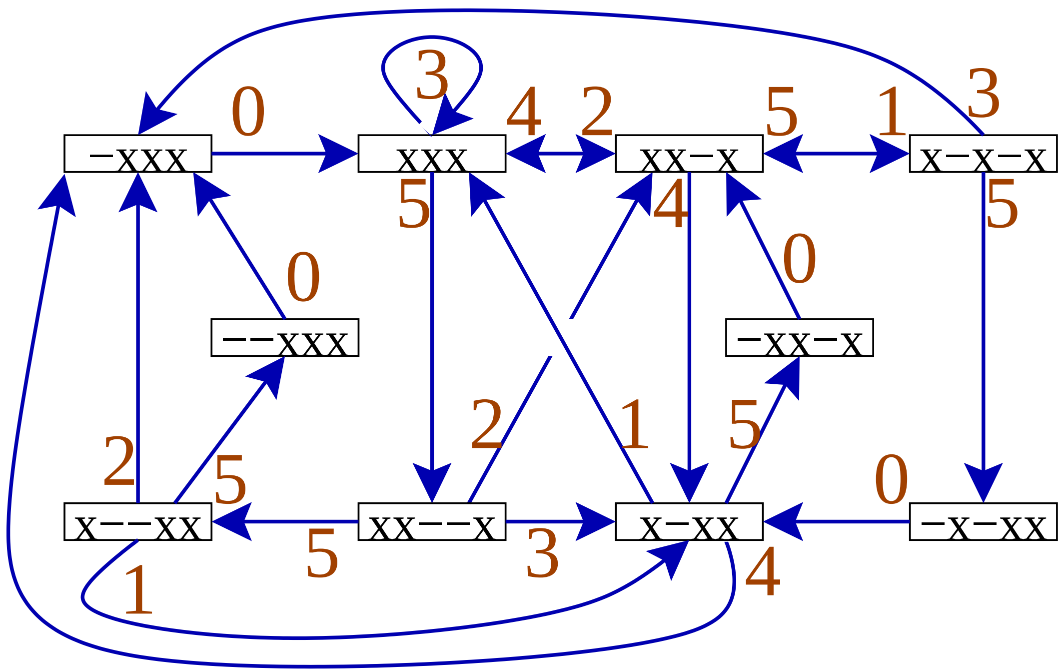
which happen to be all different. (Except initial = final, as 85340 is indeed a siteswap.)

Whereas 51414 repeats a state,

`xx-x` $\xrightarrow{5}$ `x-x-x` $\xrightarrow{1}$ `xx-x` $\xrightarrow{4}$ `x-xx` $\xrightarrow{1}$ `xxx` $\xrightarrow{4}$ `xx-x`

so 51 and 414 are individually siteswaps (as we've already seen).

All the states with 3 balls, maximum height 5.



If a state starts with “−”, the next throw must be a 0, and otherwise cannot be.
The siteswap can be reconstructed from the list of states, and it is easy to say when a list of states corresponds to a siteswap.
In 2005 work of (mathematician and 9-ball juggler) Greg Warrington, he studies random juggling, defined as the random walk on this directed graph.
In 2008 work of Ron Graham (former president of the American Mathematical Society and of the International Jugglers’ Association) and Fan Chung Graham, they count siteswaps that do not repeat their initial state.

Transitions in and out of patterns.

The 4-ball shower 71 goes between the two states `xx-x-x` and `x-x-x-x`.

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So the simplest way in is ...44444 56 717171...

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The most obvious thing to do is fill in the holes in order, ...7171 23 444...

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but this is ugly. If we disallow throws 2,3,4 on aesthetic grounds, until we get back to the ground state, the lowest possible return is

$$\boxed{\texttt{xx-x-x}} \xrightarrow{6} \boxed{\texttt{x-x-xx}} \xrightarrow{1} \boxed{\texttt{xx-xx}} \xrightarrow{5} \boxed{\texttt{x-xxx}} \xrightarrow{1} \boxed{\texttt{xxxx}}$$

Some transitions from 5 to 91: ...55 678 9191..., ...55 7892 91..., ...55 77929192 91...

Siteswaps from graphs.

Let $\Gamma = (V, E)$ be a connected graph with $b+1$ vertices V, connected by n edges E. Assume that the edges come in a prescribed order.

How can we find a minimal set $E' \subseteq E$ of edges such that (V, E') is still connected? (This is called a **spanning tree** of Γ, and necessarily has b edges.)

