Homework 14 (due 10/18): Related rate problems. Section 3.9 (1, 4, 5, 7, 9, 11, 13, 15, 27, 29)

1. \( V = x^3 \implies \frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt} = 3x^2 \frac{dx}{dt} \).

4. \( y = \sqrt{1 + x^3} \implies \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{1}{2} (1 + x^3)^{-1/2} (3x^2) \frac{dx}{dt} = \frac{3x^2}{2\sqrt{1 + x^3}} \frac{dx}{dt} \). With \( \frac{dy}{dt} = 4 \) when \( x = 2 \) and \( y = 3 \), we have \( 4 = \frac{3}{2} (4) \frac{dx}{2} \implies \frac{dx}{dt} = 2 \text{ cm/s.} \)

5. (a) Given: a plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. If we let \( t \) be time (in hours) and \( x \) be the horizontal distance traveled by the plane (in mi), then we are given that \( \frac{dx}{dt} = 500 \text{ mi/h}. \)

(b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 2 mi from the station. If we let \( y \) be the distance from the plane to the station, then we want to find \( \frac{dy}{dt} \) when \( y = 2 \) mi.

(d) By the Pythagorean Theorem, \( y^2 = x^2 + 1 \implies 2y \frac{dy}{dt} = 2x \frac{dx}{dt}. \)

(e) \( \frac{dy}{dt} = \frac{x}{y} \frac{dx}{dt} = 500 \frac{x}{y}. \) When \( y = 2 \), \( x = \sqrt{3}, \) so \( \frac{dy}{dt} = 500 \left( \frac{\sqrt{3}}{2} \right) = 250\sqrt{3} \approx 433 \text{ mi/h}. \)

9. We are given that \( \frac{dx}{dt} = 60 \text{ mi/h} \) and \( \frac{dy}{dt} = 25 \text{ mi/h}. \) \( z^2 = x^2 + y^2 \implies 2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}. \) After 2 hours, \( x = 2 \) (60) = 120 and \( y = 2 \) (25) = 50

\( \implies z = \sqrt{120^2 + 50^2} = 130, \) so

\( \frac{dz}{dt} = \frac{1}{2} \left( \frac{x}{dt} + y \frac{dy}{dt} \right) = \frac{120 \cdot 60 + 50 \cdot 25}{130} = 65 \text{ mi/h}. \)

13. \( A = \frac{1}{2} bh, \) where \( b \) is the base and \( h \) is the altitude. We are given that \( \frac{dh}{dt} = 1 \text{ cm/min} \) and \( \frac{dA}{dt} = 2 \text{ cm}^2/\text{min}. \)

Using the Product Rule, we have \( \frac{dA}{dt} = \frac{1}{2} \left( b \frac{dh}{dt} + h \frac{db}{dt} \right). \) When \( h = 10 \) and \( A = 100, \) we have \( b = 20, \) so

\( 2 = \frac{1}{2} \left( 20 \cdot 1 + 10 \frac{db}{dt} \right) \implies 4 = 20 + 10 \frac{db}{dt} \implies \frac{db}{dt} = \frac{4 - 20}{10} = -1.6 \text{ cm/min.} \)

15. We are given that \( \frac{dx}{dt} = 35 \text{ km/h}, \) \( \frac{dy}{dt} = 25 \text{ km/h}. \) \( z^2 = (x + y)^2 + 100^2 \)

\( \implies 2z \frac{dz}{dt} = 2(x + y) \left( \frac{dx}{dt} + \frac{dy}{dt} \right). \) At 4:00 p.m., \( x = 140 \) and \( y = 100 \)

\( \implies z = 260, \) so

\( \frac{dz}{dt} = \frac{x + y}{z} \left( \frac{dx}{dt} + \frac{dy}{dt} \right) = \frac{140 + 100}{260} \left( \frac{260}{260} (35 + 25) = \frac{720}{13} \approx 55.4 \text{ km/h}. \right) \)
27. With \( R_1 = 80 \) and \( R_2 = 100 \), \( \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{80} + \frac{1}{100} = \frac{9}{8000} = \frac{9}{400}, \) so \( R = \frac{400}{9}. \) Differentiating

\[
\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{with respect to} \ t, \quad \text{we have} \quad \frac{dR}{dt} = -\frac{1}{R^2} \left( \frac{dR_1}{dt} \frac{1}{R_1^2} + \frac{dR_2}{dt} \frac{1}{R_2^2} \right)
\]

\[
dR dt = R^2 \left( \frac{1}{R_1^2} \frac{dR_1}{dt} + \frac{1}{R_2^2} \frac{dR_2}{dt} \right)
\]

When \( R_1 = 80 \) and \( R_2 = 100 \),

\[
dR dt = \frac{400^2}{9^2} \left[ \frac{1}{80^2} (0.3) + \frac{1}{100^2} (0.2) \right] = \frac{107}{810} \approx 0.132 \ \Omega/s.
\]

29.

We are given that \( \frac{dx}{dt} = 2 \) ft/s. \( x = 10 \sin \theta \) \( \Rightarrow \frac{dx}{dt} = 10 \cos \theta \frac{d\theta}{dt}. \) When

\[
\theta = \frac{x}{4}, \quad \frac{d\theta}{dt} = \frac{2}{10 \left( \sqrt{2} \right)} = \frac{\sqrt{2}}{5} \ \text{rad/s}.
\]