Complex exponential functions†

A complex number is an expression of the form $z = a + ib$, where $a$ and $b$ are real numbers and $i$ is the symbol that is introduced to serve as a square root of $-1$. The real part of $z = a + ib$ is the real number $a$, and the imaginary part of $z = a + ib$ is the real number $b$. Real numbers are considered to be complex numbers with zero imaginary parts. $a + ib$ is the rectangular representation of the complex number.

Complex numbers are added and multiplied with the procedures used for real numbers with the additional rule that $i^2 = -1$. For example,

$$(2 + 3i) + (4 - 5i) = (2 + 4) + (3 - 5)i = 6 - 2i$$

$$(2 + 3i)(4 - 5i) = 2(4) + [2(-5) + 3(4)]i + 3(-5)i^2$$
$$= 8 + 2i - 15i^2 = 8 + 2i - 15(-1) = 23 + 2i$$

The complex conjugate of a complex number $z = a + ib$ is $\overline{z} = a - ib$. The magnitude or absolute value of the complex number $z = a + ib$ is $|z| = \sqrt{a^2 + b^2}$. Notice that

$$z\overline{z} = (a + ib)(a - ib) = a^2 - i^2b^2 = a^2 + b^2 = |z|^2.$$

A complex number $z = a + ib$ can be plotted in an $xy$-plane at the point with coordinates $x = a, y = b$ (Figure 1). Then the complex conjugate $\overline{z} = a - ib$ is the mirror image of $z = a + ib$ about the $x$-axis, and the magnitude of $z = a + ib$ is its distance $|z| = \sqrt{a^2 + b^2}$ to the origin.

![Figure 1](image)

To find the rectangular representation of the quotient $z_1/z_2$ of two complex numbers where $z_2 \neq 0$, you can multiply the numerator and denominator of the ratio by the conjugate $\overline{z_2}$ of the denominator to make the denominator a real number. For example,

$$\frac{1 + 2i}{3 + 4i} = \frac{(1 + 2i)(3 + 4i)}{(3 + 4i)(3 - 4i)} = \frac{1(3) + [2(3) - 1(4)]i - 2(4)i^2}{3^2 - 4^2i^2}$$
$$= \frac{3 + 2i + 8}{3^2 + 4^2} = \frac{11 + 2i}{25} = \frac{11}{25} + \frac{2}{25}i.$$

†Lecture notes to accompany a Math 20B Supplement
Example 1
Calculate (a) $z_1 + z_2$, (b) $z_1 z_2$, and (c) $z_1 / z_2$ for $z_1 = 1 - i$ and $z_2 = 3 + i$.

Answer: (a) $z_1 + z_2 = 4$ (b) $z_1 z_2 = 4 - 2i$ (c) $\frac{z_1}{z_2} = -\frac{1}{3} - \frac{2}{3}i$

Complex exponentials
As you will see later in Math 20B, the functions $y = e^x$, $y = \cos x$, and $y = \sin x$ can be given for all real $x$ by the infinite series,

\[ e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 + \cdots \]

\[ \cos x = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} = 1 - \frac{1}{2} x^2 + \frac{1}{4!} x^4 - \cdots \]

\[ \sin x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots \]

If we set $x = ib$ in the first formula, we obtain

\[ e^{ib} = \sum_{n=0}^{\infty} \frac{1}{n!} (ib)^n = 1 + ib + \frac{1}{2!} (ib)^2 + \frac{1}{3!} (ib)^3 + \frac{1}{4!} (ib)^4 + \frac{1}{5!} (ib)^5 + \cdots \]

\[ = 1 + ib - \frac{1}{2} b^2 - \frac{1}{3!} b^3 i + \frac{1}{4!} b^4 + \frac{1}{5!} b^5 i + \cdots \]

\[ = (1 - \frac{1}{2} b^2 + \frac{1}{4!} b^4 - \cdots ) + ib(-\frac{1}{3!} b^3 + \frac{1}{5!} b^5 - \cdots ) \]

which gives

\[ e^{ib} = \cos b + i \sin b. \] \hspace{1cm} (1)

This leads us to the definition of the complex exponential function,

\[ e^{a+ib} = e^a e^{ib} = e^a (\cos b + i \sin b). \] \hspace{1cm} (2)

Complex exponential functions satisfy the rules

\[ e^{z_1} e^{z_2} = e^{z_1 + z_2} \] \hspace{1cm} (3)

\[ (e^{z_1})^{z_2} = e^{z_1 z_2} \] \hspace{1cm} (4)

satisfied by real exponential functions.
**Polar forms of complex numbers**

If we set \( b = \theta \) in (1), we obtain

\[
e^{i\theta} = \cos \theta + i \sin \theta. \tag{5}
\]

This shows that \( e^{i\theta} \) is the point with polar coordinates \( r = 1 \) and \( \theta = \theta \) (Figure 2).