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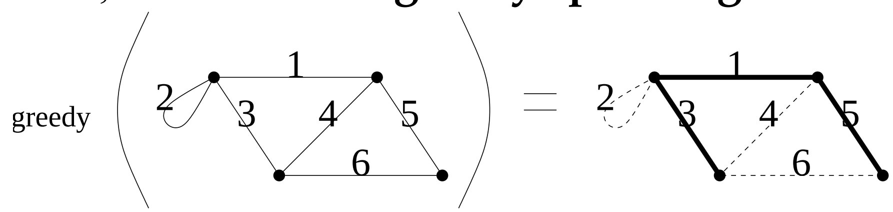
One answer: include edge $\#i$ exactly if its two endpoints aren't already connected by edges $\#1, \#2, \ldots, \#(i-1)$. Call this the **greedy spanning tree**.



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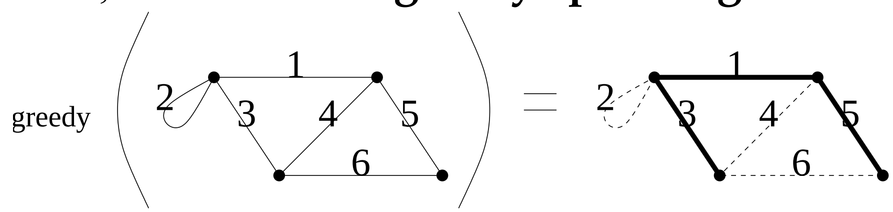
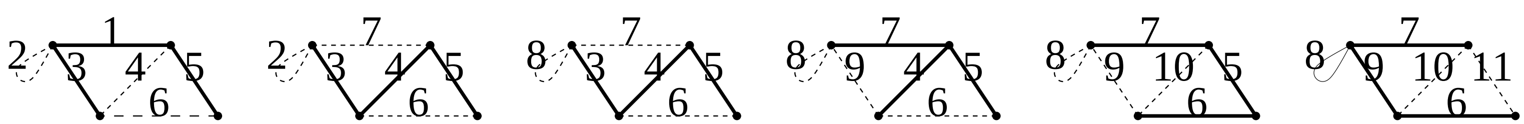


What happens if we drop edge $\#1$?
Then all the other edges in E' are still included, under this criterion, and possibly one new edge is included that wasn't before, t steps later than the dropped edge.

Instead of dropping the edge, we move edge $\#1$ to the end, and find the new greedy spanning tree. Repeat this n times and you get back where you started.



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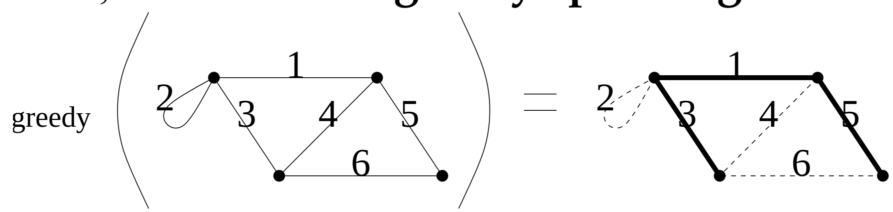

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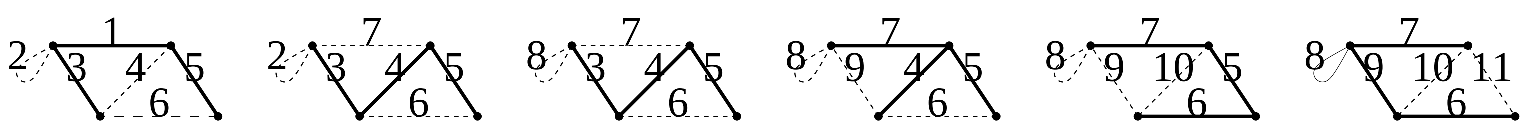


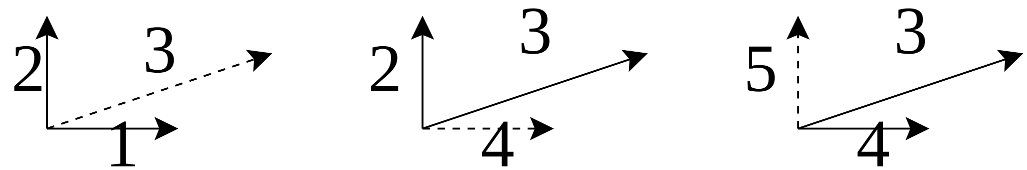
These n greedy spanning trees, thought of as subsets of $\{0, \ldots, n-1\}$ of size b, are the states of a siteswap with b balls, all throws t of height $\leq n$. Here it is 304245.

