Section 14.5

(3/23/08)

Directional derivatives and gradient vectors

OVERVIEW: *The partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are the rates of change of $z = f(x,y)$ at (x_0, y_0) in the positive x- and y-directions. Rates of change in other directions are given by directional derivatives . We open this section by defining directional derivatives and then use the Chain Rule from the last section to derive a formula for their values in terms of x- and y-derivatives. Then we study gradient vectors and show how they are used to determine how directional derivatives at a point change as the direction changes, and, in particular, how they can be used to find the maximum and minimum directional derivatives at a point.*

Topics:

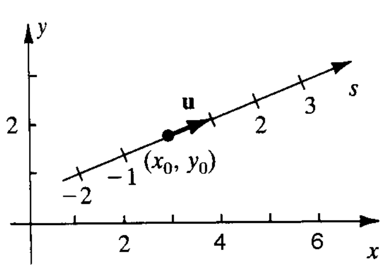
- ***Directional derivatives***
- ***Using angles of inclination***
- ***Estimating directional derivatives from level curves***
- ***The gradient vector***
- ***Gradient vectors and level curves***
- ***Estimating gradient vectors from level curves***

Directional derivatives

To find the derivative of $z = f(x,y)$ at (x_0, y_0) in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2\rangle$ in the xy-plane, we introduce an s-axis, as in Figure 1, with its origin at (x_0, y_0), with its positive direction in the direction of $\mathbf{u}$, and with the scale used on the x- and y-axes. Then the point at s on the s-axis has xy-coordinates $x = x_0 + su_1, y = y_0 + su_2$, and the value of $z = f(x,y)$ at the point s on the s-axis is

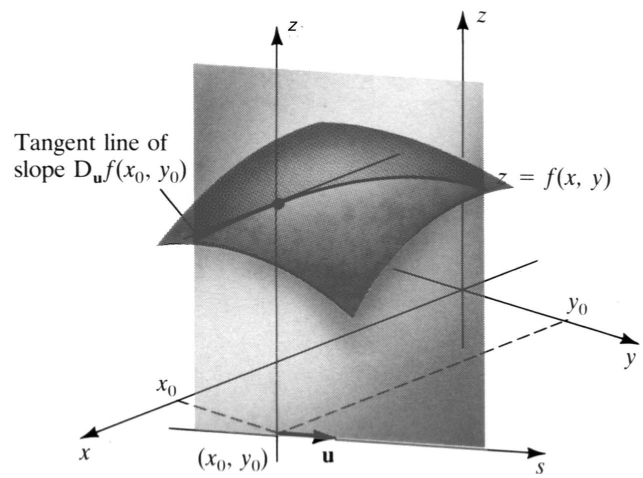
$$F(s) = f(x_0 + su_1, y_0 + su_2). \qquad \textbf{(1)}$$

We call $z = F(s)$ the CROSS SECTION through (x_0, y_0) of $z = f(x,y)$ in the direction of $\mathbf{u}$.



$$\begin{cases} x = x_0 + su_1 \\ y = y_0 + su_2 \end{cases}$$

FIGURE 1



Tangent line of slope

$F'(0) = D_{\mathbf{u}} f(x_0, y_0)$

FIGURE 2

If $(x_0, y_0) \neq (0,0)$, we introduce a second vertical z-axis with its origin at the point $(x_0, y_0, 0)$ (the origin on the s-axis) as in Figure 2. Then the graph of $z = F(s)$ the intersection of the surface $z = f(x,y)$ with the sz-plane. The directional derivative of $z = f(x,y)$ is the slope of the tangent line to this curve in the positive s-direction at $s = 0$, which is at the point $(x_0, y_0, f(x_0, y_0))$. The directional derivative is denoted $D_{\mathbf{u}} f(x_0, y_0)$, as in the following definition.

Definition 1 *The directional derivative of $z = f(x,y)$ at (x_0, y_0) in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ is the derivative of the cross section function* **(1)** *at $s = 0$:*

$$D_{\mathbf{u}} f(x_0, y_0) = \left[\frac{d}{ds} f(x_0 + su_1, y_0 + su_2)\right]_{s=0}. \tag{2}$$

The Chain Rule for functions of the form $z = f(x(t), y(t))$ (Theorem 1 of Section 14.4) enables us to find directional derivatives from partial derivatives.

Theorem 1[†] *For any unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$, the (directional) derivative of $z = f(x,y)$ at (x_0, y_0) in the direction of $\mathbf{u}$ is*

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2. \tag{3}$$

Remember formula **(3)** as the following statement: the directional derivative of $z = f(x,y)$ in the direction of $\mathbf{u}$ equals the x-derivative of f multiplied by the x-component of $\mathbf{u}$, plus the y-derivative of f multiplied by the y-component of $\mathbf{u}$.

