CHAPTER 3
GRAPHING AND OPTIMIZATION

In the first section of this chapter we see how first derivatives can be used to determine where functions have local maxima and minima and where they are increasing and decreasing. We use these principles in Section 3.2 to sketch graphs of functions. In Section 3.3 we see how second derivatives are used to find where graphs are concave up and where they are concave down. Then, in Sections 3.4 and 3.5, we use the techniques of Sections 3.1 through 3.3 to find maxima and minima of functions on intervals and to solve narrative problems involving maxima and minima.

Section 3.1

First-derivative tests

OVERVIEW: *In this section we first use the definition of the derivative to show that if a function $y = f(x)$ has a local maximum or minimum, then the local maximum or minimum must occur at a* CRITICAL POINT. *This is a value of x in the interior of the domain of f where $f'(x)$ either is zero or is not defined. Then we prove the Mean Value Theorem, which relates average rates of change to instantaneous rates of change, and use it to show that we can determine where functions are increasing and decreasing by finding where their derivatives are positive and negative.*

Topics:

- ***Local maxima and minima: a necessary condition***
- ***The Mean Value Theorem***
- ***The First-Derivative Test for increasing and decreasing functions***
- ***Local maxima and minima: a sufficient condition***

Local maxima and minima: a necessary condition

We begin with an example of water flowing into and out of a tank. We suppose that the tank is empty at time $t = 0$ (hours) and that water is added until a time t_0 and then drained out, so that the volume at time t for $0 \le t \le 3$ is

$$V(t) = 3t - \tfrac{1}{3}t^3 \text{ (gallons).} \qquad \textbf{(1)}$$

Figure 1 shows the graph of this function. It is continuous on its domain $[0, 3]$ since $y = 3t - \frac{1}{3}t^3$ is a polynomial. Because $V(3) = 3(3) - \frac{1}{3}(3^3) = 0$, the tank is empty at $t = 3$. We want to determine the time t_0 when there is the most water in the tank.

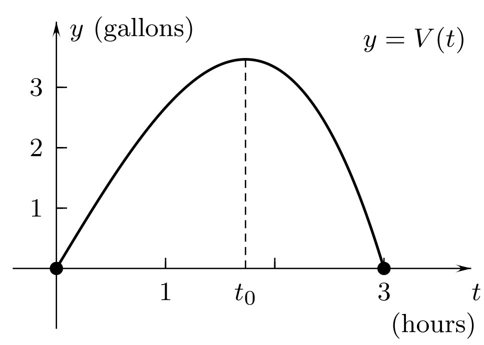


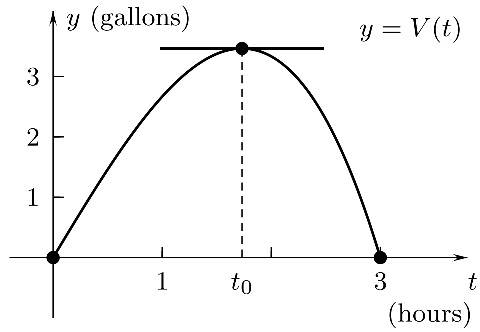
FIGURE 1

The maximum volume at $t = t_0$ is called a LOCAL MAXIMUM of $y = V(t)$ because it is the greatest value of V on an open interval $(0, 3)$ that includes points on both sides of t_0. Here is a general definition of this and the related concept of a LOCAL MINIMUM for a function with x as variable:

Definition 1 **(a)** *$y = f(x)$ has a local maximum at $x = x_0$ if $f(x_0)$ is the greatest value of f on an open interval $a < x_0 < b$ containing x_0, that is, if $f(x_0) \geq f(x)$ for all x in the interval.*

(b) *$y = f(x)$ has a local minimum at $x = x_0$ if $f(x_0)$ is the least value of f on an open interval containing x_0, that is, if $f(x_0) \leq f(x)$ for all x in the interval.*

As you might expect, the local maximum of $y = V(t)$ in Figure 2 is at the point $t = t_0$ where the tangent line to the graph is horizontal and the derivative $V'(t_0)$ is zero. This is a consequence of the following theorem.



