Prof. Allen Knutson's Math 109 Final, Dec 6, 2006, with answers

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If you have questions, come and ask me! I really really mean it!!!

"[n]" means the problem is worth n%.

1. Recall (?) from calculus that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous at** x if $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x'$ with $|x - x'| < \delta$, one has $|f(x) - f(x')| < \epsilon$.

a [5]. Write out what it means for f to *not* be continuous at x, without using the word "not" anywhere; instead the final inequality should go $\geq$.

Continue from: "A function f is not continuous at x if $\exists \epsilon > 0$ such that..."
ANSWER. $\ldots \forall \delta > 0, \exists x'$ with $|x - x'| < \delta$, but $|f(x) - f(x')| \geq \epsilon$.

b [10]. Let $h : \mathbb{R} \to \mathbb{R}$ be the function such that $h(x) = 0$ for $x \leq 0$, $h(x) = 1$ for $x > 0$. (So its graph looks like ... __—$\cdots$.) Prove that h is *not* continuous at 0, directly from the definition above.

ANSWER. Let $\epsilon = 0.4387$. (All that matters is that it's in $(0, 1)$, but it's not as convincing to say "Let $\epsilon \in (0, 1)$" as it is to actually pick one.) After that, δ is arbitrary and don't get to pick it. But we do get to pick x'; take $x' = \delta/2$. (Again, all that matters is that $x' \in (0, \delta/2)$.)
Then $|x - x'| = |0 - \delta/2| = \delta/2 < \delta$, but $|f(x) - f(x')| = |0 - 1| = 1 \geq \epsilon$.

2. Call a function f **tinconuous** at x if $\exists\epsilon > 0$ such that $\forall\delta > 0, \forall x'$ with $|x - x'| < \delta$, one has $|f(x) - f(x')| < \epsilon$.
2a [10]. Write out what it means for a function f to *not* be tinconuous at 0. Don't use "not" anywhere, instead end with "$\geq \epsilon$".

ANSWER. A function f is not tinconuous at 0 if $\forall\epsilon > 0$, $\exists\delta > 0, \exists x'$ such that $|x - x'| < \delta$, but $|f(x) - f(x')| \geq \epsilon$.

It's worth nothing that we can switch the order of the $\exists$, giving the simpler statement
A function f is not tinconuous at 0 if $\forall\epsilon > 0$, $\exists x'$ such that $|f(x) - f(x')| \geq \epsilon$. (Picking a δ at this point is easy, e.g. $1 + |x - x'|$.)

2c [10]. Let f be the function $f(x) = x$. Prove that f is not tinconuous at 0.

ANSWER. For each $\epsilon > 0$, let $x' = \epsilon$, $\delta = \epsilon + 2$. (Again note that we may choose the $\exists$ variables, and have no say over the $\forall$ variables.)
Then $|x - x'| = |0 - \epsilon| = \epsilon < \epsilon + 2 = \delta$. However $|f(x) - f(x')| = |0 - \epsilon| \geq \epsilon$.

3. Let m be even and n be odd. Let $a, b \in \mathbb{N}$. Assume $m^a = n^b$.

a [5]. One of these four numbers m, n, a, b is uniquely determined by this information. Which is it, and what is its value? Prove your answer.

ANSWER. n^b is always odd (no powers of 2 in its prime factorization). For m^a to be equal to it, it must be odd too, with no powers of 2 in its prime factorization. But m is even. So a must be zero.

4 [10]. Let $a_0 = 0$. For $n > 0$, assume that $a_n = 2a_{n-1} + 1$.
Write out a full proof by induction that that $a_n < 2^n$.
(Hint: it may be helpful to prove first that a_n is an integer.)

ANSWER. First we prove a_n is an integer by induction. $a_0 = 0$ is an integer. If a_n is an integer ($n \geq 0$), so is $2a_n + 1$, which is a_{n+1}. So we're done.
Now we prove the main result by induction on n. The $n = 0$ base case is easy: $0 < 2^0 = 1$.
Now take $n > 0$. Since $a_{n-1} < 2^{n-1}$ by induction hypothesis, and a_{n-1} is an integer (which we just proved above), in fact $a_{n-1} \leq 2^{n-1} - 1$. Therefore

$$a_n = 2a_{n-1} + 1 \leq 2 \cdot (2^{n-1} - 1) + 1 = 2 \cdot 2^{n-1} - 2 + 1 = 2^n - 1$$

as was to be shown.

There was another cool proof one person gave: show that a_n is odd for $n > 0$ (by induction), and that $a_n < 2^n + 1$ by similar algebraic manipulation as the above. Then $a_n < 2^n$, since it's an integer and not 2^n (as that's not odd).

A few people noticed that $a_n = 2^n - 1$, proved *that* by induction, then said "Therefore it's $< 2^n$."

5. Let X be a set, and $\sim_1$, $\sim_2$ be two equivalence relations on X. (So for example, $x \sim_2 y \sim_2 z$ implies $x \sim_2 z$.)

Define the relation $\sim_{12}$ on X by

$$x \sim_{12} z \iff \exists y \in X,\ x \sim_1 y \sim_2 z.$$

a [10]. Prove that $\sim_{12}$ is reflexive.

ANSWER. We need to show $\exists y \in X,\ x \sim_1 y \sim_2 x$. Take $y = x$. Then $x \sim_1 y \sim_2 x$ since $\sim_1, \sim_2$ are reflexive.

b [10]. Give an example of $X, \sim_1, \sim_2$ for which $\sim_{12}$ is not symmetric.

ANSWER.
Let $X = \{a, b, c\}$, $\sim_1$ be the equivalence relation with associated partition $\{\{a, b\}, \{c\}\}$, and $\sim_2$ be the equivalence relation with associated partition $\{\{a\}, \{b, c\}\}$.
Then $a \sim_1 b \sim_2 c$, so $a \sim_{12} c$. But $\nexists y \in X$ such that $c \sim_1 y \sim_2 a$; you can check the three possibilities for y and none of them work.

6 [10]. Let $a, b, c, d \in \mathbb{N}$ such that $gcd(a, b) = gcd(c, d)$.
Give an example where $gcd(a, c)$ and $gcd(b, d)$ are more than $gcd(a, b)$.

ANSWER. The smallest example is $a = c = 2$, $b = d = 3$. Then $gcd(a, b) = 1 = gcd(c, d)$, but $gcd(a, c) = 2$, $gcd(b, d) = 3$, both more than 1.

7. Let $f : X \to Y$ be a function that is not $1:1$, and $g : Y \to X$ a function that is not onto.
a [10]. Prove that the composite $g \circ f : X \to X$ is neither $1:1$ nor onto.

ANSWER. Since f is not $1:1$, $\exists x_1, x_2$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Hence $g(f(x_1)) = g(f(x_2))$, so $(g \circ f)(x_1) = (g \circ f)(x_2)$, meaning $g \circ f$ is not $1:1$.
Since g is not onto, $\exists x \in X$ such that $\forall y \in Y$, $g(y) \neq x$. Hence $\forall x' \in X$, $g(f(x')) \neq x$. Therefore x is not in the image of $g \circ f$, so that function is not onto.

7b [10]. Give an example of X, Y, f, g satisfying those conditions, but where $f \circ g$ is both $1 : 1$ and onto.

Many people took $|X| = 1$, $|Y| = 2$. Then f and g are essentially unique, and do the job.

(In fact I had meant to ask a much harder question, in which $X = Y$. Then for 7b to occur, X would have to be infinite.)