Proof of Theorem 1: Definition **(2)** and the Chain Rule from the last section give

$$\begin{aligned} F'(s) &= \frac{d}{ds}[f(x_0 + u_1 s, y_0 + u_2 s)] \\ &= f_x(x_0 + u_1 s, y_0 + u_2 s)\frac{d}{ds}(x_0 + u_1 s) + f_y(x_0 + u_1 s, y_0 + u_2 s)\frac{d}{ds}(y_0 + u_2 s) \\ &= f_x(x_0 + u_1 s, y_0 + u_2 s)u_1 + f_y(x_0 + u_1 s, y_0 + u_2 s)u_2. \end{aligned}$$

We set $s = 0$ to obtain **(3)**:

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2. \quad \textbf{QED}$$

Example 1 Find the directional derivative of $f(x,y) = -4xy - \frac{1}{4}x^4 - \frac{1}{4}y^4$ at $(1, -1)$ in the direction of the unit vector $\mathbf{u} = \langle \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2} \rangle$ (Figure 3).

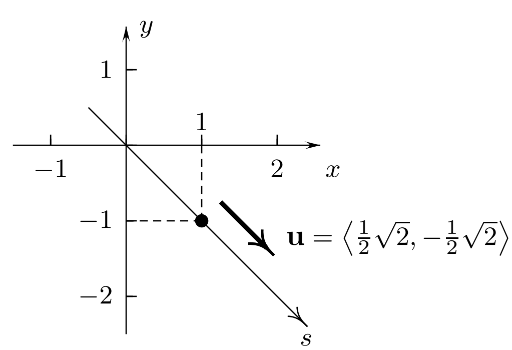


FIGURE 3

[†] We assume in Theorems 1 through 5 of this section and their applications that the functions involved have continuous first-order partial derivatives in open circles centered at all points (x, y) that are being considered.

SOLUTION We first find the partial derivatives,

$$f_x(x,y) = \frac{\partial}{\partial x}(-4xy - \tfrac{1}{4}x^4 - \tfrac{1}{4}y^4) = -4y - x^3$$
$$f_y(x,y) = \frac{\partial}{\partial y}(-4xy - \tfrac{1}{4}x^4 - \tfrac{1}{4}y^4) = -4x - y^3.$$

We set $(x,y) = (1,-1)$ to obtain $f_x(1,-1) = -4(-1) - 1^3 = 3$ and $f_y(1,-1) = -4(1) - (-1)^3 = -3$. Then formula **(3)** with $\langle u_1, u_2\rangle = \langle \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\rangle$ gives

$$\begin{aligned} D_{\mathbf{u}}f(1,-1) &= f_x(1,1)u_1 + f_y(1,1)u_2 \\ &= 3(\tfrac{1}{2}\sqrt{2}) + (-3)(-\tfrac{1}{2}\sqrt{2}) = 3\sqrt{2}. \ \square \end{aligned}$$

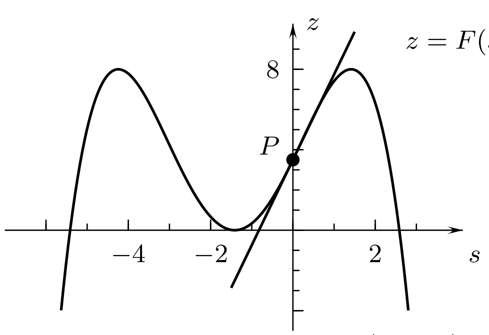
Figures 4 and 5 show the geometric interpretation of Example 1. The line in the xy-plane through $(1,-1)$ in the direction of the unit vector $\mathbf{u} = \langle \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\rangle$ has the equations

$$x = 1 + \tfrac{1}{2}\sqrt{2}\, s, y = -1 - \tfrac{1}{2}\sqrt{2}\, s$$

with distance s as parameter and $s = 0$ at $(1,-1)$. Since $f(x,y) = -4xy - \frac{1}{4}x^4 - \frac{1}{4}y^4$, the cross section of $z = f(x,y)$ through $(1,1)$ in the direction of $\mathbf{u}$ is