Horizontal tangent line

$V'(t_0) = 0$

FIGURE 2

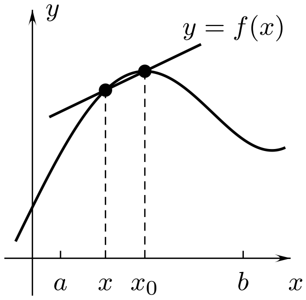
Theorem 1 (A necessary condition for a local maximum or minimum) *If $y = f(x)$ has a local maximum or local minimum at $x = x_0$, then either* **(a)** *$f'(x_0) = 0$ or* **(b)** *$f'(x_0)$ does not exist.*

If $y = f(x)$ is defined on an open interval containing x_0 and either $f'(x_0)$ is zero or does not exist, then x_0 is called a CRITICAL POINT (or CRITICAL NUMBER) of the function. Consequently, Theorem 1 can be rephrased as the statement that if $y = f(x)$ has a local maximum or minimum at x_0, then x_0 is a critical point of the function.

Proof of Theorem 1: Suppose that f is a function, as in Figure 3, that has a local maximum at a point $x = x_0$ and that the derivative $f'(x_0)$ exists. Then the function f is defined on an open interval (a, b) containing x_0 such that $f(x) \leq f(x_0)$ for $a < x < b$. We need to show that the derivative $f'(x_0)$ is 0. This derivative is the two-sided limit as $x \to x_0$ of the difference quotient,

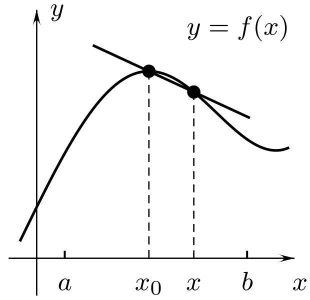
$$\frac{f(x) - f(x_0)}{x - x_0} \tag{2}$$

which is the slope of the secant line through the points at $x = x$ and $x = x_0$ on the graph of $y = f(x)$.



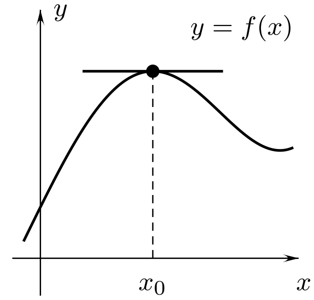
Slope ≤ 0

FIGURE 3



Slope ≥ 0

FIGURE 4



Slope $= 0$

FIGURE 5

For $a < x < x_0$, as in Figure 3, the slope **(2)** is ≥ 0 because $f(x) - f(x_0)$ is ≤ 0 and $x - x_0$ is < 0. Consequently, $f'(x_0)$, which is the limit of **(2)** as x approaches x_0 from the left, is ≥ 0.

For $a < x < x_0$, as in Figure 4, the slope **(2)** is ≤ 0 because $f(x) - f(x_0)$ is ≤ 0 and $x - x_0$ is > 0. This implies that, $f'(x_0)$, which is also the limit of **(2)** as x approaches x_0 from the right, is ≤ 0. Since $f'(x_0)$ is both ≥ 0 and ≤ 0, it equals 0. (Figure 5) This shows that if $f'(x_0)$ exists, then it is zero. A similar argument shows that $f'(x_0)$ is zero or does not exist if f has a local minimum at x_0. **QED**

Example 1 Find the local maximum of the volume function $V = 3t - \frac{1}{3}t^3$ of Figures 1 and 2.

SOLUTION The derivative of V is

$$V'(t) = \frac{d}{dt}(3t - \tfrac{1}{3}t^3) = 3 - t^2. \tag{3}$$

The polynomial $3 - t^2$ on the right of **(3)** is zero at $t = \pm\sqrt{3}$, so the only point in the domain $[0, 3]$ of V where $V'(t)$ is 0 is $t = \sqrt{3}$ and this is the only critical point of V. By Theorem 1, the local maximum of V is at $t_0 = \sqrt{3}$ and is $V(\sqrt{3}) = 3(\sqrt{3}) - \frac{1}{3}(\sqrt{3})^3 = 2\sqrt{3}$ gallons. □

The local maximum $V(\sqrt{3}) = 2\sqrt{3}$ of the volume in Example 1 is also the function's GLOBAL MAXIMUM (or ABSOLUTE MAXIMUM) because it is the greatest value of the function in its domain $[0, 3]$. The function's GLOBAL MINIMUM is its value $V = 0$ at $t = 0$ and $t = 3$.

