**The Multi-Period Binomial Model (Cox-Ross-Rubinstein Model)**
**Mathematical Finance Lecture Notes**
**Winter 2006**


**Doob's stopping theorem**. If $\{Z_t, t = 0, \ldots, T\}$ is a $\mathbf{P}^*$-martingale, then for any stopping time $\tau$, $\mathbf{E}^*[Z_\tau] = Z_0$.
**Proof**. We have

$$Z_\tau = \sum_{t=0}^{T} 1_{\{\tau=t\}} Z_t.$$

Note that for $0 \le t \le T-1$, $1_{\{\tau=t\}} = 1_{\{\tau\ge t\}} - 1_{\{\tau\ge t+1\}}$. Also, $1_{\{\tau=T\}} = 1_{\{\tau\ge T\}}$. By substituting and rearranging,

$$\begin{aligned} Z_\tau &= \sum_{t=0}^{T-1} \left(1_{\{\tau\ge t\}} - 1_{\{\tau\ge t+1\}}\right) Z_t + 1_{\{\tau\ge T\}} Z_T \\ &= \sum_{t=0}^{T} 1_{\{\tau\ge t\}} Z_t - \sum_{t=0}^{T-1} 1_{\{\tau\ge t+1\}} Z_t \\ &= \sum_{t=0}^{T} 1_{\{\tau\ge t\}} Z_t - \sum_{t=1}^{T} 1_{\{\tau\ge t\}} Z_{t-1} \\ &= Z_0 + \sum_{t=1}^{T} 1_{\{\tau\ge t\}} (Z_t - Z_{t-1}). \end{aligned}$$

Note that $1_{\{\tau\ge t\}}$ is $\mathcal{F}_{t-1}$ measurable. Therefore,

$$\begin{aligned} \mathbf{E}^*\left[1_{\{\tau\ge t\}}(Z_t - Z_{t-1})\right] &= \mathbf{E}^*\left[\mathbf{E}^*\left[1_{\{\tau\ge t\}}(Z_t - Z_{t-1}) \mid \mathcal{F}_{t-1}\right]\right] \\ &= \mathbf{E}^*\left[1_{\{\tau\ge t\}} \mathbf{E}^*\left[Z_t - Z_{t-1} \mid \mathcal{F}_{t-1}\right]\right] \\ &= \mathbf{E}^*\left[1_{\{\tau\ge t\}} \left(\mathbf{E}^*\left[Z_t \mid \mathcal{F}_{t-1}\right] - Z_{t-1}\right)\right] \\ &= \mathbf{E}^*\left[1_{\{\tau\ge t\}} \left(Z_{t-1} - Z_{t-1}\right)\right] \\ &= 0. \end{aligned}$$

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