Section 14.4

(3/23/08)

Chain Rules with two variables

OVERVIEW: *In this section we discuss procedures for differentiating composite functions with two variables. Then we consider second-order and higher-order derivatives of such functions.*

Topics:

- ***Using the Chain Rule for one variable***
- ***The general Chain Rule with two variables***
- ***Higher order partial derivatives***

Using the Chain Rule for one variable

Partial derivatives of composite functions of the forms $z = F(g(x, y))$ can be found directly with the Chain Rule for one variable, as is illustrated in the following three examples.

Example 1 Find the x-and y-derivatives of $z = (x^2y^3 + \sin x)^{10}$.

SOLUTION To find the x-derivative, we consider y to be constant and apply the one-variable Chain Rule formula $\dfrac{d}{dx}(f^{10}) = 10\, f^9\, \dfrac{df}{dx}$ from Section 2.8. We obtain

$$\begin{aligned}\frac{\partial}{\partial x}[(x^2y^3 + \sin x)^{10}] &= 10(x^2y^3 + \sin x)^9 \frac{\partial}{\partial x}(x^2y^3 + \sin x)\\ &= 10(x^2y^3 + \sin x)^9(2xy^3 + \cos x).\end{aligned}$$

Similarly, we find the y-derivative by treating x as a constant and using the same one-variable Chain Rule formula with y as variable:

$$\begin{aligned}\frac{\partial}{\partial y}[(x^2y^3 + \sin x)^{10}] &= 10(x^2y^3 + \sin x)^9 \frac{\partial}{\partial y}(x^2y^3 + \sin x)\\ &= 10(x^2y^3 + \sin x)^9(3x^2y^2). \ \square\end{aligned}$$

Example 2 The radius (meters) of a spherical balloon is given as a function $r = r(P, T)$ of the atmospheric pressure P (atmospheres) and the temperature T (degrees Celsius). At one moment the radius is ten meters, the rate of change of the radius with respect to atmospheric pressure is -0.01 meters per atmosphere, and the rate of change of the radius with respect to the temperature is 0.002 meter per degree. What are the rates of change of the volume $V = \frac{4}{3}\pi r^3$ of the balloon with respect to P and T at that time?

SOLUTION We first take the P-derivative with T constant and then take the T-derivative with P constant, using the Chain Rule for one variable in each case to differentiate r^3. We obtain

$$\begin{aligned}\frac{\partial V}{\partial P} &= \frac{\partial}{\partial P}\left(\tfrac{4}{3}\pi r^3\right) = \tfrac{1}{3}\pi r^2 \frac{\partial r}{\partial P}\\ \frac{\partial V}{\partial P} &= \frac{\partial}{\partial T}\left(\tfrac{4}{3}\pi r^3\right) = \tfrac{1}{3}\pi r^2 \frac{\partial r}{\partial T}.\end{aligned}$$

Setting $r = 10, \partial r/\partial P = -0.01$, and $\partial r/\partial T = 0.002$ then gives

$$\begin{aligned}\frac{\partial V}{\partial P} &= \tfrac{1}{3}\pi(10^2)(-0.01) = -\tfrac{1}{3}\pi \doteq -1.05\ \frac{\text{cubic meters}}{\text{atmosphere}}\\ \frac{\partial V}{\partial T} &= \tfrac{1}{3}\pi(10^2)(0.002) = \tfrac{1}{15}\pi \doteq 0.21\ \frac{\text{cubic meters}}{\text{degree}}. \ \square\end{aligned}$$

Example 3 What are the x- and y-derivatives of $z = F(g(x,y))$ at $x = 5, y = 6$ if $g(5,6) = 10, F'(10) = -7, g_x(5,6) = 3$, and $g_y(5,6) = 11$?

SOLUTION By the Chain Rule formula $\frac{d}{dt}[F(u(t))] = F'(u(t))\, u'(t)$ for one variable with first x and then y in place of t, we obtain

$$\begin{aligned}\left[\frac{\partial}{\partial x}\{F(g(x,y))\}\right]_{x=5,y=6} &= F'(g(5,6))\, g_x(5,6)\\ &= F'(10)\, g_x(5,6) = (-7)(3) = -21\\ \left[\frac{\partial}{\partial y}\{F(g(x,y))\}\right]_{x=5,y=6} &= F'(g(5,6))\, g_y(5,6)\\ &= F'(10)\, g_y(5,6) = (-7)(11) = -77. \ \square\end{aligned}$$

Partial derivatives of composite functions of the forms $F(t) = f(x(t), y(t))$ and $F(s,t) = f(x(s,t), y(s,t))$ can be found directly with the Chain Rule for one variable if the "outside" function $z = f(x,y)$ is given in terms of power functions, exponential functions, logarithms, trigonometric functions, and inverse trigonometric functions rather than just by a letter name. This is illustrated in the following example.