Question 1 Why can we not use Theorem 1 to conclude that the derivative of $V = V(t)$ from Example 1 is zero at $x = 0$ and $x = 3$, where it has a global minimum?

The function $f(x) = 3 - |x|$, whose graph is in Figure 6, illustrates case (b) of Theorem 1. The function has a local maximum at $x = 0$, and this is a critical point because $f(x) = 3 - |x|$ does not have a derivative at $x = 0$.

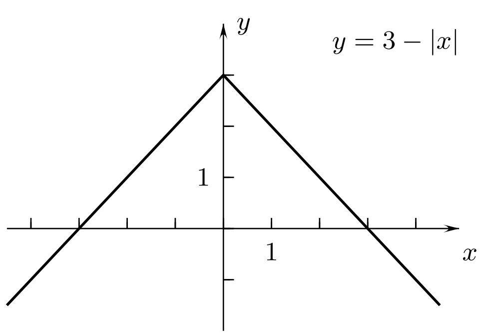


FIGURE 6

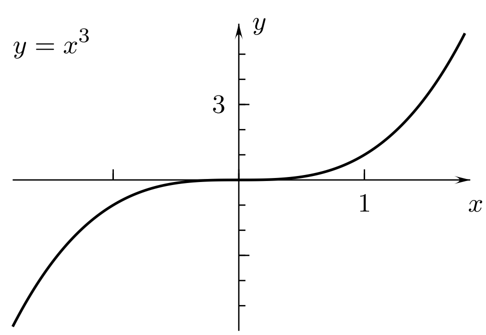


FIGURE 7

It is important to realize that the converse to Theorem 1 is not valid: A function can have a critical point without having a local maximum or minimum there. This is the case for the function $y = x^3$ of Figure 7. The derivative $y' = \frac{d}{dx}(x^3) = 3x^2$ is zero at $x = 0$, and the function is defined for all x, so $x = 0$ is a critical point. Nevertheless, the value $0^3 = 0$ of the function at $x = 0$ is not a local maximum or local minimum because the function's value x^3 at x is less than 0 for $x < 0$ and greater than 0 for $x > 0$.

Example 2 The function $y = \frac{1}{2}x^2 - 2\sqrt{x}$, whose graph is shown in Figure 8, has a local minimum. Find its value.

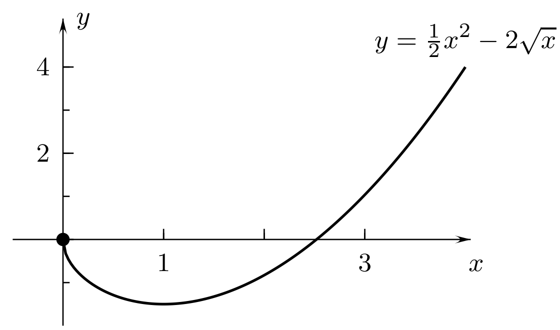


FIGURE 8

SOLUTION The derivative

$$y'(x) = \frac{d}{dx}(\tfrac{1}{2}x^2 - 2x^{1/2}) = x - x^{-1/2} = x - \frac{1}{x^{1/2}} = \frac{x^{3/2} - 1}{x^{1/2}}$$

exists for all $x > 0$ and is zero at $x = 1$, where $x^{3/2} = 1$. Consequently $x = 1$ is the function's one critical point. Since we are told that $y = y(x)$ has a local minimum, it is at $x = 1$, by Theorem 1. The value of the local minimum is $y(1) = \frac{1}{2}(1)^2 - 2\sqrt{1} = -\frac{3}{2}$. (Figure 8 shows that the local minimum is, in fact, the global minimum of the function.) □

The Mean Value Theorem

The next theorem, which relates average rates of change of functions to their derivatives, will be used below to derive criteria for determining where functions are increasing and decreasing. It will also be used in later chapters in proofs of other fundamental results.