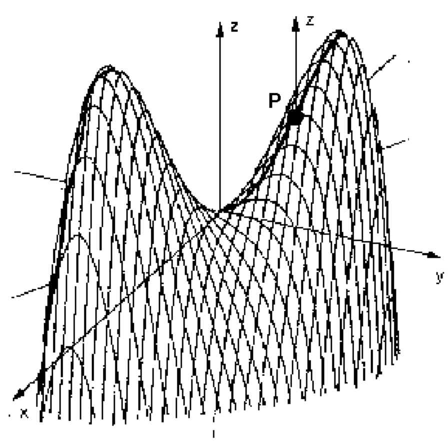
$$\begin{aligned} F(s) &= -4\Big(1 + \tfrac{1}{2}\sqrt{2}\, s\Big)\Big(-1 - \tfrac{1}{2}\sqrt{2}\, s\Big) - \tfrac{1}{4}\Big(1 + \tfrac{1}{2}\sqrt{2}\, s\Big)^4 - \tfrac{1}{4}\Big(-1 - \tfrac{1}{2}\sqrt{2}\, s\Big)^4 \\ &= 4\Big(1 + \tfrac{1}{2}\sqrt{2}\, s\Big)^2 - \tfrac{1}{2}\Big(1 + \tfrac{1}{2}\sqrt{2}\, s\Big)^4. \end{aligned}$$

The graph of this function is shown in the sz-plane of Figure 4. The slope of its tangent line at $s = 0$ is the directional derivative from Example 1. The corresponding cross section of the surface $z = f(x,y)$ is the curve over the s-axis drawn with a heavy line in Figure 5, and the directional derivative is the slope of this curve in the positive s-direction at the point $P = (1,-1,f(1,-1))$ on the surface.



Cross section of $z = -4xy - \frac{1}{4}x^4 - \frac{1}{4}y^4$ through $(1,-1)$ in the direction of $\mathbf{u} = \langle \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\rangle$

FIGURE 4



$z = -4xy - \frac{1}{4}x^4 - \frac{1}{4}y^4$

FIGURE 5

Example 2 What is the derivative of $f(x,y) = x^2y^5$ at $P = (3,1)$ in the direction toward $Q = (4,-3)$?

SOLUTION We first calculate the partial derivatives at the point in question. For $f(x,y) = x^2y^5$, we have $f_x = 2xy^5$ and
$f_y = 5x^2y^4$, so that $f_x(3,1) = 2(3)(1^5) = 6$, and $f_y(3,1) = 5(3^2)(1^4) = 45$.

To find the unit vector $\mathbf{u}$ in the direction from $P = (3,1)$ toward $Q = (4,-3)$, we first find the displacement vector $\overrightarrow{PQ} = \langle 4-3, -3-1\rangle = \langle 1,-4\rangle$. Next, we divide by its length $|\overrightarrow{PQ}| = \sqrt{1^2+(-4)^2} = \sqrt{17}$ to obtain $\mathbf{u} = \langle u_1, u_2\rangle = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \dfrac{\langle 1,-4\rangle}{\sqrt{17}}$.

This formula shows that, $u_1 = \dfrac{1}{\sqrt{17}}$ and $u_2 = \dfrac{-4}{\sqrt{17}}$. Consequently,

$$\begin{aligned} D_{\mathbf{u}}f(3,1) &= f_x(3,1)u_1 + f_y(3,1)u_2 \\ &= 6\left(\frac{1}{\sqrt{17}}\right) + 45\left(\frac{-4}{\sqrt{17}}\right) = -\frac{174}{\sqrt{17}}. \ \square \end{aligned}$$

Using angles of inclination

If the direction of a directional derivative is described by giving the angle α of inclination of the unit vector $\mathbf{u}$, then we can use the expression

$$\mathbf{u} = \langle \cos\alpha, \sin\alpha\rangle \qquad \textbf{(4)}$$

for $\mathbf{u}$ in terms of α to calculate the directional derivative (Figure 6).

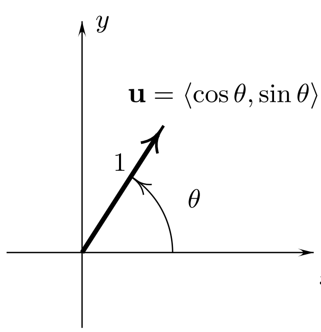


FIGURE 6

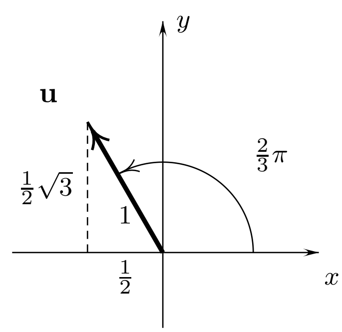


FIGURE 7

Example 3 What is the derivative of $h(x,y) = e^{xy}$ at $(2,3)$ in the direction at an angle of $\frac{2}{3}\pi$ from the positive x-direction?

