Section 3.2

Drawing graphs using first-derivative tests

Overview: In this section we sketch graphs of functions constructed from powers, linear combinations, products, and quotients by studying formulas for the functions and then applying procedures from Section 3.1 to analyze their first derivatives.

Topics:
- Sketching graphs of polynomials
- Graphs of polynomials near \( x = 0 \)
- Sketching graphs of rational functions
- Functions with restricted domains

Sketching graphs of polynomials

As we saw in Section 1.3, the limits of a polynomial as \( x \to \pm \infty \) are determined by its highest degree term. This is because, when \(|x|\) is large, the lower degree terms are relatively small compared to the highest-degree term. The polynomial \( y = \frac{1}{3}x^3 - 2x^2 + 20 \), for example, has the same limits as \( x \to \pm \infty \) as its term of highest degree \( \frac{1}{3}x^3 \):

\[
\lim_{x \to \infty} \left( \frac{1}{3}x^3 - 2x^2 + 20 \right) = \lim_{x \to \infty} \left( \frac{1}{3}x^3 \right) = \infty \\
\lim_{x \to -\infty} \left( \frac{1}{3}x^3 - 2x^2 + 20 \right) = \lim_{x \to -\infty} \left( \frac{1}{3}x^3 \right) = -\infty.
\]

(1)

This is because, for \( x \neq 0 \),

\[
\frac{1}{3}x^3 - 2x^2 + 8 = x^3 \left( \frac{1}{3} - \frac{2}{x} + \frac{8}{x^2} \right)
\]

and the expression in parentheses on the right is approximately equal to \( \frac{1}{3} \) for large positive or negative \( x \).

To determine the overall shape of the graph, we study its first derivative.

Example 1  Sketch the graph of \( y = \frac{1}{3}x^3 - 2x^2 + 20 \). Show where the function is increasing and decreasing and plot any local or global maxima and minima.

Solution  The polynomial \( y = \frac{1}{3}x^3 - 2x^2 + 20 \) is continuous on \((-\infty, \infty)\). Its derivative

\[
y'(x) = \frac{d}{dx} \left( \frac{1}{3}x^3 - 2x^2 + 20 \right) = x^2 - 4x = x(x - 4)
\]

(2)

is zero at \( x = 0 \) and \( x = 4 \) and has constant sign on each of the open intervals \((-\infty, 0), (0, 4), \) and \((4, \infty)\) set off by the zeros. We can determine the signs by calculating the value of the derivative at one point in each interval, and use this information with Theorem 3 of Section 3.1 to see where the function is increasing and decreasing:\footnote{If we allow nonopen intervals in applying Theorem 3 of Section 3.1, we would conclude that the function \( y = \frac{1}{3}x^3 - 2x^2 + 20 \) of Example 1 is, in fact, increasing on the closed intervals \((-\infty, 0]\) and \([4, \infty)\) and is decreasing on the closed interval \([0, 4]\). We only consider open intervals in the solutions of examples, tune-up problems, and basic exercises in this section, but consider nonopen intervals in the solutions of some exploratory exercises.}

\[
y'(-1) = (-1)(-1 - 4) = 5 > 0 \quad \Rightarrow \quad y'(x) > 0 \text{ and } y \nearrow \text{ on } (-\infty, 0) \\
y'(1) = (1)(1 - 4) = -3 < 0 \quad \Rightarrow \quad y'(x) < 0 \text{ and } y \searrow \text{ on } (0, 4) \\
y'(5) = (5)(5 - 4) = 5 > 0 \quad \Rightarrow \quad y'(x) > 0 \text{ and } y \nearrow \text{ on } (4, \infty)
\]
We display this information with an $x$-axis in Figure 1. We use it with the facts, established above, that the function tends to $\infty$ as $x \to \infty$ and tends to $-\infty$ as $x \to -\infty$, and plot the values $y(-3) = \frac{1}{3}(3^3) - 2(3^2) + 20 = -7$, $y(0) = 20$, $y(4) = \frac{1}{3}(4^3) - 2(4^2) + 20 = 9\frac{1}{3}$, and $y(6) = \frac{1}{3}(6^3) - 2(6^2) + 20 = 20$ to obtain the graph in Figure 2. The function has a local maximum of 20 at $x = 0$, and a local minimum of $9\frac{1}{3}$ at $x = 4$. It has no global maxima or minima. □

<table>
<thead>
<tr>
<th>$y' &gt; 0$</th>
<th>$y' &lt; 0$</th>
<th>$y' &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>4</td>
</tr>
</tbody>
</table>

$y \downarrow$ $y \uparrow$ $y \uparrow$

**FIGURE 1**

**FIGURE 2**

**Question 1** Use the factorization (2) to explain why the derivative changes sign
(a) at $x = 0$ and (b) at $x = 4$.

**Graphs of polynomials near $x = 0$**

Just as we can determine the limits of a polynomial as $x \to \pm \infty$ by looking at its term of highest degree, we can anticipate that its graph near $x = 0$ will look like the graph of its lowest degree term or terms. We use the following result.