Theorem 2 (The Mean Value Theorem) *Suppose that $y = f(x)$ is continuous on the finite closed interval $[a, b]$ and that $f'(x)$ exists for all x with $a < x < b$. Then there is at least one number c with $a < c < b$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{4}$$

Theorem 2 has the geometric interpretation shown in Figure 9. The difference quotient $[f(b) - f(a)]/(b - a)$ on the right of **(4)** is the slope of the secant line between the points at $x = a$ and $x = b$ on the graph $y = f(x)$. Under the conditions of the theorem, there is a number c between a and b, as in the drawing, such that the slope $f'(c)$ of the tangent line at $x = c$ is equal to the slope to the secant line and consequently the tangent line is parallel to the secant line.

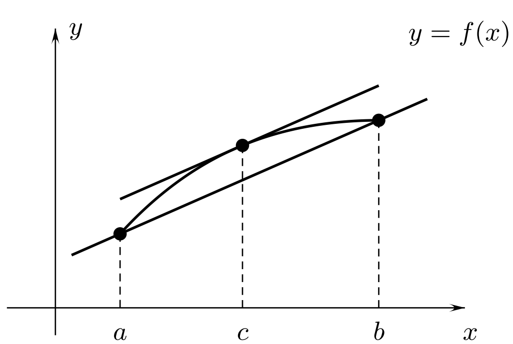


FIGURE 9

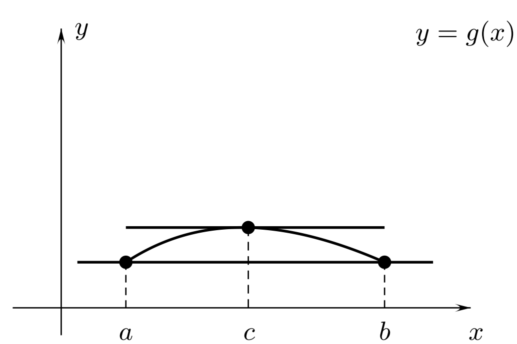


FIGURE 10

Proof of the Mean Value Theorem: We let m denote the slope $[f(b) - f(a)]/(b - a)$ of the secant line and define a new function $g(x) = f(x) - mx$ (Figure 10). This function is also continuous on $[a, b]$. It has the same value at $x = a$ as at $x = b$ because

$$g(b) - g(a) = [f(b) - mb] - [f(a) - ma][f(b) - f(a)] - m(b - a)$$
$$= [f(b) - f(a)] - [f(b) - f(a)] = 0.$$

Consequently, the secant line through the points at $x = a$ and $x = b$ on the graph of $y = g(x)$ is horizontal.

According to the Extreme Value Theorem, which we will discuss in Section 3.4, g has a maximum value and a minimum value on $[a, b]$. If the maximum and minimum both equal $g(a)$, then $g(x)$ is constant and $g'(c) = 0$ for all x with $a < c < b$. If the maximum is greater than $g(a)$, then it occurs at a point c with $a < c < b$ and is a local maximum so that $g'(c) = 0$ by Theorem 1 above. If neither of these conditions holds, then the minimum is less than $g(a)$ and it occurs at a c with $a < c < b$. It is a local minimum, so that $g'(c) = 0$, again by Theorem 1.