![Figure 2](image)

Moreover, if the complex number \( z = a + ib \) has polar coordinates \([r, \theta]\), then its \( xy \)-coordinates are \( a = r \cos \theta, b = r \sin \theta \) and it can be written (Figure 3)

\[
a + ib = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta) = re^{i\theta}.
\]

This is the **polar form** of the complex number. Recall that \( r = |z| \) is the “magnitude” of the number. The angle \( \theta \) is called its **argument**.

![Figure 3](image)

**Example 2**  
Plot the point \( z = -2 - 2i \) and find a polar form for it.

**Answer:** Figure A2  
One answer: \( r = 2\sqrt{2} \)  
\( \theta = \frac{5}{4}\pi \)
Example 3

Give the rectangular representation of the complex number $e^{3+4i}$.

**Answer:** $e^{3+4i} = e^3 \cos(4) + ie^3 \sin(4)$

With the polar representations, we can write for $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

$$z_1 z_2 = \left( r_1 e^{i\theta_1} \right) \left( r_2 e^{i\theta_2} \right) = r_1 r_2 e^{i(\theta_1 + \theta_2)} \quad (6)$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left( \frac{r_1}{r_2} \right) e^{i(\theta_1 - \theta_2)} \quad (7)$$

The first equation shows that to multiply two complex numbers, you multiply their moduli and add their arguments. The second shows that to divide one complex number by another, you divide the magnitude of the second by the magnitude of the first and subtract the argument of the second from the argument of the first.

Example 4

(a) Calculate $z^2$ for $z = 3\sqrt{2} + 3\sqrt{2}i$. (b) Find the magnitudes and arguments of $z$ and $z^2$ and explain the results.

**Answer:** (a) $z^2 = 36i$ (b) $|z| = 6 \bullet |z^2| = 36$ [Argument of $z$] = $\frac{1}{4} \pi \bullet$ [Argument of $z^2$] = $\frac{1}{2} \pi \bullet$

The magnitude of $z^2$ is the square of the magnitude of $z$. ● The argument of $z^2$ is 2 times the argument of $z$.

**Trigonometric identities**

We can replace $\theta$ by $-\theta$ in formula (5) to obtain

$$e^{-i\theta} = \cos(-\theta) + i \sin(-\theta) = \cos \theta - i \sin \theta.$$  

Then we have the two equations,

$$\begin{cases}
  e^{i\theta} = \cos \theta + i \sin \theta \\
  e^{-i\theta} = \cos \theta - i \sin \theta
\end{cases}$$

which can be solved for $\cos \theta$ and $\sin \theta$. (Do it.) The result is formulas,

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad (6)$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad (7)$$

that express $\cos \theta$ and $\sin \theta$ in terms of complex exponential function. These can be used to derive properties of the trigonometric functions from properties of the exponential functions.

Example 5

Use complex exponential functions to derive the trigonometric identity,

$$\sin^2 \theta = \frac{1}{2} [1 - \cos(2\theta)].$$

**Answer:** Omitted. The answer is the solution.

**Integrals**

The differentiation and integration formulas for complex exponential functions are the same as for real exponential functions:

$$\frac{d}{dx} \left( e^{(a+ib)x} \right) = (a + ib)e^{(a+ib)x} \quad (8)$$

and for nonzero $a + ib$,

$$\int e^{(a+ib)x} \, dx = \frac{1}{a+ib} e^{(a+ib)x} + C \quad (9)$$
Example 6  Use complex exponential functions to find the derivative,

\[ \frac{d}{dx}(\sin x). \]

Answer: \[ \frac{d}{dx}(\sin x) = \cos x \]

Example 7  Use complex exponential functions to find the antiderivative,

\[ \int \sin(3x) \, dx. \]

Answer: \[ \int \sin(3x) \, dx = -\frac{1}{3}\cos(3x) + C \]

Example 8  Use complex exponential functions to find the perform the integration,

\[ \int e^x \cos x \, dx. \]

Answer: \[ \int e^x \cos x \, dx = \frac{1}{2}e^x (\cos x + \sin x) + C \]