**Theorem 1** The graph of a polynomial $y = a + bx^{n_0} + \text{[terms of degree} > n_0\text{]}$ with $n_0$ a positive integer and $b \neq 0$ is closely approximated by the graph $y = a + bx^{n_0}$ of its lowest-order term or terms for $x$ sufficiently close to 0.

The approximating polynomial $y = a + bx^{n_0}$ in Theorem 1 has two terms if $a \neq 0$ and one term if $a = 0$. This result is a consequence of the fact that for very small $|x|$, $x^2$ is relatively small compared to $x$, $x^3$ is relatively small compared to $x^2$, $x^4$ is relatively small compared to $x^3$, and so forth (see Exercises 1 and 2). The theorem is illustrated in Figure 3, which shows the graph of the polynomial $y = \frac{1}{3}x^3 - 2x^2 + 20$ of Example 1 with the parabola $y = -2x^2 + 20$ that is the graph of its lowest order terms. The curve $y = \frac{1}{3}x^3 - 2x^2 + 20$ is closely approximated by the parabola $y = -2x^2 + 20$ near $x = 0$ because

$$\frac{1}{3}x^3 - 2x^2 + 20 = \left(\frac{1}{3}x - 2\right)x^2 + 20$$

and for small $|x|$, $\frac{1}{3}x - 2$ is close to $-2$.

$y = \frac{1}{3}x^3 - 2x^2 + 20$
and $y = -2x^2 + 20$

**FIGURE 3**
Drawing graphs of rational functions

Recall that the rational functions are constructed from polynomials by taking linear combinations, products, and quotients. To sketch the graph of a rational function, we first learn what we can by studying its formula. This often involves finding its limits as \( x \to \pm \infty \), studying its behavior near any vertical asymptotes, and noting whether the function is even or odd. In some cases, we can also learn something about the graph for small \( x \) by applying Theorem 1 to polynomials that are parts of the formula for the rational function. Then we study the function’s derivative.

Example 2

Use the formulas for the function and its first derivative to sketch the graph of \( y = \frac{x^2}{x-3} \).

Solution

**STUDYING THE FUNCTION:** Recall that the limits as \( x \to \pm \infty \) of a quotient of two polynomials is the limit of the quotient of their terms of highest degree. Therefore,

\[
\lim_{x \to \infty} \frac{x^2}{x-3} = \lim_{x \to \infty} \frac{x}{x} = \lim_{x \to \infty} x = \infty
\]

\[
\lim_{x \to -\infty} \frac{x^2}{x-3} = \lim_{x \to -\infty} \frac{x^2}{x} = \lim_{x \to -\infty} x = -\infty.
\]

This can be shown by dividing the numerator or denominator by \( x^n \) where \( n \) is the lower of the degrees of the numerator and denominator. In this case, we divide the numerator and denominator by \( x \) to obtain for \( x \neq 0 \),

\[
y = \frac{x^2}{x-3} = \frac{x}{1-3/x}.
\]

(3)

This shows that \( y \to \infty \) as \( x \to \infty \) and \( y \to -\infty \) as \( x \to -\infty \) since \( 1-3/x \) is close to 1 for large positive or negative \( x \). Equation (3) also shows that the graph looks like the line \( y = x \) for large positive and negative \( x \) since \( 1-3/x \approx 1 \) for large \( |x| \).

The function \( y = \frac{x^2}{x-3} \) can change sign either at \( x = 0 \), where its numerator is zero and the function is zero, or at \( x = 3 \), where the denominator is zero and the graph has a vertical asymptote. These two points set off three open intervals \((-\infty, 0), (0, 3), \) and \((3, \infty)\) on which the function is continuous and nonzero has therefore has constant sign. We could find the signs by calculating sample values. Instead, we determine them in this case by finding the signs of the numerator and denominator, as in the following table.

