Prof. Allen Knutson's Math 109 Midterm #2, Nov 13, 2006

Name: ________________ SID: ______________________

If you have questions, come and ask me! I really really mean it!!!

1. For each of the relations listed below, write **R**, **S**, **T** if the relation is **r**eflexive ($\forall x, x \sim x$), **s**ymmetric ($\forall x, y,\ x \sim y \implies y \sim x$), or **t**ransitive ($\forall x, y, z,\ x \sim y\ \&\ y \sim z \implies x \sim z$). You aren't required to prove that the relations have these properties.
For each such property that a relation does *not* have, give an example showing that the relation lacks the property. (That is to say, you *are* required to prove that they *don't* have those properties.) Part A is done as an example:

A. Let $X = \mathbb{R}$, and write $a \sim b$ if $a \leq b$.
R, T. This $\sim$ is not symmetric because $55 \sim 89$ but $89 \not\sim 55$.

B. Let $X = \mathbb{R}$, and write $a \sim b$ if $b - a$ is an integer multiple of 2π.
R,S,T. (Some people seemed to think that 0 is not an integer multiple of 2π, but it is.)

C. Let $X = \mathbb{R}$, and write $a \sim b$ if a is a real-number multiple of b, i.e. $\exists x \in \mathbb{R}$ s.t. $a = xb$.
R,T. While 0 is a multiple of 6 billion, 6 billion is not a multiple of 0. ($\nexists x \in \mathbb{R}$ s.t. 6 billion $= x \cdot 0$.)

D. Let $X = \mathbb{N}$, and write $a \sim b$ if $5 \mid ab$ (i.e. if ab is an integer multiple of 5).
S. $1 \not\sim 1$ so no **R**, and $1 \sim 5 \sim 1$ but $1 \not\sim 1$ so no **T**.

2. Let $a, b \in \mathbb{Z}$. Say $a \equiv b \bmod 10$ and $a \equiv b \bmod 15$.

a. Is it automatically true that $a \equiv b \bmod 150$? If yes, give a proof; if no, give a counterexample.

No; $0 \equiv 30 \bmod 10$ and $0 \equiv 30 \bmod 15$ but $0 \not\equiv 30 \bmod 150$.
(You could look at the prime factorization of $a-b$ and notice that it contains 2 and 5, and 3 and 5, but that doesn't force it to contain 5 twice.)

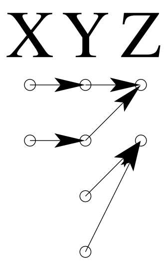
b. Show that $a \equiv b \bmod 6$.

Look at the prime factorization $a - b = 2^{d_2} 3^{d_3} 5^{d_5} \cdots$.
Since $10 | a - b$, we know $2 | a - b$, so $d_2 \geq 1$.
Since $15 | a - b$, we know $3 | a - b$, so $d_3 \geq 1$.
Hence $a - b = 2 \cdot 3 \cdot 2^{d_2 - 1} 3^{d_3 - 1} 5^{d_5} \cdots$ is a multiple of 6.

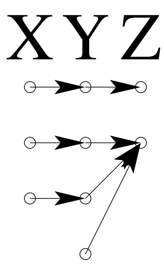
3. On the sets below, you are to draw six functions, with arrows pointing from w to $h(w)$. Note that X, Y, Z are of different sizes in the various pictures. Each $f : X \to Y$ should be $1:1$, and each $g : Y \to Z$ should be onto.
When you're done, you should have examples of

- $g \circ f$ is 1:1 but not onto
- $g \circ f$ is onto but not 1:1
- $g \circ f$ is neither 1:1 nor onto

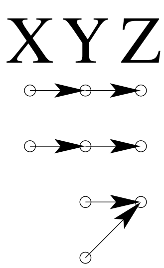
not necessarily in that order.
Answer. Well, for $g \circ f$ to be 1:1 but not onto, we need $|Z| \geq |X|$. For onto but not 1:1, we need $|X| \geq |Z|$. That nails down which picture should be turned into the answer to which question.



neither



onto, not 1:1



1:1 not onto

4. Recall (?) from calculus that a function $f : \mathbb{R} \to \mathbb{R}$ is **continuous at** x if $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x'$ with $|x - x'| < \delta$, one has $|f(x) - f(x')| < \epsilon$.

a. Write out what it means for f to *not* be continuous at x, without using the word "not" anywhere; instead the final inequality should go $\geq$.

Continue from: "A function f is not continuous at x if $\exists \epsilon > 0$ such that..."

Answer. ...$\forall \delta > 0$, $\exists x'$ such that $|x - x'| < \delta$ but $|f(x) - f(x')| \geq \epsilon$.

(Move the "not" past $\forall$, $\exists$, $\forall$; finally you get "not $|f(x) - f(x')| < \epsilon$", which is the same as "$|f(x) - f(x')| \geq \epsilon$".)