Prof. Allen Knutson's Math 20D Final, Dec 5, 2005

Each number in parentheses is the point score for that question.

For each question 1–5 it's best to *read it entirely* before starting any part of it.

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1. Let M be the matrix

$$M = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \qquad \text{where } \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

are two eigenvectors to get you started. ("Eigen" = "characteristic" for those people who never ever came to class.)

a) What are the eigenvalues of those two eigenvectors? [2+2 points]

ANSWER. Apply the matrix to them:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ -2 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

So the eigenvalues are 0 and -2.

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b) Find all eigenvectors of M the form $\begin{bmatrix} 1 \\ a \\ -a \\ -1 \end{bmatrix}$ and give their eigenvalues.

[3+3 points]

ANSWER. If the eigenvalue is s,

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ a \\ -a \\ -1 \end{bmatrix} = \begin{bmatrix} -1+a \\ 1-3a \\ 3a-1 \\ -a+1 \end{bmatrix} = s \begin{bmatrix} 1 \\ a \\ -a \\ -1 \end{bmatrix}$$

So $-1+a = s$, $1-3a = sa$, so $1-3a = (-1+a)a = a^2-a$, $a^2+2a-1 = 0$, and solving the quadratic equation we get $a = -1 \pm \sqrt{2}$, $s = -2 \pm \sqrt{2}$.

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c) Solve $\vec{y}' = M\vec{y}$ with the initial conditions $\vec{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$.

"Solve" means that your answer should only have exponentials like e^{3t}, no visible exponentials of matrices. [10 points]

ANSWER. We know how to solve such systems if the initial conditions are linear combinations of eigenvectors. In the case at hand, $\vec{y}(0)$ is visibly a linear combination of the two eigenvectors given.

$$\vec{y}(0) = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Solving, we get $c_1 = 3/2, c_2 = 1/2$. Then

$$\vec{y}(t) = \frac{3}{2} e^{0t} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} e^{-2t} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} e^{-2t} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

(which can be rewritten several equivalent ways).

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d) Solve $\vec{y}' = M\vec{y}$ with the initial conditions $\vec{y}(0) = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

"Solve" means that your answer should only have exponentials like e^{4t}, no visible exponentials of matrices. [10 points]

ANSWER. This is harder (indeed, one of the two hardest questions), because this $\vec{y}(0)$ is not a linear combination of the two eigenvectors given; we have to use the ones from part 1b) also.

$$\vec{y}(0) = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 - \sqrt{2} \\ 1 + \sqrt{2} \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ -1 + \sqrt{2} \\ 1 - \sqrt{2} \\ -1 \end{bmatrix}$$

$$\begin{aligned}
(1) \qquad 3 &= c_1 + c_2 + c_3 + c_4 \\
(2) \qquad 0 &= c_1 - c_2 + (-1-\sqrt{2})c_3 + (-1+\sqrt{2})c_4 \\
(3) \qquad 0 &= c_1 - c_2 + (1+\sqrt{2})c_3 + (+1-\sqrt{2})c_4 \\
(4) \qquad 0 &= c_1 + c_2 - c_3 - c_4
\end{aligned}$$

The first and last equation can be summed and differenced to give $3 = 2c_1 + 2c_2 = 2c_3 + 2c_4$. Similarly, the second and third can be summed to give $0 = 2c_1 - 2c_2$. So $c_1 = c_2 = 3/4$, $c_3 = 3/4 + d$, $c_4 = 3/4 - d$ for some d. Finally, the second and third can be differenced to give

$$0 = 2(1+\sqrt{2})c_3 + 2(1-\sqrt{2})c_4 = 2(1+\sqrt{2})(3/4+d) + 2(1-\sqrt{2})(3/4-d) = 3 + 4\sqrt{2}d$$

so $c_3 = 3/4(1 - 1/\sqrt{2})$, $c_4 = 3/4(1 + 1/\sqrt{2})$. Then

$$\vec{y}(t) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - e^{-2t} c_2 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + e^{(-2-\sqrt{2})t} c_3 \begin{bmatrix} 1 \\ -1-\sqrt{2} \\ 1+\sqrt{2} \\ -1 \end{bmatrix} + e^{(-2+\sqrt{2})t} c_4 \begin{bmatrix} 1 \\ -1+\sqrt{2} \\ 1-\sqrt{2} \\ -1 \end{bmatrix}$$

with these c_1, c_2, c_3, c_4.