<table>
<thead>
<tr>
<th></th>
<th>( x^2 )</th>
<th>( x-3 )</th>
<th>( \frac{x^2}{x-3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x &lt; 0 )</td>
<td>&gt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>( 0 &lt; x &lt; 3 )</td>
<td>&gt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>( x &gt; 3 )</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
</tr>
</tbody>
</table>

The information we have obtained about the function is displayed in Figure 4, where we have written \( |y| \to \infty \) above 3 to indicate that the graph has a vertical asymptote at \( x = 3 \). Then we can see from Figure 4 that \( y \) tends to \( \infty \) as \( x \to 0^+ \) and tends to \( -\infty \) as \( x \to 0^- \) because \( y > 0 \) just to the right of 0 and \( y < 0 \) just to the left of 0.
STUDYING THE DERIVATIVE: The Quotient Rule gives the formula

\[ y' = \frac{d}{dx} \left( \frac{x^2}{x-3} \right) = \frac{(x-3) \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(x-3)}{(x-3)^2} \]

\[ = \frac{(x-3)(2x) - x^2(1)}{(x-3)^2} = \frac{2x^2 - 6x - x^2}{(x-3)^2} = \frac{x^2 - 6x}{(x-3)^2} = \frac{x(x-6)}{(x-3)^2}. \]  

(4)

The derivative (4) can change sign at \( x = 0 \) and \( x = 6 \), where the numerator is zero and the function has critical points, or at \( x = 3 \), where the denominator is zero and the graph has a vertical asymptote. We could find the signs of the derivative by studying the factors \( x \) and \( x - 3 \) of its numerator and its denominator \( x - 3 \). This time, we calculate sample values instead:

\[ y'(-1) = \frac{(1)(1-6)}{(1-3)^2} = \frac{-5}{4} > 0 \quad \Rightarrow \quad y' > 0 \text{ and } y \nearrow \text{ in } (-\infty, 0) \]

\[ y'(2) = \frac{(2)(2-6)}{(2-3)^2} = -8 < 0 \quad \Rightarrow \quad y' < 0 \text{ and } y \searrow \text{ in } (0,3) \]

\[ y'(4) = \frac{(4)(4-6)}{(4-3)^2} = -8 < 0 \quad \Rightarrow \quad y' < 0 \text{ and } y \searrow \text{ in } (3,6) \].

\[ y'(7) = \frac{(7)(7-6)}{(7-3)^2} = \frac{7}{16} > 0 \quad \Rightarrow \quad y' > 0 \text{ and } y \nearrow \text{ in } (6,\infty) \]

This information is displayed in Figure 5.

DRAWING THE GRAPH: To obtain the graph of the function in Figure 6, we draw the asymptote \( x = 3 \) and have the curve go up on its right side and down on its left side. We plot the values \( y(-6) = (-6)^2/(-6-3) = 36/(-9) = -4, y(0) = 0, \)

\( y(6) = 6^2/(6-3) = 36/3 = 12, \) and \( y(9) = 9^2/(9-3) = 81/6 = 13\frac{1}{2}, \) and use the information in Figures 4 and 5. The function has a local maximum of 0 at \( x = 0, \)

and a local minimum of 12 at \( x = 6. \) Notice that the graph looks like \( y = -\frac{1}{3}x^2 \) near \( x = 0. \) This is because the term \( x \) in the denominator of \( y = \frac{x^2}{x-3} \) is small relative to 3 for small \(|x|\). □
Question 2  Use formula (4) to explain why the derivative in Example 2 changes sign at \( x = 0 \) and at \( x = 6 \), but not at \( x = 3 \).

Example 3  Sketch the curve \( y = \frac{x^2}{x^2 + 1} \) by analyzing the formulas for the function and its first derivative.

Solution  Studying the function: The function \( y = \frac{x^2}{x^2 + 1} \) is even and is positive for all \( x \). Its limits as \( x \to \pm \infty \) are the same as the limits of the highest power terms in its numerator and denominator:

\[
\lim_{x \to \pm \infty} \frac{x^2}{x^2 + 1} = \lim_{x \to \pm \infty} \left(1 - \frac{1}{x^2} \right) = 1.
\]

We could also find these limits by dividing the numerator and denominator by \( x^2 \): For \( x \neq 0 \),

\[
y = \frac{x^2}{x^2 + 1} = \frac{1}{1 + 1/x^2}.
\]

This formula shows that \( \frac{x^2}{x^2 + 1} \to 1 \) as \( x \to \pm \infty \), as we concluded above, so that the line \( y = 1 \) is a horizontal asymptote of the graph.

Studying the derivative: By the Quotient Rule,

\[
y'(x) = \frac{d}{dx} \left( \frac{x^2}{x^2 + 1} \right) = \frac{d}{dx} \left( 1 - \frac{1}{x^2} \right) = \frac{2x}{(x^2 + 1)^2}
\]

Since the denominator is positive for all \( x \), the derivative is positive and the function is increasing on \( (0, \infty) \) and the derivative is negative and the function is decreasing on \( (-\infty, 0) \).

