Prof. Allen Knutson's Math 20b final exam

1 [10 pts]. Compute the indefinite integral

$$\int \sin(4x)\sqrt{1+\cos(4x)}\ dx$$

and put a $\boxed{\text{box around your answer}}$.

Answer. Let $u = 1+\cos(4x)$, so $du = -4\sin(4x)\ dx$. Then

$$\begin{aligned}
\int \sin(4x)\sqrt{1+\cos(4x)}\ dx &= \int \frac{1}{-4}\sqrt{u}\ du \\
&= \frac{1}{-4}\frac{u^{3/2}}{3/2} + C \\
&= \frac{-1}{6}(1+\cos(4x))^{3/2} + C
\end{aligned}$$

A few people used the double-angle formula, which works out nicely too.

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2 [5 pts]. Let n be a natural number, i.e. $0, 1, 2, 3, \ldots$. Compute $\int_{x=0}^{n\pi} |\sin x|\ dx$. $\boxed{\text{Box}}$ your answer.

Answer.

It is not true that $\int |\sin x|\ dx = |\cos x|$. Rather, one should look at the graph of $|\sin x|$, which is always nonnegative; the question is asking for the area under n of its humps. (If you integrated this positive function and got a negative number you should have been worried about that.)

Each hump has area

$$\int_{x=0}^{\pi} |\sin x|\ dx = \int_{x=0}^{\pi} \sin x\ dx = -\cos x|_{x=0}^{x=\pi} = -(-1) - (-1) = 2$$

so the answer is $2n$.

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3 [10 pts]. Let x be a positive real number, and consider the infinite series

$$\sum_{n=0}^{\infty} \frac{x^n}{n + x^{2n}}.$$

For which positive values of x does it converge?

(Your answer should say "for these x, it converges because of this test, and for all other x, it diverges because of this other test.")

Answer. Let's try the ratio test:

$$\frac{\frac{x^{n+1}}{n+1+x^{2(n+1)}}}{\frac{x^n}{n+x^{2n}}} = x \frac{n + x^{2n}}{n + 1 + x^{2(n+1)}}$$

What happens as $n \to \infty$?

If $|x| < 1$, then the terms $x^{2n} \to 0$, and the ratio goes to

$$x \frac{n + x^{2n}}{n + 1 + x^{2(n+1)}} = x \frac{1 + x^{2n}/n}{1 + 1/n + x^{2(n+1)}/n}$$

which goes to $x \cdot 1$. So by the ratio test, it converges.

If $|x| > 1$, then the terms $x^{2n} \to \infty$. So let's divide top and bottom by them.

$$x \frac{n + x^{2n}}{n + 1 + x^{2(n+1)}} = \frac{1}{x} \frac{nx^{-2n} + 1}{(n+1)x^{-2n-2} + 1}$$

Now the ratio goes to $1/x$. Which is less than 1, so again, by the ratio test, it converges.

If $x = 1$, then the sum is $\sum 1/(n + 1)$, which diverges by comparison to the sum $\sum 1/n$.

In sum, for $x = 1$ the series diverges, for $x \neq 1$ it converges.

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4. Say

$$\int_{x=b}^{c} f(x)\, dx = \sqrt{c} - 3$$

for all positive c.

a [5 pts]. What is the function $f(x)$?

Answer. Differentiate both sides w.r.t. c, and get

$$f(c) = (\sqrt{c} - 3)' = \frac{1}{2\sqrt{c}}$$

or $f(x) = 1/2\sqrt{x}$.

b [5 pts]. What is the number b?

Answer. When $b = c$, the left side is 0. So $0 = \sqrt{c} - 3$, hence $c = 9$. So $b = 9$.

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5 [10 pts]. Compute

$$\int_{x=0}^{x=b} \frac{dx}{x^4 - 1}, \qquad |b| < 1.$$

Answer. Do partial fractions and get

$$\frac{1}{x^4 - 1} = \frac{1/4}{x - 1} - \frac{1/4}{x + 1} - \frac{1/2}{x^2 + 1}$$

(or break it up further using $x^2 + 1 = (x - i)(x + i)$ if you want). Then the integral is

$$\int_{x=0}^{x=b} \frac{dx}{x^4 - 1} = \int_{x=0}^{x=b} \left(\frac{1/4}{x - 1} - \frac{1/4}{x + 1} - \frac{1/2}{x^2 + 1} \right) dx$$

$$= \left(\frac{1}{4} \ln(x - 1) - \frac{1}{4} \ln(x + 1) - \frac{1}{2} arctan(x) \right) |_{x=0}^{x=b} = \frac{1}{4} \ln(b - 1) - \frac{1}{4} \ln(b + 1) - \frac{1}{2} arctan(b).$$

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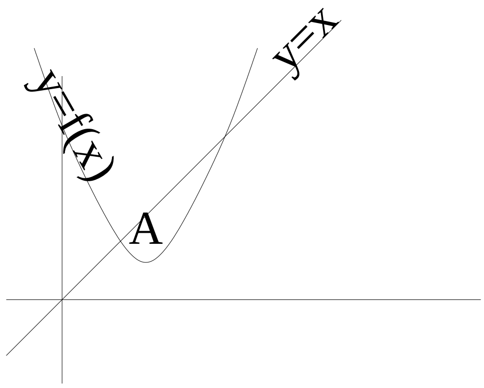
6 [10 pts]. Let $f(x) = x^2 - 3x + 3$. Compute the area of the region in the picture. Your answer should look like $\boxed{A = \text{this}}$, including the box.