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2. Assume that

$$f(y)\,dx + (x+y)\,dy = 0$$

is exact, where $f(y)$ is a function of y but not x.

a) Find all possibilities for the function f. [5 points]

ANSWER. The exactness condition is $\frac{\partial}{\partial y} f(y) = \frac{\partial}{\partial x}(x+y)$, so $\frac{d}{dy} f(y) = 1$, hence $f(y) = y + C$ for some unknown constant C.

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b) For each possible f, give all solutions to the ODE, **in the form** $y = g(x)$. Partial credit available if you leave them in some other form. [20 points]

ANSWER. Since it's exact, we can determine a function $F(x,y)$ of which the above is the total differential, by the recipe

$$F(x,y) = \int_{t=0}^{x} P(t,0)\,dt + \int_{s=0}^{y} Q(x,s)\,ds = \int_{t=0}^{x} C\,dt + \int_{s=0}^{y} (x+s)\,ds$$
$$= Cx + xy + 1/2\; y^2$$

and the solutions to $dF = 0$ are given by $F = D$. So one way to express the solutions is $Cx + xy + 1/2y^2 = D$.

However, the question asks for the solution in the form $y = g(x)$. So we have to use the quadratic equation to solve for y:

$$y = -x \pm \sqrt{x^2 - 2(Cx - D)}$$

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3. Consider the ODE

$$t^2 \frac{dy}{dt} = 2t^2 y - 2t - 1$$

for $t > 0$.
Hint given during the test: there is a solution of the form t^k for some number k.
a) Find the general solution to this ODE. [10 points]
ANSWER. Dividing by t^2 (which we can do since $t > 0$), this becomes a linear inhomogeneous (read: "easy") ODE.

$$\frac{dy}{dt} = 2y - 2/t - 1/t^2$$

We learned two ways to solve such an ODE. Both use the general solution to the homogeneous equation

$$\frac{dy}{dt} = 2y$$

which is $y = Ce^{2t}$. One is the "variation of constants" method, letting C be a function of t. This leads to an integral that one can do by integrating by parts twice. Some people did that successfully.
The other way we learned was to add the general solution to the homogeneous to *any* solution of the inhomogeneous. We're partly given one in the hint, t^k. What is k? Plugging it into the ODE,

$$t^2 \frac{dy}{dt} = t^2 k t^{k-1} = 2t^2 t^k - 2t - 1$$

At $t = 1$, this is $k = 2 - 2 - 1 = -1$. So $y = t^{-1}$ is a solution, and the general solution is $Ce^{2t} + t^{-1}$.

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b) Find $y(\pi)$, assuming the initial conditions $\lim_{t\to\infty} y(t) = 0$. [5 points]
ANSWER. The limit as $t \to \infty$ of $Ce^{2t} + t^{-1}$ is $\pm$infinity unless $C = 0$. In that case, $y(t) = 1/t$, and $y(\pi) = 1/\pi$.

