Prof. Allen Knutson's Math 109 Midterm #1, Oct 16, 2006
with answers

Name: ____________________ SID: ___________________________

If you have questions, come and ask me! I really really mean it!!!

1 [30%]. Let S be the set of students in the Knutson section of Math 109. Let T be the set of all teaching assistants at UCSD. For $s \in S, t \in T$, write

$$t \text{ helps } s$$

if the student s can seek help from the TA t (in t's official capacity).
Next to each of the following statements, write True or False.

1. $\exists s \in S$ such that $\exists t \in T$ such that t helps s **True**
2. $\forall s \in S, \forall t \in T, t$ helps s **False**
3. $\forall s \in S, \exists t \in T$ such that t helps s **True**
4. $\exists t \in T$ such that $\forall s \in S, t$ helps s **True**
5. $\exists s \in S$ such that $\forall t \in T, t$ helps s **False**

For each of the False statements above, explain in English what they're asserting, expressed in a way that makes it obvious that they're false.

#2. Any student in this 109 class can legitimately seek help from any TA at UCSD. *Not.*
#5. There's a student in this class that can legitimately seek help from any TA at UCSD. *Not.*

There were several people who misread the definition of S as "all students at UCSD", and got #4 wrong because of that. (#4 is true: the TA in question is Ameera.) Many, many people did not properly follow the underlined text above (but got most credit anyway).

2 [15%]. Give three ways of negating each of these propositions, according to the three places the "not" might go. Proposition #1 below is an example.

1. Negating "$\forall s \in S, \exists t \in T$ such that t helps s":

   $\not\forall s \in S, \exists t \in T$ such that t helps s

   $\exists s \in S$ such that $\nexists t \in T$ such that t helps s

   $\exists s \in S$ such that $\forall t \in T, t$ doesn't help s

2. Negating "$\forall s \in S, \forall t \in T, t$ helps s":

3. Negating "$\exists s \in S$ such that $\forall t \in T, t$ helps s":

Just follow the template:

- negate the first quantifier,
- negate the second, while switching $\forall \leftrightarrow \exists$ in the first,
- change "t helps s" to "t doesn't help s", while switching $\forall \leftrightarrow \exists$ in the first and second.

3 [20%]. The following proof by (weak) induction concerns **factorials**: for each natural number $n \in \mathbb{N}$, let $n! = 1 \cdot 2 \cdot \ldots \cdot n$ be the product of the first n numbers, pronounced "n factorial". But it "proves" something false!

"**Theorem.** For each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, either $n = 0$, or $n! = 0$.

Proof. By weak induction on n.
Let $P(n)$ denote the proposition "either $n = 0$, or $n! = 0$".

Base case. If $n = 0$, then any statement that starts "either $n = 0$..." is certainly true. This proves $P(0)$.

Induction step. Now let $n > 0$. We will show that $P(n) \implies P(n+1)$.

If $n > 0$, then $(n+1)! = (n+1) \cdot n!$. By induction, we know $P(n)$, so $n! = 0$. Hence $(n+1)! = (n+1) \cdot 0 = 0$, establishing $P(n+1)$.

So by induction, $P(n)$ holds for all $n \in \mathbb{N}$. □"

Question. In what way is it not a proper proof by induction?

Note 1: it is *not enough* to say "it proves something false, that's pretty improper!" You must point out some important way in which it fails to conform to the templates you know of proofs by induction.
Note 2: don't waste your time looking through it for incorrect steps – each statement (other than the very last one, "So by induction, ...") is true.

Answer. If the base case is at $n = 0$ (as it is here), and you're writing your proof in the "$P(n) \implies P(n+1)$" form, then you must start at $n \geq 0$ not $n > 0$ as stated above.

Basically, the problem is that there is no linkage between the base case and the other numbers.

(In retrospect, this was a lousy question. Probably it should have been multiple choice. "Here are a dozen possible objections to this 'proof'. Which of them is actually valid?")

To address some of the wrong answers given: it's okay to start induction at $n = 0$. Induction proofs don't go $P(n+1) \implies P(n)$. It's okay to assume $P(n)$ – that's the power of an induction proof.

4 [25%]. **Theorem.** Let $n \in \mathbb{N}$. Assume that n^2 is one more than a multiple of 2 (i.e., n^2 is odd). Then n^2 is actually one more than a multiple of 4.

Proof.

[Do **not** put scratchwork here; use backs of other pages for that!
Only start writing here when you have a complete proof.
If by test's end you have only an idea, not a full proof, *say so* and then state your idea.]

First we claim that since n^2 is odd, n itself is odd. (We did this in class, but many people reproved it anyway, by contradiction: if n were even, then $n = 2k$ for some $k \in \mathbb{Z}$, so $n^2 = (2k)^2 = 2(2k^2)$ is even, contradiction.) So we can write $n = 2m + 1$ for some $m \in \mathbb{Z}$.

Now $n^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 4(m^2 + m) + 1$.
So n^2 is indeed one more than a multiple of 4, the multiple being $m^2 + m$.

This was a lousy thing to try to prove by induction. The statement is only about odd n. All the people who tried to prove it by induction tried to do weak induction on n^2 (though without saying so). But if you add two to a square it's not again a square, so this doesn't work very well.

Question [10%]. Is n^2 necessarily one more than a multiple of 8?

Yes. (And if that's all you said, you got half credit.)

To continue the above, $n^2 = 4m(m+1) + 1 = 8\frac{m(m+1)}{2} + 1$. We observed before in class that $\frac{m(m+1)}{2} = \binom{m+1}{2}$ is an integer; the proof split into cases according to m even or odd.

So n^2 is indeed one more than a multiple of 8, the multiple being $\frac{m(m+1)}{2}$.

Most of the false proofs given of this would also show n^2 is one more than a multiple of 16, which is false.