Example 4 Find the t-derivative of $z = f(x(t), y(t))$, where $f(x,y) = x^5y^6, x(t) = e^t$, and $y(t) = \sqrt{t}$.

SOLUTION Because $f(x,y)$ is a product of powers of x and y, the composite function $f(x(t), y(t))$ can be rewritten as a function of t. We obtain

$$f(x(t), y(t)) = [x(t)]^5[y(t)]^6 = (e^t)^5(t^{1/2})^6 = e^{5t}t^3.$$

Then the Product and Chain Rules for one variable give

$$\begin{aligned}\frac{d}{dt}[f(x(t), y(t))] &= \frac{d}{dt}(e^{5t}t^3) = e^{5t}\frac{d}{dt}(t^3) + t^3\frac{d}{dt}(e^{5t})\\ &= 3t^2e^{5t} + t^3e^{5t}\frac{d}{dt}(5t) = 3t^2e^{5t} + 5t^3e^{5t}. \ \square\end{aligned}$$

The general Chain Rule with two variables

We the following general Chain Rule is needed to find derivatives of composite functions in the form $z = f(x(t), y(t))$ or $z = f(x(s,t), y(s,t))$ in cases where the outer function f has only a letter name. We begin with functions of the first type.

Theorem 1 (The Chain Rule) *The t-derivative of the composite function $z = f(x(t), y(t))$ is*

$$\frac{d}{dt}[f(x(t), y(t))] = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t). \qquad \textbf{(1)}$$

We assume in this theorem and its applications that $x = x(t)$ and $y = y(t)$ have first derivatives at t and that $z = f(x,y)$ has continuous first-order derivatives in an open circle centered at $(x(t), y(t))$.

Learn equation **(1)** as the following statement: the t-derivative of the composite function equals the x-derivative of the outer function $z = f(x,y)$ at the point $(x(t), y(t))$ multiplied by the t-derivative of the inner function $x = x(t)$, plus the y-derivative of the outer function at $(x(t), y(t))$ multipled by the t-derivative of the inner function $y = y(t)$.

Proof of Theorem 1: We fix t and set $(x, y) = (x(t), y(t))$. We consider nonzero Δt so small that $(x(t+\Delta t), y(t+\Delta t))$ is in the circle where f has continuous first derivatives and set $\Delta x = x(t+\Delta t) - x(t)$ and $\Delta y = y(t + \Delta t) - y(t)$. Then, by the definition of the derivative,

$$\begin{aligned} \frac{d}{dt}\left[f(x(t), y(t))\right] &= \lim_{\Delta t \to 0} \frac{f(x(t+\Delta t), y(t+\Delta t)) - f(x(t), y(t))}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{f(x+\Delta x, y+\Delta y) - f(x, y)}{\Delta t}. \end{aligned} \tag{2}$$

We express the change $f(x + \Delta x, y + \Delta y) - f(x, y)$ in the value of $z = f(x, y)$ from (x, y) to $(x + \Delta x, y + \Delta y)$ as the change in the x-direction from (x, y) to $(x + \Delta x, y)$ plus the change in the y-direction from $(x + \Delta x, y)$ to $(x + \Delta x, y + \Delta y)$, as indicated in Figure 1:

$$f(x + \Delta x, y + \Delta y) - f(x, y) = [f(x + \Delta x, y) - f(x, y)] + [f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)]. \tag{3}$$

(Notice that the terms $f(x + \Delta x, y)$ and $-f(x + \Delta x, y)$ on the right side of **(3)** cancel to give the left side.)