Siteswaps from lists of vectors (e.g. matrices).

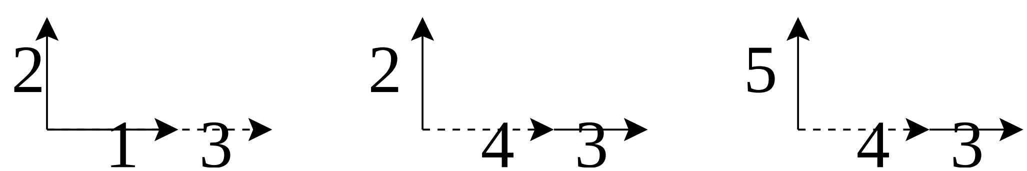
Question. Which siteswaps arise, in this way, from graphs?
Are there more conditions, beyond that the maximum throw is $\leq$ the length?

Theorem [Knutson-Lam-Speyer 2008, building on Postnikov 2006]. Given an ordered list of n vectors spanning a b-dimensional space, include vector $\#i$ exactly if it is not a linear combination of the vectors $\#1, \ldots, \#(i-1)$. Call this the **greedy basis**.
Then the same trick, "rotate vector $\#1$ to the end and construct the new greedy basis, then repeat n times", similarly produces n greedy bases that are the states of a siteswap with all throws $\leq n$, **and every siteswap does arise in this manner.**

Example. Consider the list $\big((1,0) = \rightarrow, (0,1) = \uparrow, (2,t) = \nearrow\big)$. Then for $t \neq 0$ we get



with siteswap 222, whereas for $t = 0$ we get the more excited siteswap 231:



In fact the siteswap smoothing operation is related to perturbations of matrices!

Multiplex and synchronous siteswaps.

<u>Multiplexing</u>

It is easy to extend the theory to include multiple throws from one hand;
just indicate *all* the throws being made, e.g.

33[33], [23], 23[43], 21[54], 25[75]51, 25[56]2, ...

Instead of only *one* ball landing at a time, only *the right number* of balls should.

The juggling state should now say *how many* balls are landing at any future time, e.g.

$$\boxed{\texttt{xx-xx}} \xrightarrow{2} \boxed{\texttt{xxxx}} \xrightarrow{1} \boxed{\texttt{2xx}} \xrightarrow{[54]} \boxed{\texttt{xx-xx}}$$

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Synchronous throws

Now instead of writing the throw made by the only hand throwing, we must list the throws made by each hand, and say whether the throw crosses or not.
By convention, all the throw numbers are even, to better compare with throws in asynchronous siteswaps.

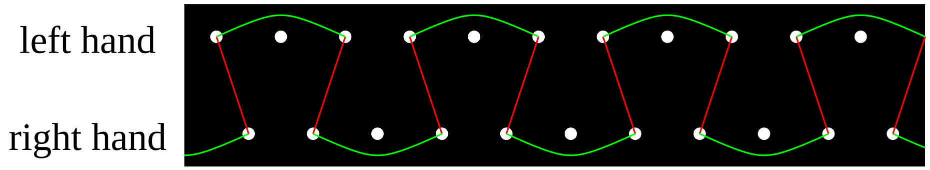
(4,4)	synchronous fountain	(4x,4x)	synchronous half-shower
(6x,4x)	five-ball half-shower	(4x,2x)	synchronous shower
(4,2x) (2x,4)	the shower-box again	(6x,0) (0,6x)	a three-ball cascade

Some *very* strange throw numbers.

What is a negative throw?

The sequence ... 4 -1 0 4 -1 0 ... has a perfectly nice space-time diagram:

4 −1 0:



What would you see, watching someone juggle this?

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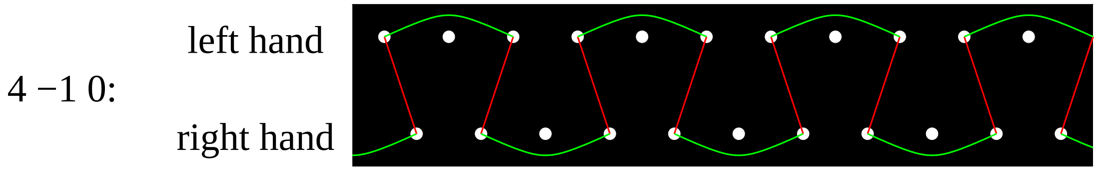