In any case, there is a c with $a < c < b$ such that $g'(c) = 0$. Since $g'(x) = \frac{d}{dx}[f(x) - mx]$ $= f'(x) - m$, this implies $f'(c) = m$ and gives **(4)** since $m = [f(b) - f(a)]/(b - a)$. **QED**

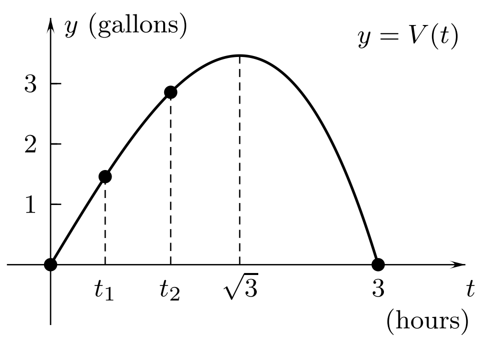
Question 2 Sketch the graph of a differentiable, nonlinear function $y = f(x)$ such that $f(1) = 1$ and $f(4) = 4$. Find the approximate value of a number c with $1 < c < 4$ such that $f'(c) = 1$ and draw the tangent line at $x = c$. Then explain how your drawing relates to the Mean Value Theorem.

The First-Derivative Test for increasing and decreasing functions

We return to the volume function $V = 3t - \frac{1}{3}t^3$ of Figure 1 to describe our next result. As can be seen from the graph in Figure 11, the volume $V(t_1)$ at time t_1 is less than the volume $V(t_2)$ at time t_2 for any t_1 and t_2 with $0 \le t_1 < t_2 \le \sqrt{3}$. We say that $y = V(t)$ is INCREASING ON THE INTERVAL $[0, \sqrt{3}]$. Similarly, Figure 12 shows that the volume $V(t_1)$ at time t_1 is less than the volume $V(t_2)$ at time t_2 for any t_1 and t_2 with $\sqrt{3} \le t_1 < t_2 \le 3$. Accordingly, we say that $y = V(t)$ is DECREASING ON THE INTERVAL $[\sqrt{3}, 3]$. This illustrates the following general definition.

Definition 2 **(a)** *$y = f(x)$ is* INCREASING ON AN INTERVAL *if $f(x)$ increases as x increases across the interval. This means that $f(x_1) < f(x_2)$ for all x_1 and x_2 in the interval with $x_1 < x_2$.*

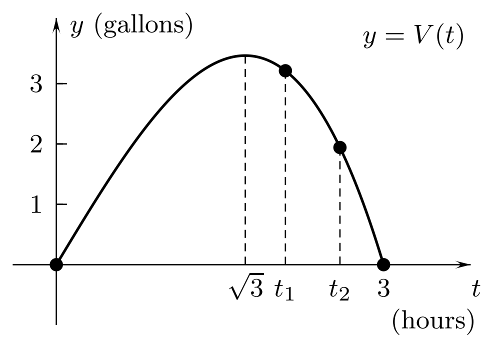
(b) *$y = f(x)$ is* DECREASING ON AN INTERVAL *if $f(x)$ decreases as x increases across the interval. This means that $f(x_1) > f(x_2)$ for all x_1 and x_2 in the interval with $x_1 < x_2$.*



$V(t_1) < V(t_2)$ if $0 \le t_1 < t_2 \le \sqrt{3}$

$y = V(t)$ is increasing on $[0, \sqrt{3}]$.

FIGURE 11



$V(t_1) > V(t_2)$ if $\sqrt{3} \le t_1 < t_2 \le 3$

$y = V(t)$ is decreasing on $[\sqrt{3}, 3]$.

FIGURE 12

We can show that $y = V(t)$ of Figure 1 is increasing on $[0, \sqrt{3}]$ and decreasing on $[\sqrt{3}, 3]$, as is indicated in Figures 11 and 12, by applying the next theorem.

Theorem 3 (The First-Derivative Test for increasing and decreasing functions)

(a) *If $y = f(x)$ is continuous on an interval and $f'(x)$ exists and is positive at all points in the interior of the interval, then $y = f(x)$ is increasing on the interval.*†

(b) *If $y = f(x)$ is continuous on an interval and $f'(x)$ exists and is negative at all points in the interior of the interval, then $y = f(x)$ is decreasing on the interval.*

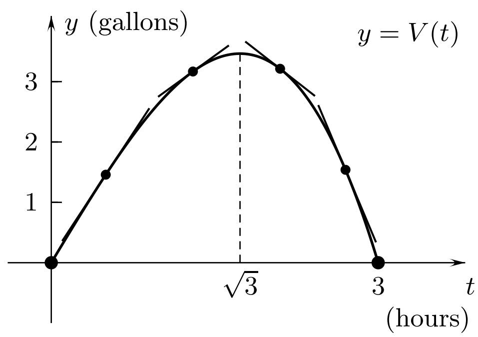
We will often use this theorem with open intervals. Then we only have to determine whether the derivative is positive or negative on the interval, since, by Theorem 1 of Section 2.5, functions are automatically continuous on any open intervals where they have derivatives.