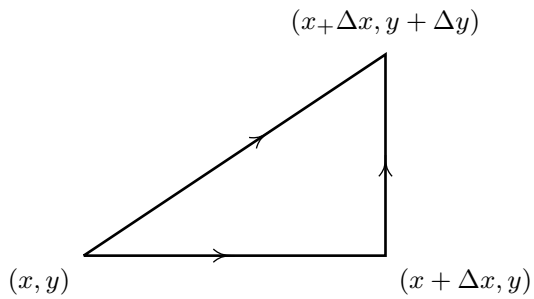

FIGURE 1

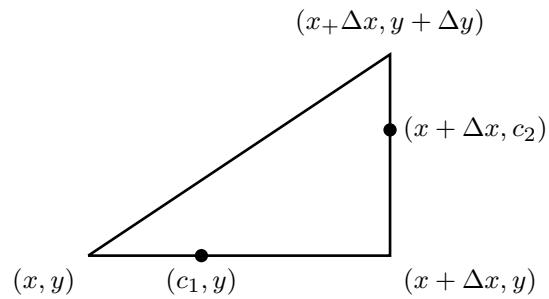

FIGURE 2

We can apply the Mean Value Theorem from Section 3.3 to the expression in the first set of square brackets on the right of **(3)** where y is constant and to the expression in the second set of square brackets where x is constant. We conclude that there is a number c_1 between x and $x + \Delta x$ and a number c_2 between y and $y + \Delta y$ (see Figure 2) such that

$$\begin{aligned} f(x + \Delta x, y) - f(x, y) &= f_x(c_1, y)\Delta x \\ f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y) &= f_y(x + \Delta x, c_2)\Delta y. \end{aligned} \tag{4}$$

We combine equations **(3)** and **(4)** and divide by Δt to obtain

$$\frac{f(x + \Delta x, y + \Delta y) - f(x, y)}{\Delta t} = f_x(c_1, y_0)\left[\frac{\Delta x}{\Delta t}\right] + f_y(x + \Delta x, c_2)\left[\frac{\Delta y}{\Delta t}\right]. \tag{5}$$

The functions $x = x(t)$ and $x = y(t)$ are continuous at t because they have derivatives at that point. Consequently, as $\Delta t \to 0$, the numbers Δx and Δy both tend to zero and the triangle in Figure 2 collapses to the point (x, y). Because the partial derivatives of f are continuous, the term $f_x(c_1, y + \Delta y)$ in **(5)** tends to $f_x(x, y)$ and the term $f_y(x, c_2)$ tends to $f_y(x, y)$ as $\Delta t \to 0$. Moreover $\Delta x/\Delta t \to x'(t)$ and $\Delta y/\Delta t \to y'(t)$ as $\Delta t \to 0$, so equation **(5)** with **(2)** gives

$$\frac{d}{dt}\left[f(x(t), y(t))\right] = f_x\left(x(t), y(t)\right) x'(t) + f_y\left(x(t), y(t)\right) y'(t)$$

to establish the theorem. **QED**

Example 5 What is the t-derivative of $z = f(x(t), y(t))$ at $t = 1$ if $x(1) = 2, y(1) = 3$, $x'(1) = -4$, $y'(1) = 5$, $f_x(2,3) = -6$, and $f_y(2,3) = 7$?

SOLUTION By formula **(1)** with $t = 1$,

$$\begin{aligned}\left[\frac{d}{dt}\{f(x(t), y(t))\}\right]_{t=1} &= f_x(x(1), y(1))\, x'(1) + f_y(x(1), y(1))\, y'(1)\\ &= f_x(2,3)\, x'(1) + f_y(2,3)\, y'(1)\\ &= (-6)(-4) + (7)(5) = 59. \ \square\end{aligned}$$

Example 6 Find $G'(2)$ where $G(t) = h(t^2, t^3)$ and $h = h(x, y)$ is such that $h_x(4,8) = 10$ and $h_y(4,8) = -20$.

SOLUTION Formula **(1)** gives

$$\begin{aligned}G'(t) &= \frac{d}{dt}[h(t^2, t^3)] = h_x(t^2, t^3)\frac{d}{dt}(t^2) + h_y(t^2, t^3)\frac{d}{dt}(t^3)\\ &= 2th_x(t^2, t^3) + 3t^2 h_y(t^2, t^3).\end{aligned}$$

Therefore,

$$\begin{aligned}G'(2) &= 2(2)h_x(2^2, 2^3) + 3(2^2)h_y(2^2, 2^3)\\ &= 4h_x(4,8) + 12h_y(4,8) = 4(10) + 12(-20) = -200. \ \square\end{aligned}$$

In applications it often helps to interpret the Chain Rule formula **(1)** in terms of rates of change. We write it in the form

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \qquad \textbf{(6)}$$

without reference to where the derivatives are evaluated. Equation **(6)** states that the rate of change of F with respect to t equals the rate of change of F with respect to x multiplied by the rate of change of x with respect to t, plus the rate of change of F with respect to y multiplied by the rate of change of y with respect to t.