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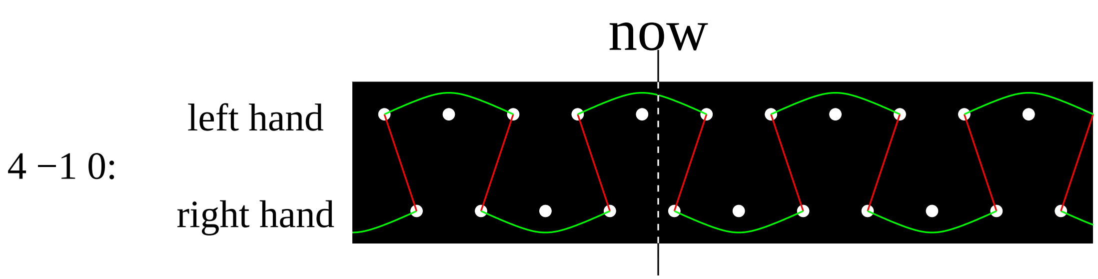
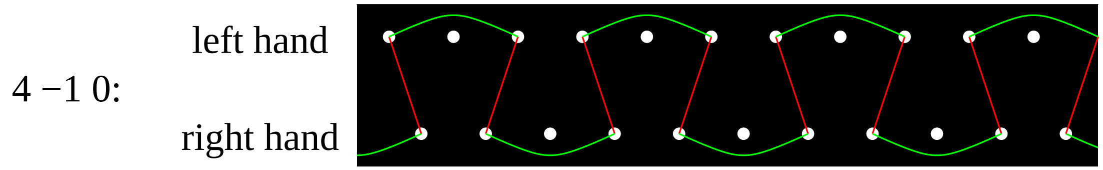


Right **now**, you see one ball.

Some *very* strange throw numbers.

What is a negative throw?

The sequence ... 4 -1 0 4 -1 0 ... has a perfectly nice space-time diagram:



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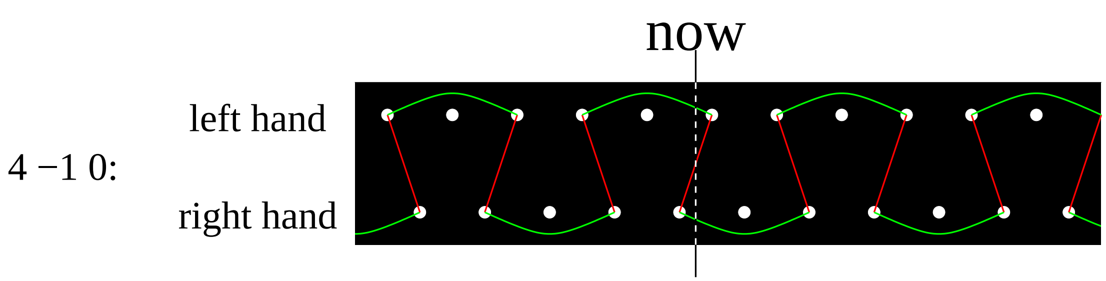

Right **now**, you see one ball.

But **now**, two balls have appeared out of nowhere in the right hand.

Some *very* strange throw numbers.

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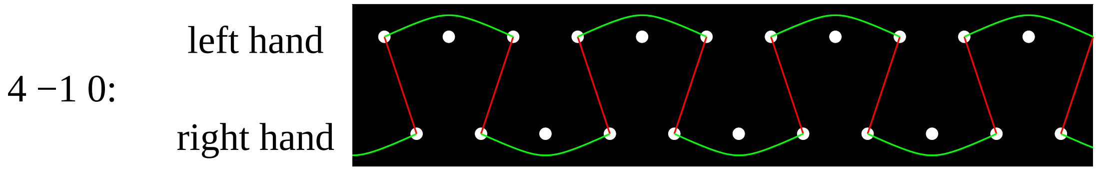

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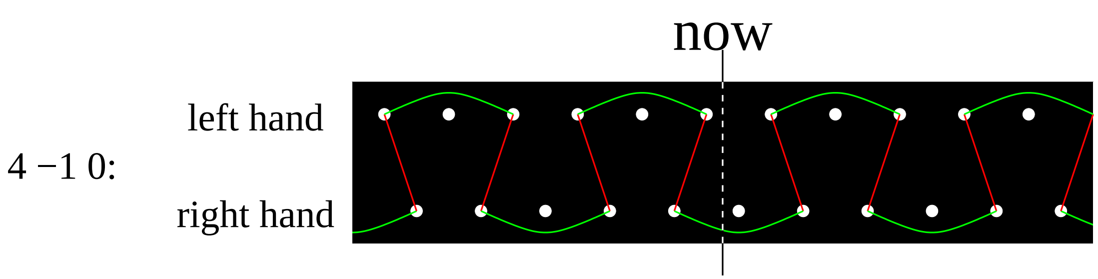


Right now, you see one ball.