To apply the theorem to the volume function $y = V(t)$, which is continuous on $[0, 3]$, we note that its derivative,

$$V'(t) = 3 - t^2 \qquad \textbf{(5)}$$

is positive for $0 < t < \sqrt{3}$ and negative for $\sqrt{3} < t < 3$, so that the tangent lines to the graph of the volume have positive slopes for $0 < t < \sqrt{3}$ and have negative slopes for $\sqrt{3} < t < 3$, as shown in Figure 13. By Theorem 3, V is increasing on $[0, \sqrt{3}]$ and decreasing on $[\sqrt{3}, 3]$, as we saw earlier from its graph.

FIGURE 13



Proof of Theorem 3: Suppose first that f is continuous on an interval I and that $f'(x)$ exists and is positive on the interior of I. Consider any points a and b in I with $a < b$. The function f is continuous on $[a, b]$ and $f'(x)$ exists for $a < x < b$, so that by the Mean Value Theorem, there is a number c with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \qquad \textbf{(6)}$$

Since $f'(c)$ and $b - a$ are positive, equation **(4)** shows that $f(b) - f(a)$ is positive. Hence, $f(b) > f(a)$ and, since a and b are arbitrary points in I with $a < b$, f is increasing on I, as asserted in part (a) of the theorem.

If $f'(x)$ is negative for x on the interior of I, then the number **(6)** is negative. This implies that $f(b) < f(a)$ for any a and b in I with $a < b$, so that f is decreasing on I, as stated in part (b). **QED**

†Recall that the interior of an open interval is the interval itself, and the interior of a nonopen interval is the open interval obtained by removing its endpoint or endpoints.

Example 3 **(a)** Find the open intervals on which the function $f(x) = x + 4/x^2$ is increasing and in which it is decreasing. **(b)** Are there any larger nonopen intervals on which f is increasing and decreasing?[†]

SOLUTION **(a)** The derivative of $f(x) = x + 4/x^2$ is

$$f'(x) = \frac{d}{dx}(x + 4x^{-2}) = 1 - 8x^{-3} = 1 - \frac{8}{x^3} = \frac{x^3 - 8}{x^3}. \tag{7}$$

The derivative is defined and continuous on the open intervals $(-\infty, 0)$ and $(0, \infty)$ and is zero at $x = 2$, where $x^3 = 8$. It, therefore, is either positive or negative in each of the open intervals $(-\infty, 0), (0, 2)$ and $(2, \infty)$.[‡] We can find the sign of $f'(x)$ on these intervals by calculating its value at one point in each. If we use $x = -1$ in $(-\infty, 0)$, $x = 1$ in $(0, 2)$, and $x = 3$ in $(2, \infty)$, and apply Theorem 3, we obtain the following:

$$f'(-1) = \frac{(-1)^3 - 8}{(-1)^3} = \frac{-9}{-1} = 9 > 0 \Longrightarrow f' > 0 \quad \text{and } f \nearrow \text{ on } (-\infty, 0)$$

$$f'(1) = \frac{1^3 - 8}{1^3} = -7 < 0 \Longrightarrow f' < 0 \quad \text{and } f \searrow \text{ on } (0, 2)$$

$$f'(3) = \frac{3^3 - 8}{3^3} = \tfrac{19}{27} > 0 \Longrightarrow f' > 0 \quad \text{and } f \nearrow \text{ on } (2, \infty).$$

This information is displayed in Figure 14. The function f is increasing on the open intervals $(-\infty, 0)$ and $(2, \infty)$ and is decreasing on the open interval $(0, 2)$. This is corroborated by its graph in Figure 14.