Answer. First figure out the limits of integration:

$$f(x) = y = x \quad \text{at} \quad x^2 - 3x + 3 = x$$

so

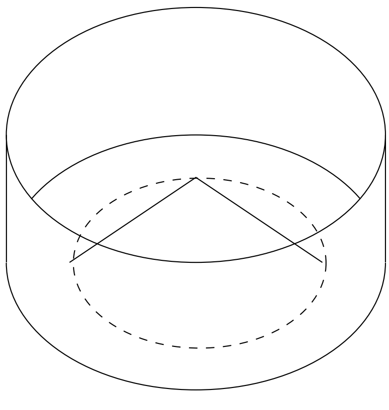
$$0 = x^2 - 3x + 3 - x = (x-1)(x-3).$$

Then compute

$$\int_{x=1}^{x=3} (x-(x^2-3x+3))\,dx = (-\frac{x^3}{3}+2x^2-3x)|_{x=1}^{x=3} = (-9+2*3^2-3*3)-(-\frac{1}{3}+2-3) = \frac{4}{3}.$$

7.

You have a cylindrical kiddie pool with a built-in mountain in the center (for playing "King of the Kiddie Pool," of course). *BONUS:* You can turn it over, put water in the the cone, and use it as a water table, too!
The height of the conical mountain is h, the height of the cylinder is c, the radius of the cone is r, and of the cylinder is s.



a [5 pts]. When full of water (this side up), the water forms a volume of revolution. Of what? Draw an area that, if we revolve it around the vertical axis, sweeps this out. Label the edges with their lengths (or at least, label the easy ones). If you need to assume something about c vs. h, explain in English why you're justified in doing so.

Answer. Consider a rectangle of width s, height c, but missing a triangle in its lower left of width r, height h.

For this to make sense we need $h \leq c$. But we know the mountain isn't taller than the kiddie pool, or else we wouldn't be able to turn it upside down as advertised. (Some people insisted that $c = h$, because standing on a tiny mountain wasn't any fun. Full marks for that too of course.)

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b [5 pts]. How much water can this hold (this side up)? Box your answer.

Answer.

One way to do it is to take the volume of the cylinder, $\pi s^2 c$, minus the volume of the cone, $\frac{1}{3}\pi r^2 h$. Or you could treat it as a volume of revolution, or slice it in horizontal slices and add those up.

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8 [5 pts]. Indefinitely integrate $\int x \cos x \, dx$.

Answer. If $u = x$ and $dv = \cos x \, dx$, then $du = dx$ and $v = \sin x$, giving

$$\int x \cos x \, dx = \int u \, dv = uv - \int v \, du = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

Easy to check, and most of you had plenty of time to do so:

$$(x \sin x + \cos x)' = x \cos x + \sin x - \sin x = x \cos x.$$

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9 [5 pts]. Consider the set of curves $\{y = 1/(C + x^2)\}$, where C is any real number. Find a first-order differential equation that they all satisfy, and box it.

Answer. We need to get rid of the C. The easiest way is to differentiate it and make it go away. So expose it first:

$$C + x^2 = 1/y$$

then differentiate with respect to x, so $2x = y'/y^2$.

If you'd rather write $y' = 2xy^2$ feel free.

Again, it's easy to check. If $y = 1/(C + x^2)$, then

$$y' = \frac{2x}{(C + x^2)^2} = 2x(\frac{1}{C + x^2})^2 = 2xy^2.$$

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10 [10 pts]. For which values of b does

$$\int_{x=0}^{\infty} (e^{bx} - e^{-bx})\ dx$$

converge? Your answer should say "for these b it does, by this test, whereas for these b it doesn't, by this other test".

Answer. This was just like the practice problem. If $b > 0$, then the first term doesn't go to 0 (it goes to ∞) as $x \to \infty$, while the second one does, so the difference doesn't go to 0. If $b < 0$ the reverse is true, and the integrand still doesn't go to 0.

At $b = 0$, however, we're integrating the 0 function, and the integral *does* converge.

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11 [5 pts]. Compute the first five terms (up to x^4) of the Taylor series of x^2e^x. Then box them. Feel free to use Taylor series you may already know.

Answer. We did

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots$$

in class. Multiplying by x^2:

$$x^2 e^x = x^2 + x^3 + \frac{x^4}{2!} + \dots$$

12a [5 pts]. For which b does $y = e^{bx}$ satisfy $y'' + 15y = 8y'$? $\boxed{\text{Answer in a box.}}$

Answer. Plug it in:

$$(e^{bx})'' + 15e^{bx} = 8(e^{bx})'$$

$$b^2 e^{bx} + 15e^{bx} = 8be^{bx}$$

$$b^2 + 15 = 8b$$

$$b^2 - 8b + 15 = 0$$

Now the quadratic formula gets you $b = 3, 5$. (Easy to check: $3^2 + 15 = 8 * 3$, $5^2 + 15 = 8 * 5$.)

12b [5 pts].

Write down ($\boxed{\text{inside a box}}$) infinitely many solutions of $y'' + 15y = 8y'$ that are *not* of the form Ce^{bx}.

Answer. This is a linear homogeneous equation. We learned how to make new solutions by scaling and adding old solutions. So $C(e^{3x} + e^{5x})$ is a solution, and unless $C = 0$ it's not of the forbidden form.