Drawing the graph: The graph is symmetric about the \( y \)-axis because the function is even. To draw the graph in Figure 7, we use the information obtained above and plot the values \( y(0) = 0^2/(1 + 0^2) = 1 \) and \( y(\pm2) = 2^2/(1 + 2^2) = \frac{4}{5} \). Notice that the graph looks like \( y = x^2 \) near \( x = 0 \). This because the \( x^2 \) in the denominator of \( y = \frac{x^2}{x^2 + 1} \) is small relative to the 1 for small \( |x| \). □
Functions with restricted domains
If a function's formula involves even roots or irrational powers, then it is not defined where the expressions whose roots or powers are taken are negative.

Example 4  Sketch the graph of $h(x) = \sqrt{x^2 - 3}$ by analyzing the formulas for the function and its first derivative.

Solution  Studying the function: $h(x) = \sqrt{x^2 - 3}$ is even and is defined, nonnegative, and continuous where $x^2 \geq 3$, which is on the intervals $(-\infty, -\sqrt{3}]$ and $[\sqrt{3}, \infty)$. Also, $\lim_{x \to \pm\infty} h(x) = \infty$ because $x^2 - 3 \to \infty$ as $x \to \pm\infty$.

Studying the derivative: By the rule for finding derivatives of powers of functions,

$$h'(x) = \frac{d}{dx}[(x^2 - 3)^{1/2}] = \frac{1}{2}(x^2 - 3)^{-1/2} \frac{d}{dx}(x^2 - 3)$$

$$= \frac{1}{2} \frac{2x}{(x^2 - 3)^{1/2}} = \frac{x}{\sqrt{x^2 - 3}} \quad (5)$$

The derivative is defined for $x < -\sqrt{3}$ and for $x > \sqrt{3}$. It is negative for $x < -\sqrt{3}$ and positive for $x > \sqrt{3}$, so the function is decreasing on $(-\infty, -\sqrt{3})$ and increasing on $(\sqrt{3}, \infty)$. We plot the values $h(\pm\sqrt{3}) = 0$ and $h(\pm4) = \sqrt{4^2 - 3} = \sqrt{13} \approx 3.6$ to draw its graph in Figure 8. The graph is symmetric about the $y$-axis because the function is even. □

Question 3  Show that the tangent line at $x$ to the graph of $h(x) = \sqrt{x^2 - 3}$ approaches the vertical line $x = \sqrt{3}$ as $x$ approaches $\sqrt{3}$ from the right and approaches the vertical line $x = -\sqrt{3}$ as $x$ approaches $-\sqrt{3}$ from the left.

Question 4  The slope (5) of the tangent line at $x$ to the graph of $h(x) = \sqrt{x^2 - 3}$ in Figure 8 tends to 1 as $x \to \infty$ and tends to $-1$ as $x \to -\infty$. Illustrate this by calculating
the approximate decimal values of the slope (a) at \( x = 5, 10 \), and 50 and (b) at \( x = -5, -10 \), and -50.

### Responses 3.2

**Response 1**

(a) The derivative \( y' = \frac{1}{2}x(4 - x) \) changes sign at \( x = 0 \) because the factor \( x \) changes sign and the factor \( 4 - x \) does not.

(b) The derivative \( y' = \frac{1}{2}x(4 - x) \) changes sign at \( x = 4 \) because the factor \( 4 - x \) changes sign and the factor \( x \) does not.

**Response 2**

(a) The derivative \( y' = \frac{x(x - 6)}{(x - 3)^2} \) changes sign at \( x = 0 \) because \( x \) changes sign but \( x - 6 \) and \( (x - 3)^2 \) do not. • The derivative changes sign at \( x = 6 \) because \( x - 6 \) changes sign but \( x \) and \( (x - 3)^2 \) do not. • The derivative does not change at \( x = 3 \) because \( x, x - 6 \) and \( (x - 3)^2 \) do not change sign there.

**Response 3**

The tangent line at \( x \) to the graph \( y = h(x) \) in Figure 8 approaches the vertical line \( x = \sqrt{3} \) as \( x \) approaches \( \sqrt{3} \) from the right because the derivative \( h'(x) = x/\sqrt{x^2 - 3} \) tends to \( \infty \). • The tangent line at \( x \) approaches the vertical line \( x = -\sqrt{3} \) as \( x \) approaches \( -\sqrt{3} \) from the left because the derivative \( h'(x) = x/\sqrt{x^2 - 3} \) tends to \( -\infty \).

**Response 4**

(a) \( h'(5) = \frac{5}{5^2 - 3} = 1.0660 \) • \( h'(10) = \frac{10}{10^2 - 3} = 1.0153 \) •

\[ h'(50) = \frac{50}{50^2 - 3} \approx 1.0006 \]

(b) \( h'(-5) = \frac{-5}{(-5)^2 - 3} = -1.0660 \) • \( h'(-10) = \frac{-10}{10^2 - 3} \approx -1.0153 \) •

\[ h'(50) = \frac{-50}{50^2 - 3} \approx -1.0006 \]