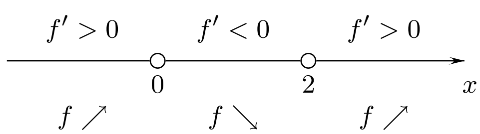


FIGURE 14

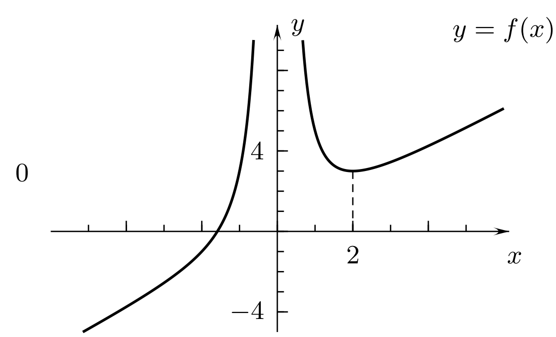


FIGURE 15

(b) We saw in part (a) that f is increasing on the open interval $(-\infty, 0)$. It is not increasing on any larger interval since it is not defined at $x = 0$.

f is decreasing on the interval $(0, 2]$ because it is continuous on this interval and its derivative is positive in its interior $(0, 2)$. It is not decreasing on any larger interval because it is not defined at $x = 0$ and is not decreasing for any $x > 2$.

f is increasing on $[2, \infty)$ since it is continuous in this interval and its derivative is positive in the interior $(2, \infty)$. It is not increasing on any larger interval because it is not decreasing for any x with $0 < x < 2$. □

Question 3 Use the last formula in **(7)** to explain why $f'(x)$ changes sign at $x = 0$ and at $x = 2$.

[†]Recall that an interval I_1 is said to be larger than an interval I_2 if I_1 contains I_2 and is not equal to it.

[‡]This is a consequence of the Intermediate Value Theorem from Section 1.3 since if $f'(x)$ is positive at one point and negative at another, it must be zero or discontinuous at some point in between.

Local maxima and minima: a sufficient condition

Theorem 3 yields the following criteria for locating local maxima and minima.

Theorem 4 (Sufficient conditions for local maxima and minima) *Suppose that $y = f(x)$ is continuous on an open interval (a, b) containing x_0.*

(a) *If $f'(x)$ is positive on (a, x_0) and negative on (x_0, b), then f has a local maximum at x_0.*

(b) *If $f'(x)$ is negative on (a, x_0) and positive on (x_0, b), then $f)$ has a local minimum at x_0.*

(c) *If $f'(x)$ is positive on (a, x_0) and on (x_0, b) or is negative on (a, x_0) and on (x_0, b), then f does not have a local maximum or mimumm at x_0.*

Proof: Under the conditions of part (a), f is increasing on $(a, x_0]$ and decreasing on $[x_0, b)$ by Theorem 3 and therefore has a local maximum at x_0.

In case (b), Theorem 3 implies that f is decreasing on $(a, x_0]$ and increasing on $[x_0, b)$ and therefore has a local minimum at x_0.

In case (c), Theorem 3 implies that f is either increasing on $(a, x_0]$ and on $[x_0, b)$ or is decreasing on both of these intervals, so that it does not have a local maximum or minimum at x_0. **QED**

In applying Theorem 4, visualize a curve whose tangent lines have positive slopes where $f'(x)$ is positive and have negative slopes where $f'(x)$ is negative.

Example 4 Find the local maxima and minima of $g(x) = (x^2 - 1)^2$.

SOLUTION By the Chain Rule for differentiating powers of functions,

$$\begin{aligned} g'(x) &= \frac{d}{dx}[(x^2-1)^2] = 2(x^2-1)\frac{d}{dx}(x^2-1) \\ &= 4x(x^2-1) = 4x(x+1)(x-1). \end{aligned} \tag{8}$$

This derivative is zero at $x = -1, x = 0$, and $x = 1$ and is, therefore, of constant sign for $x < -1$, for $-1 < x < 0$, for $0 < x < 1$, and for $x > 1$. We could determine the signs by calculating the value of the derivative **(8)** at one point in each interval, as we did in solving Example 3. Instead, we find the signs of each of the factors in **(8)** in each of the intervals, as in the next table.