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4. Let $c_0, c_1, \ldots$ be a sequence of real numbers satisfying two properties:

$$\text{for all } n \geq 0, \qquad c_{n+2} = c_{n+1} + 6\, c_n$$

$$\text{there exists some } a > 0 \text{ such that for all } n \geq 0, \qquad c_n = a^n.$$

Answer the following questions with numbers (not "x", "$c_0 + a$", etc.).
a) Find the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$. [5 points]
b) Find the possible values of a. [5 points]

ANSWERS. This was the question for which "read it entirely" was included in the instructions. If we do a) first, the ratio test gives us

$$\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{a^n}{a^{n+1}} \right| = \lim_{n\to\infty} \frac{1}{|a|} = \frac{1}{|a|}$$

so the radius of convergence can only be found by determining a.
In part (b), we do that. Combining the two things we know about c_n,

$$a^{n+2} = a^{n+1} + 6a^n.$$

Since $a > 0$, we can divide by it (n times):

$$a^2 = a + 6.$$

Solving the quadratic equation, we get $a = 3$ or $a = -2$. Since we were told $a > 0$, we know $a = 3$ and not -2.
So the answer to (a) is 1/3, and to (b) is 3.

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5. Let M be the matrix

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Compute the matrix $\exp(tM)$. Your answer should be a 4×4 matrix where each entry is a function of t. None of those entries should be written as (or left in the form of) an infinite sum. If you do write down the infinite sums correctly you'll get partial credit.
Suggestion: you might want to start calculating on some other page and only put your line of reasoning here when you've finished figuring it out. [15 points]

ANSWER. The first several powers of M (the 0th through the 4th) are

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&0&-4&0\\0&1&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&-4&0&0\\0&0&-4&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&0&-16&0\\0&-4&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&16&0&0\\0&0&16&0\\0&0&0&0\end{bmatrix}$$

Adding up the exponential series $e^{tM} = \sum_{n=0}^{\infty}(tM)^n/n!$, the outer part of the matrix is simple:

$$\begin{bmatrix}1&0&0&t\\0&&&0\\0&&&0\\0&0&0&1\end{bmatrix}$$

The central 2×2 is more complicated:

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}+t\begin{bmatrix}0&-4\\1&0\end{bmatrix}+t^2/2!\begin{bmatrix}-4&0\\0&-4\end{bmatrix}+t^3/3!\begin{bmatrix}0&16\\-4&0\end{bmatrix}+t^4/4!\begin{bmatrix}16&0\\0&16\end{bmatrix}+\dots$$

(The reason these behave so simply is that M^2 is diagonal, and multiplying by it just rescales the central square by -4.) Adding all these up, we get

$$\begin{bmatrix}\sum_{n=0}^{\infty}(-4)^n t^{2n}/(2n)! & -4\sum_{n=0}^{\infty}(-4)^n t^{2n+1}/(2n+1)!\\ \sum_{n=0}^{\infty}(-4)^n t^{2n+1}/(2n+1)! & \sum_{n=0}^{\infty}(-4)^n t^{2n}/(2n)!\end{bmatrix}$$

or

$$\begin{bmatrix}\sum_{n=0}^{\infty}(-1)^n (2t)^{2n}/(2n)! & -4\sum_{n=0}^{\infty}(-1)^n (2t)^{2n+1}/(2n+1)!\\ \sum_{n=0}^{\infty}(-1)^n (2t)^{2n+1}/(2n+1)! & \sum_{n=0}^{\infty}(-1)^n (2t)^{2n}/(2n)!\end{bmatrix}$$

These are power series we've seen before, and the total answer is

$$\begin{bmatrix}1&0&0&t\\0&\cos 2t&-4\sin 2t&0\\0&\sin 2t&\cos 2t&0\\0&0&0&1\end{bmatrix}$$

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b) Solve $\vec{y}' = M\vec{y}$ for $\vec{y}(0) = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$. [5 points]

ANSWER. The general solution is $\vec{y}(t) = e^{tM}\vec{y}(0)$, which in this case is

$$\vec{y}(t) = \begin{bmatrix} 1 & 0 & 0 & t \\ 0 & \cos 2t & -4\sin 2t & 0 \\ 0 & \sin 2t & \cos 2t & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+t \\ \cos 2t - 4\sin 2t \\ \sin 2t + \cos 2t \\ 1 \end{bmatrix}.$$