Example 7 A small plane uses gasoline at the rate of $r = r(h, v)$ gallons per hour when it is flying at an elevation of h feet above the ground and its air speed is v knots (nautical miles per hour). At a moment when the plane has an altitude of 8000 feet and a speed of 120 knots, its height is increasing 500 feet per minute and it is accelerating 3 knots per minute. At what rate is its gasoline consumption increasing or decreasing at that moment if at $h = 8000$ and $v = 120$ the function r and its derivatives have the values $r = 7.2$ gallons per hour, $\partial r/\partial h = -2 \times 10^{-4}$ gallons per hour per foot, and $\partial r/\partial v = 0.13$ gallons per hour per knot?[1]

[1] Data adapted from *Cessna 172N Information Manual* , Wichita Kansas: Cessna Aircraft Company, 1978, p.5-16.

SOLUTION At the moment in question the plane's rate of gas consumption r is changing at the rate

$$\begin{aligned}
\frac{dr}{dt} &= \left[\frac{\partial r}{\partial h}\ \frac{\text{gallons per hour}}{\text{foot}}\right]\left[\frac{dh}{dt}\ \frac{\text{feet}}{\text{minute}}\right] \\
&\quad + \left[\frac{\partial r}{\partial v}\ \frac{\text{gallons per hour}}{\text{knot}}\right]\left[\frac{dv}{dt}\ \frac{\text{knots}}{\text{minute}}\right] \\
&= \left[-2\times 10^{-4}\ \frac{\text{gallons per hour}}{\text{foot}}\right][500 \text{ feet per minute}] \\
&\quad + \left[0.13\ \frac{\text{gallons per hour}}{\text{knot}}\right]\left[3\ \frac{\text{knots}}{\text{minute}}\right] \\
&= (-2\times 10^{-4})(500) + (0.13)(3)\ \frac{\text{gallons per hour}}{\text{minute}} \\
&= 0.29\ \frac{\text{gallons per hour}}{\text{minute}}.
\end{aligned} \tag{7}$$

The plane's rate of fuel consumption is increasing 0.29 gallons per hour per minute. □

Theorem 1 can be applied to find the s- and t-derivatives of a function of the form $z = f(x(s,t), y(s,t))$ because in taking the derivative with respect to s or t, the other variable is constant. We obtain the following.

Theorem 2 (The Chain Rule) *The s- and t-derivatives of the composite function $z = f\,(x(s,t), y(s,t))$ are*

$$\begin{aligned}
\frac{\partial}{\partial s}[f(x(s,t),y(s,t))] &= f_x(x(s,t),y(s,t))x_s(s,t) + f_y(x(s,t),y(s,t))y_s(s,t) \\
\frac{\partial}{\partial t}[f(x(s,t),y(s,t))] &= f_x(x(s,t),y(s,t))x_t(s,t) + f_y(x(s,t),y(s,t))y_t(s,t).
\end{aligned} \tag{8}$$

We assume in this theorem and its applications that the functions involved have continuous first derivatives in the open sets where they are considered.

Formulas **(8)** are easier to remember without the values of the variables in the form,

$$\begin{aligned}
f_s &= f_x\, x_s + f_y\, y_s \\
f_t &= f_x\, x_t + f_y\, y_t
\end{aligned} \tag{9}$$

or with Leibniz notation as

$$\begin{aligned}
\frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial s} \\
\frac{\partial f}{\partial t} &= \frac{\partial f}{\partial x}\,\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\,\frac{\partial y}{\partial t}.
\end{aligned} \tag{10}$$

Example 8 What are the s- and t-derivatives of $z = f(st^2, te^s)$ at $s = 0, t = 2$ if the derivatives of $z = f(x, y)$ have the values $f_x(0, 2) = 10$ and $f_y(0, 2) = -5$?