	$x+1$	x	$x-1$	$g'(x) = 4(x+1)x(x-1)$	
$x < -1$	< 0	< 0	< 0	< 0	$g \searrow$
$-1 < x < 0$	> 0	< 0	< 0	> 0	$g \nearrow$
$0 < x < 1$	> 0	> 0	< 0	< 0	$g \searrow$
$x > 1$	> 0	> 0	> 0	> 0	$g \nearrow$

This information is displayed in Figure 16. Because g is continuous on $(-\infty, \infty)$ it has, by Theorem 4, local minima at $x = -1$ and at $x = 1$ and a local maximum at $x = 0$. The graph of the function in Figure 17 shows that the local minima are, in fact, the absolute minimum of the function and that the local maximum is not a global maximum. □

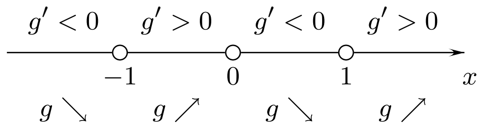


FIGURE 16

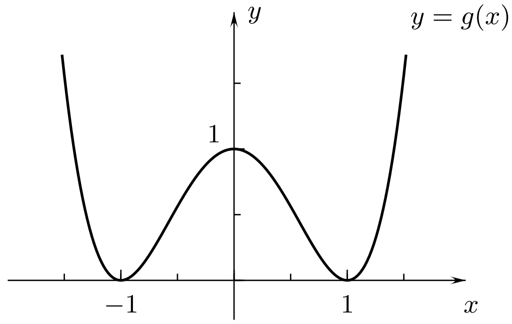


FIGURE 17

Question 4 The function $f(x) = x + 4/x^2$ of Figure 15 is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$, but does not have a local maximum at $x = 0$. Explain.

Responses 3.1

Response 1 Theorem 1 does not apply to the global minimum of $V = V(t)$ at $t = 0$ and $t = 2$ because the function does not have local minima at those points.

Response 2 Two answers: Figures R2a with $c \approx 2.5$ and R2b with $c \approx 3.4$ • Because the slope of the secant line from $x = 1$ to $x = 4$ is 1, the Mean Value Theorem implies that in each case there is a number c with $1 < c < 4$ such that the tangent line at $x = c$ has slope 1. This is shown in the drawings.

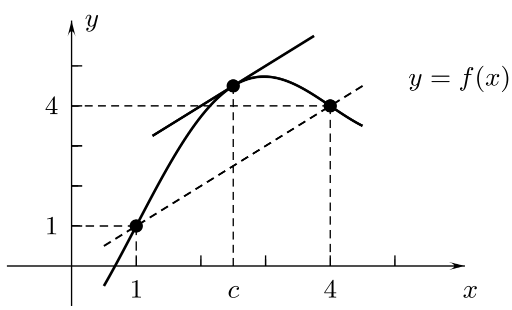


FIGURE R2a

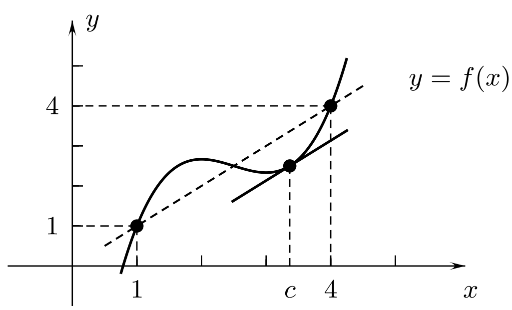


FIGURE R2b

Response 3. $f'(x) = \dfrac{x^3 - 8}{x^3}$ changes sign at $x = 0$ because the denominator x^3 changes sign there and the numerator $x^3 - 8$ does not. • $f'(x)$ changes sign at $x = 2$ because the numerator changes sign there and the denominator does not.

Response 4 $f(x) = x + 4/x^2$ does not have a local maximum at $x = 0$ because it is not defined there and because $f(x) \to \infty$ as $x \to 0$.