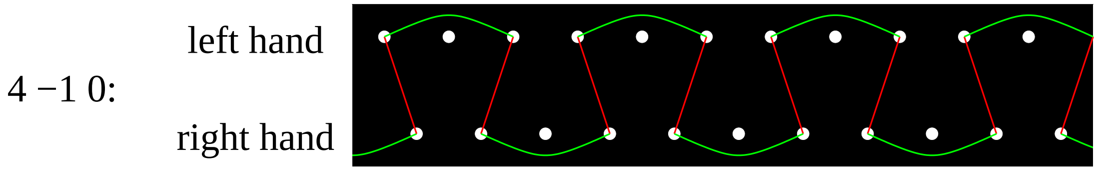
But now, two balls have appeared out of nowhere in the right hand.

And just **now**, two balls in the left hand simultaneously disappeared.

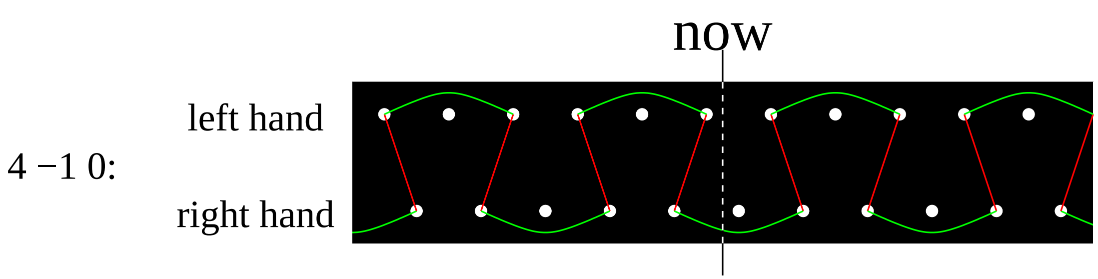
Some *very* strange throw numbers.

What is a negative throw?

The sequence ... 4 -1 0 4 -1 0 ... has a perfectly nice space-time diagram:



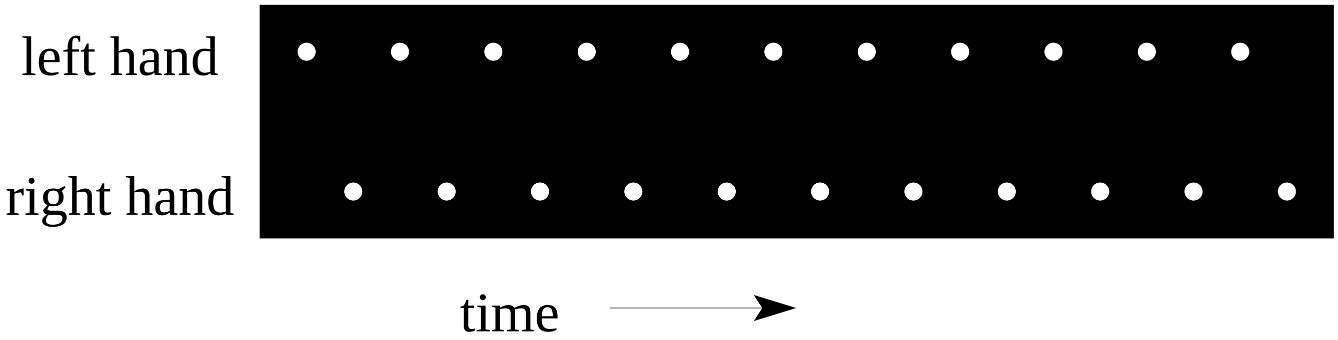
What would you see, watching someone juggle this?