SOLUTION Equations **(8)** give

$$\begin{aligned}
\frac{\partial}{\partial s}[f(st^2, te^s)] &= f_x(st^2, te^s)\frac{\partial}{\partial s}(st^2) + f_y(st^2, te^s)\frac{\partial}{\partial s}(te^s). \\
&= t^2 f_x(st^2, te^s) + te^s f_y(st^2, te^s) \\
\frac{\partial}{\partial t}[f(st^2, te^s)] &= f_x(st^2, te^s)\frac{\partial}{\partial t}(st^2) + f_y(st^2, te^s)\frac{\partial}{\partial t}(te^s) \\
&= 2st f_x(st^2, te^s) + e^s f_y(st^2, te^s).
\end{aligned}$$

We set $s = 0$ and $t = 2$ and use the given values of the derivatives of $z = f(x, y)$ to obtain

$$\begin{aligned}
\left[\frac{\partial}{\partial s}\{f(st^2, te^s)\}\right]_{s=0,t=2} &= 2^2 f_x(0, 2) + 2e^0 f_y(0, 2) = 4(10) + 2(-5) = 30 \\
\left[\frac{\partial}{\partial t}\{f(st^2, te^s)\}\right]_{s=0,t=2} &= 2(0)(2) f_x(0, 2) + e^0 f_y(0, 2) = 0(10) + 1(-5) = -5. \ \square
\end{aligned}$$

Higher order partial derivatives

The first-order partial derivatives $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$ of $z = f(x, y)$ can be differentiated with respect to x and y to obtain the second x-derivative

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

the second y-derivative

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

and the MIXED SECOND DERIVATIVES

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \text{ and } f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right). \tag{11}$$

In all cases that we will encounter, the mixed second derivatives **(11)** are equal, so this derivative may be calculated either by differentiating first with respect to x and then with respect to y or by differentiating first with respect to y and then with respect to x. This is a consequence of the following theorem, whose proof is outlined in Problem 34.

Theorem 3 *If $f_{xy}(x, y)$ and $f_{yx}(x, y)$ are defined and continuous in an open set, then these derivatives are equal in the set.*

Because the mixed partial derivatives **(11)** are equal, you can use both orders of differentiation to check your work.

Example 9 What are the second-order derivatives of $f(x, y) = xy^2 + x^3y^5$?

SOLUTION We begin with the first derivatives:

$$f_x = \frac{\partial}{\partial x}(xy^2 + x^3y^5) = y^2 + 3x^2y^5$$
$$f_y = \frac{\partial}{\partial y}(xy^2 + x^3y^5) = 2xy + 5x^3y^4.$$

Then we find the second derivatives:

$$f_{xx} = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}(y^2 + 3x^2y^5) = 6xy^5$$
$$f_{xy} = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(y^2 + 3x^2y^5) = 2y + 15x^2y^4$$
$$f_{yx} = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}(2xy + 5x^3y^4) = 2y + 15x^2y^4$$
$$f_{yy} = \frac{\partial}{\partial y}(f_y) = \frac{\partial}{\partial y}(2xy + 5x^3y^4) = 2x + 20x^3y^3.$$

Notice that f_{xy} and f_{yx} are equal, as should be the case because these second-order derivatives are continuous for all (x, y). $\square$

Higher-order derivatives are defined similarly and mixed derivatives can generally be taken in any order for functions given by formulas.

Example 10 Find g_{yxx} for $g(x, y) = x^4 \sin(3y) + 5x - 6y$.

SOLUTION We can take the one y-derivative and the two x-derivatives in any order. We start with the y-derivative:

$$g_y = \frac{\partial}{\partial y}[x^4 \sin(3y) + 5x - 6y] = x^4 \cos(3y)\frac{\partial}{\partial y}(3y) - 6 = 3x^4 \cos(3y) - 6.$$

Then we take two x-derivatives:

$$g_{yx} = \frac{\partial}{\partial x}(g_y) = \frac{\partial}{\partial x}[3x^4 \cos(3y) - 6] = 12x^3 \cos(3y)$$
$$g_{yxx} = \frac{\partial}{\partial x}(g_{yx}) = \frac{\partial}{\partial x}[12x^3 \cos(3y)] = 36x^2 \cos(3y). \ \square$$

Interactive Examples 14.4

Interactive solutions are on the web page http//www.math.ucsd.edu/~ashenk/.†

1. Express the x- and y-derivatives of $W(x^3y^5)$ in terms of x, y, and $W'(t)$.

2. What is $\frac{d}{dt}[F(x(t), y(t))]$ at $t = 0$ if $x(0) = 3, y(0) = 7, x'(0) = -4, y'(0) = 6, F_x(3, 7) = 8$, and $F_y(3, 7) = 2$?

