Prof. Allen Knutson's Math 20D Midterm #1, Oct 17, 2005, answers

1a. $\sum_{n=1}^{\infty} n!/2^{(n^2)}$
Converges. The ratio test gives

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!/2^{((n+1)^2)}}{n!/2^{(n^2)}} = \frac{(n+1)!}{n!}\frac{2^{n^2}}{2^{n^2+2n+1}} = \frac{n+1}{2^{2n+1}}$$

The limit of this ratio is 0. As long as the limit exists and is < 1, the series is (absolutely) convergent.

1b. $\sum_{n=1}^{\infty}(2n)!/n^n$
Diverges; the terms don't go to zero. The ratio test gives

$$\frac{a_{n+1}}{a_n} = \frac{(2(n+1))!/(n+1)^{n+1}}{(2n)!/n^n}$$

$$= \frac{(2n+2)!}{(2n)!}\frac{n^n}{(n+1)^{n+1}} = (2n+2)(2n+1)\frac{1}{n+1}\left(\frac{n}{n+1}\right)^{n+1}$$

The limit of this is ∞. (The limit of $\left(\frac{n}{n+1}\right)^{n+1}$ is $1/e$, in fact.) So the terms don't go to 0.

1c. $\sum_{n=1}^{\infty}(\log n)/(n\sqrt{n})$
Converges. These terms are *larger* than $1/n^{3/2}$, so comparing to that series will *not* tell you that this converges. They're smaller than $1/n$, but that series diverges, so that comparison also tells you nothing. Instead, compare to $1/n^{5/4}$ or some other exponent between 1 and 3/2. The ratio goes to 0, so the p-series test with $p = 5/4$ says this converges.

1d. $\sum_{n=1}^{\infty}(\log 2^n)/(n^2 + \sin n)$
Diverges.
Rewriting, $a_n = \frac{n \log 2}{n^2+\sin n} = \log 2 \frac{1}{n+n^{-1}\sin n}$. Because of the $\sin n$, these terms are *not* always larger than $(\log 2)\frac{1}{n}$. So you have to compare to e.g. half that; these terms *are* eventually larger than $\frac{\log 2}{2}\frac{1}{n}$. (Take the ratio and compute the limit, which is 1/2.)

Compute the radius of convergence.

$\sum_{n=1}^{\infty} \sin(n\pi/2)x^n$

$R = 1$. The values of $\sin(n\pi/2)$ are $0, 1, 0, -1$ repeating. This is most closely approximated by the geometric series (1^n), not say (2^n). One way to be completely rigorous about it would be to divide by x, rewrite with $y = -x^2$, and now the coefficients are all 1.

$\sum_{n=1}^{\infty}(-n)^n x^n$

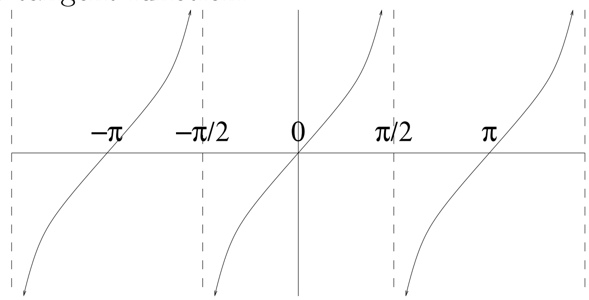
$R = 0$, by the root test.

$\sum_{n=1}^{\infty} n^{n/\log n} x^n$

$R = 1/e$.

$n^{n/\log n} = (e^{\log n})^{n/\log n} = e^{(\log n)n/\log n} = e^n$. So this series is just $\sum_n (en)^n$.

3. Let $f(x) = \tan x$. Though it *shouldn't* be necessary, here's a picture of the tangent function:



a. What is the second Taylor polynomial for $\tan x$ around $x = 0$? (I.e. including the 1, x, and x^2 terms.)

Answer: $0 + 1x + 0x^2$. The value at 0 is 0, the slope is 1 and it's an inflection point so the second derivative is 0.

One way to do this is look at the ratio

$$\sin x / \cos x = (x - x^3/3 + \ldots)\big/(1 - x^2/2 + \ldots).$$

Replace dividing by $1 - u$ by multiplying by $1 + u + u^2 + \ldots$, and this becomes $(x - \ldots)(1 + x^2/2 + \ldots) = x$+cubic terms we can ignore.

Or you could take the derivatives using the ratio formula on $\sin / \cos$.

b. Let $\sum_{n=0}^{\infty} c_n x^n$ be the Taylor series for $\tan x$. Assume that $\tan x = \sum_{n=0}^{\infty} c_n x^n$ for $|x| < \pi/2$. Show that the radius of convergence is at most 2.

We're told it converges for $|x| < \pi/2$. Since tangent blows up at $\pi/2$, it can't converge wider than that. So the radius of convergence is exactly $\pi/2$. Which is less than 2.