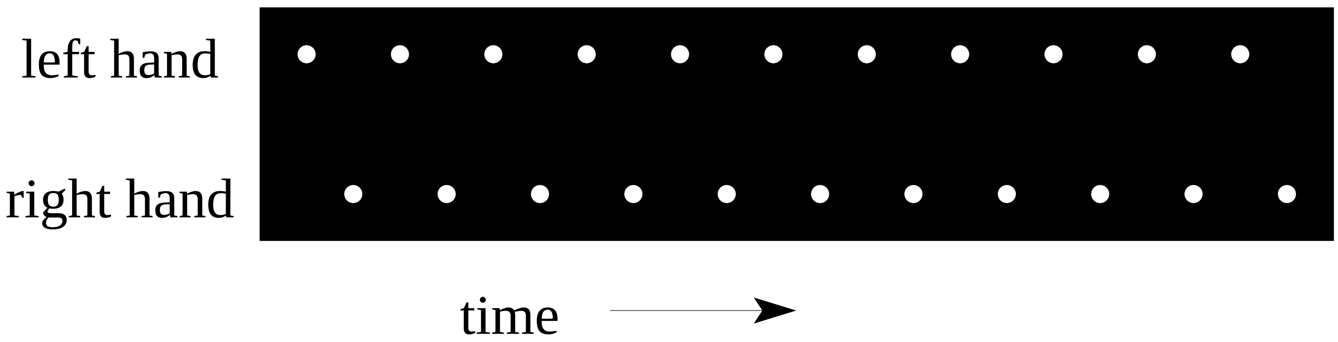


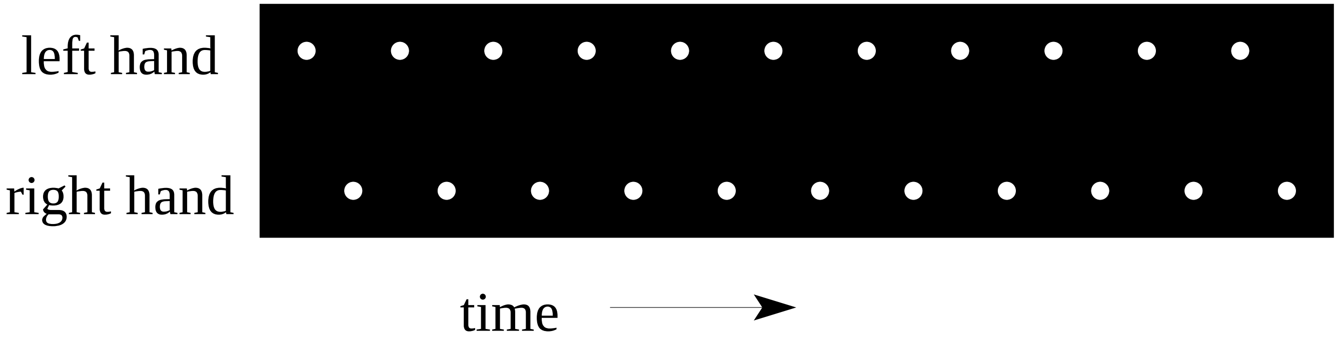
But now, two balls have appeared out of nowhere in the right hand.

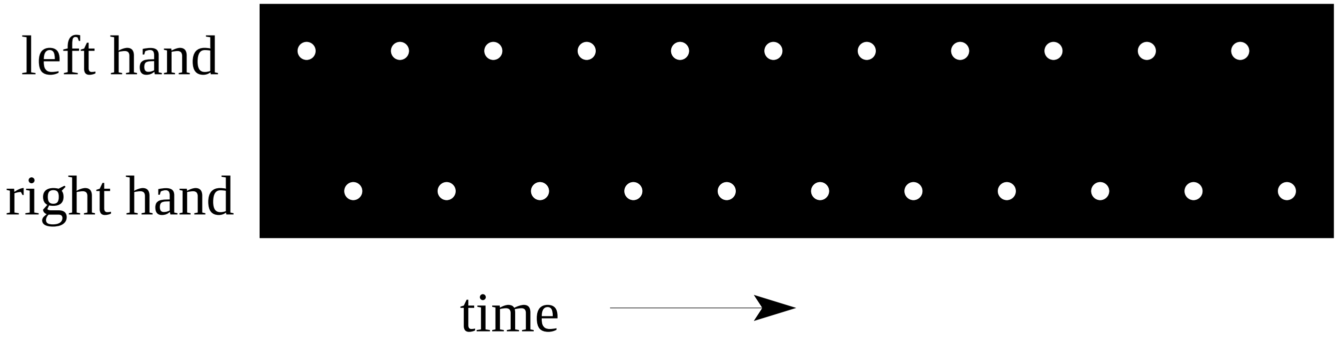
And just **now**, two balls in the left hand simultaneously disappeared.

Feynman's answer: a ball moving backwards is time is an *antiball*.


left hand
right hand
time


left hand
right hand
time


left hand
right hand
time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