3. What is $W'(t)$ where $W(t) = V(x(t), y(t)), x(t) = t^3, y(t) = t^5$, and $V(x, y) = \sqrt{x + y}$?

4. Find the values of **(a)** $g(2)$ and **(b)** $g'(2)$ for $g(t) = f(t^3, t^4)$, where $z = f(x, y)$ satisfies $f(8, 16) = 3, f_x(8, 16) = 5$, and $f_y(8, 16) = -7$.

5. What are **(a)** $T_x(1, 2)$ and **(b)** $T_y(1, 2)$ for $T(x, y) = U(v(x, y))$ if $v(1, 2) = 3, v_x(1, 2) = 5$, $v_y(1, 2) = 7$, and $U'(3) = 9$?

6. What is $P_u(3, 2)$ for $P(u, v) = R(x(u, v), y(u, v))$ if $x(3, 2) = 1, y(3, 2) = 0, x_u(3, 2) = 5$, $y_u(3, 2) = 7, R_x(1, 0) = 9$ and $R_y(1, 0) = 10$?

Exercises 14.4

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

1. If $A = A(w, h)$ is the area of a rectangle of width w and height h, and w and h are function of the time t, then by Theorem 1, $\frac{dA}{dt} = \frac{\partial A}{\partial w}\frac{dw}{dt} + \frac{\partial A}{\partial h}\frac{dh}{dt}$. Put formulas for the w- and h-derivatives of $A(w, h) = wh$ in this equation and describe the result verbally.

2. If $V = V(r, h)$ is the volume of a right circular cylinder of height h and whose base has radius r, then by Theorem 1, $\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$. Put formulas for the r- and h-derivatives of $V(r, h) = \pi r^2 h$ in this equation and describe the result verbally.

3. If you have a pig farm and $v = v(W, r)$ is the value of your herd when its total weight is W (pounds) and it could be sold for r dollar per pound, then by Theorem 1, $\frac{dv}{dt} = \frac{\partial v}{\partial W}\frac{dW}{dt} + \frac{\partial v}{\partial r}\frac{dr}{dt}$. The value is the product $v = Wr$ of the weight and price. Put formulas for the W- and r-derivatives of v in this equation and describe the result verbally.

BASICS:

4.[O] Express $F(t) = f(x(t), y(t))$ in terms of t for $f(x, y) = x\sin(xy), x(t) = t^5$, and $y(t) = t^{-3}$.

5.[A] Express $P(u, v) = f(x(u, v), y(u, v))$ in terms of u and v where $f(x, y) = x\ln(xy + 1), x(u, v) = ue^v$, and $y(u, v) = u^2v^3$.

6. Express $Q(u, v) = f(x(u, v), y(u, v))$ in terms of u and v where $f(x, y) = \sqrt{x^2 + y^2}$, $x(u, v) = u^2v^2$, and $y(u, v) = 2u - 3v$.

7. Express $f(x(t), y(t))$ in terms of t where $f(x, y) = x\sin(xy), x(t) = t^5$, and $y(t) = t^{-3}$.

Find the derivatives in Exercises 8 through 11.

8.[O] $Q'(t)$ where $Q(t) = Q(x(t), y(t)), x(t) = \sin t, y(t) = \cos t$, and $V(x, y) = e^{x+y}$

9.[O] $K_x(x, y)$ and $K_y(x, y)$, where $K(x, y) = L(M(x, y)), L(z) = \sin z$, and $M(x, y) = 3x - 4y$

10.[A] $P'(t)$ where $P(t) = Q(x(t), y(t)), x(t) = e^t, y(t) = \ln t$, and $Q(x, y) = x^2 + y^2$

11. $S_x(x, y)$ and $S_y(x, y)$, where $S(x, y) = T(A(x, y), B(x, y)), T(A, B) = A - B, A(x, y) = x^2y$, and $B(x, y) = xy^2$

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†In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

Find the second order derivatives of the functions in Problems 12 through 15.

12.º $f(x,y) = x^4y^5$

13.^{A} $k(x,y) = \ln(2x-3y)$

14. $M(x,y) = (1+xy)^{10}$

15.^{A} $N(x,y) = \tan^{-1}(xy)$

16.^{A} Find $\dfrac{\partial f^3}{\partial y \partial x^2}$ for $f(x,y) = x^3y^4$.

17. What is $\dfrac{\partial g^3}{\partial x \partial y^2}$ for $g(x,y) = y^3e^{-4x}$?

18.º Find the second-order derivatives of $R(x,y) = x^{1/2}y^{1/3}$.

EXPLORATION:

19.º Find the values of **(a)** $g(\pi)$ and **(b)** $g'(\pi)$ for $g(t) = f(\sin t, \cos t)$, where $z = f(x,y)$ satisfies $f(0,1) = 10$, $f_x(0,1) = -20$, and $f_y(0,-1) = 30$.

20.^{A} What are **(a)** $h(3)$ and **(b)** $h'(3)$ for $h(t) = g(t^3-5t, 11t-1)$ if $g(12,32) = 0$, $g_x(12,32) = -3$, and $g_y(12,32) = 2$?

21. What are **(a)** $K(0)$ and **(b)** $K'(0)$ for $K(t) = L(\sin t, \cos t)$ if $L(0,1) = 50$, $L_x(0,1) = 10$, and $L_y(0,1) = -7$?

22. Find **(a)** $Z(0)$ and **(b)** $Z'(0)$ for $Z(t) = W(\ln(1+t), e^{5t})$, where $W(0,1) = -3$, $W_x(0,1) = 2$, and $W_y(0,1) = 3$.

23.º What are **(a)** $W_x(0,0)$ and **(b)** $W_y(0,0)$ for $W(x,y) = q(v(x,y))$ if $v(0,0) = 5$, $v_x(0,0) = 10$, $v_y(0,0) = -1$, and $q'(5) = 6$?

24.^{A} Find **(a)** $J_x(1,1)$ and **(b)** $J_y(1,1)$ for $J(x,y) = \sqrt{K(x,y)}$, where $K(1,1) = 9$, $K_x(1,1) = 25$, and $K_y(1,1) = 16$.

25. Calculate **(a)** $W_u(5,10)$ and **(b)** $W_v(5,10)$ for $W(u,v) = u\ln[H(u,v)]$, where $H(5,10) = e$, $H_u(5,10) = 6$, and $H_v(5,10) = -7$.

26.º Give the values of **(a)** $F(0,2)$, **(b)** $F_u(0,2)$ and **(c)** $F_v(0,2)$ where $F(u,v) = f(v\sin u, u\sin v)$ with $f(0,0) = 4$, $f_x(0,0) = 10$, and $f_y(0,0) = 2$.

27.^{A} What are **(a)** $G(1,2)$, **(b)** $G_u(1,2)$ and **(c)** $G_v(1,2)$ where $G(u,v) = g(u^2v^3, 3u+7v)$ with $g(8,17) = 10$, $g_x(8,17) = 5$, and $g_y(8,17) = -1$.

28.º Find **(a)** $\dfrac{\partial}{\partial x}[P(q(x,y))]$ and **(b)** $\dfrac{\partial}{\partial y}[P(q(x,y))]$ at $x = 3, y = 7$ where $q(3,7) = 9$, $q_x(3,7) = 6$, $q_y(3,7) = 10$, and $P'(9) = -11$.

29.^{A} What are **(a)** $Q(0,0)$, **(b)** $Q_u(0,0)$, and **(c)** $Q_v(0,0)$ for $Q(u,v) = S(x(u,v), y(u,v))$ where $x(0,0) = 10$, $y(0,0) = 20$, $x_u(0,0) = -5$, $x_v(0,0) = 8$, $y_u(0,0) = 10$, $y_v(0,0) = 5$, $S(10,20) = -5$, $S_x(10,20) = -3$ and $S_y(10,20) = 4$?

30. Find **(a)** $W(1,10)$, **(b)** $W_u(11,10)$, and **(c)** $W_v(11,10)$ for $W(u,v) = Z(x(u,v), y(u,v))$ where $x(11,10) = 6$, $y(1,10) = 8$, $x_u(11,10) = 7$, $x_v(11,10) = 6$, $y_u(11,10) = 5$, $y_v(11,10) = 4$, $Z(9,8) = 3$, $Z_x(9,8) = 2$ and $Z_y(9,8) = 1$.

31.º What is $U_t(2,6)$ if $U(s,t) = V(x(s,t), y(s,t))$, $x(2,6) = 5$, $y(2,6) = -7$, $x_t(2,6) = 8$, $y_t(2,6) = 10$, $V_x(5,-7) = 11$, and $V_y(5,-7) = 12$?

32.^{A} A farmer will earn $P = P(x,t)$ dollars profit if he sell x pounds of cattle t weeks after the cattle market opens for the season. The total weight of his herd is a function $x = x(t)$ of the time. Express the rate of change of his profit with respect to time in terms of the derivatives of $P(x,t)$ and $x(t)$ with appropriate units.

33. The temperature $T = T(x, y)^\circ$F at the point (x, y) on a metal plate does not change with time, and an ant crosing the plate is at $x = t^2, y = 4t+1$ feet at time t minutes. What is the temperature at the ant's position at $t = 3$ and what is the rate of change of the temperature at the ant's position with respect to time at $t = 3$ if $T(9, 13) = 50$ degrees, $T_x(9, 13) = 5$ degrees per foot, and $T_y(9, 13) = -1$ degree per foot?

34.[A] Wool contains 17.5% water (measured by weight) when the ambient temperature is 96°F and the relative humidity is 75%. Moreover, at this temperature and humidity the rates of change with respect of the percent of water in wool with respect to the temperature and relative humidity are -0.06 percent per degree and 0.025 percent per percent, respectively.[(2)] **(a)** What is the approximate percent of water in wool when the temperature is 96°F and the relative humidity is 73%? (Use a tangent-line approximation.) **(b)** What is the approximate percent of water in wool when the temperature is 98°F and the relative humidity is 75%? **(c)** At what rate is the percent of water in the wool increasing or decreasing if the ambient temperature is 96°F and is increasing 3 degrees per hour while the relative humidity is 75% and is increasing 0.4% per hour?

35. At a temperature of 40°C and pressure of 2 atmospheres, the density of hydrogen gas is 1.5×10^{-4} grams per cubic centimter, the rate of change of the density with respect to temperature is -5×10^{-7} grams per cubic centimeter per degree, and the rate of change of the density with respect to atmospheric pressure is 7.7×10^{-5} grams per cubic centimeter per atmosphere.[(3)] **(a)** What is the approximate density of hydrogen when the temperature is 50°C and the pressure is 2 atmospheres? **(b)** What is the approximate density of hydrogen when the temperature is 40°F and the pressure is 1.8 atmospheres? **(c)** At what rate is the density of hydrogen increasing or decreasing if the temperature is 40°C and is decreasing 0.2 degrees per hour while the pressure is 2 atmospheres and is decreasing 0.06 atmospheres per hour?

36. A company sells two products, A and B. When it has sold 2000 units of product A and 3000 units of product B, it earns \$3 per unit profit on the sale of item A and \$4.50 per unit profit on the sale of item B. At what rate is its total profit from the two items increasing at a time when it has sold 2000 units of product A and 3000 units of product B and it is selling item A at the rate of 200 units per month and item B at the rate of 250 units per month?

37. (Equality of mixed partial derivatives) Show that $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$ for any function $f(x, y)$ that has continuous second-order derivatives in an open circle centered at (x_0, y_0) by the following argument: Use the Mean-Value Theorem for functions of one variable four times to show that for any point (x, y) in the circle, the quantity $f(x, y) + f(x_0, y_0) - f(x, y_0) - f(x_0, y)$ can be expressed either in the form $f_{xy}(c_1, c_2)(x - x_0)(y - y_0)$ or $f_{yx}(c_3, c_4)(x - x_0)(y - y_0)$, where the points (c_1, c_2) and (c_3, c_4) are in a rectangle with (x_0, y_0) and (x, y) as opposite vertices. This shows that $f_{xy}(c_1, c_2) = f_{yx}(c_3, c_4)$. Complete the derivation by having (x, y) tend to (x_0, y_0).

(End of Section 14.4)

[(2)] Data adapted from *Handbook of Engineering Materials*, Ibid. p. 3-167.

[(3)] Data adapted from *CRC Handbook of Tables for Applied Engineering Science*, 2nd Edition, R. Bolz and G. Tuve, editors, Boca Raton, FL: CRC Press, 1973, . pp. 65–67.