SOLUTION The partial derivatives are $h_x = e^{xy}\dfrac{\partial}{\partial x}(xy) = ye^{xy}$ and $h_y = e^{xy}\dfrac{\partial}{\partial y}(xy) = xe^{xy}$ and their values at $(2,3)$ are $h_x(2,3) = 3e^6$ and $h_y(2,3) = 2e^6$.

The unit vector $\mathbf{u}$ with angle of inclination $\frac{2}{3}\pi$ forms the hypotenuse of the 30°-60°-right triangle in Figure 7 whose base is $\frac{1}{2}$ and height is $\frac{1}{2}\sqrt{3}$. Therefore, $\mathbf{u} = \langle u_1, u_2\rangle$ with $u_1 = \cos\left(\frac{2}{3}\pi\right) = -\frac{1}{2}$ and $u_2 = \sin\left(\frac{2}{3}\pi\right) = \frac{1}{2}\sqrt{3}$, so that

$$\begin{aligned} D_{\mathbf{u}}h(2,3) &= f_x(2,3)u_1 + f_y(2,3)u_2 \\ &= 3e^6\left(-\tfrac{1}{2}\right) + 2e^6\left(\tfrac{1}{2}\sqrt{3}\right) = (-\tfrac{3}{2}+\sqrt{3})e^6. \ \square \end{aligned}$$

Estimating directional derivatives from level curves

We could find approximate values of directional derivatives from level curves by using the techniques of the last section to estimate the x- and y-derivatives and then applying Theorem 1. It is easier, however, to estimate a directional derivative directly from the level curves by estimating an average rate of change in the specified direction, as in the next example.

Example 4 Figure 8 shows level curves of a temperature reading $T = T(x, y)$ (degrees Celsius) of the surface of the ocean off the west coast of the United States.[1] **(a)** Express the rate of change toward the northeast of the temperature at point P in the drawing as a directional derivative. **(b)** Find the approximate value of this rate of change.

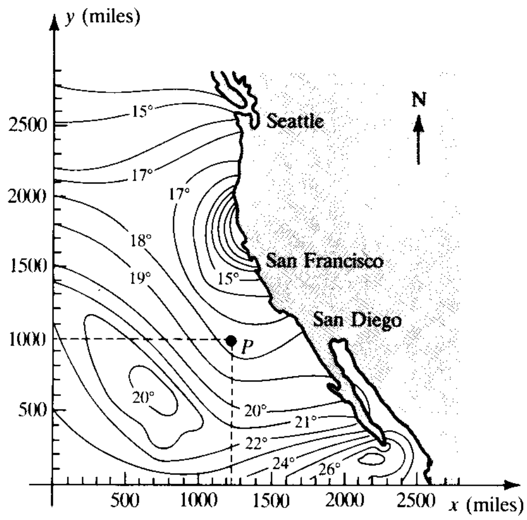


FIGURE 8

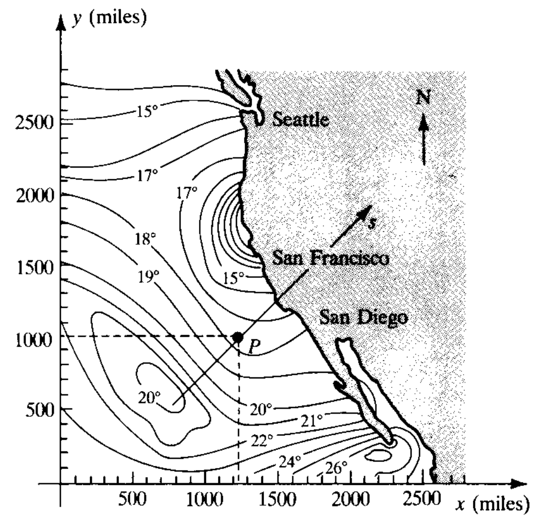


FIGURE 9

SOLUTION **(a)** If we suppose that the point P has coordinates $(1240, 1000)$, as suggested by Figure 8, and denote the unit vector pointing toward the northeast as $\mathbf{u}$, then the rate of change of the temperature toward the northeast at P is $D_{\mathbf{u}}T(1240, 1000)$.

(b) We draw an s-axis toward the northeast in the direction of $\mathbf{u}$ with its origin at P and with the same units as used on the x- and y-axes (Figure 9). This axis crosses the level curve $T = 18^\circ\text{C}$ at a point just below P and crosses the level curve $T = 17^\circ\text{C}$ at a point just above it. The change in the temperature from the lower to the upper point is $\Delta T = 17^\circ - 18^\circ = -1^\circ$. We use the scales on the x- and y-axes to determine that s increases by approximately $\Delta s = 200$ miles from the lower point to the upper point. Consequently, the rate of change of T at P in the direction of the positive s-axis is approximately $\dfrac{\Delta T}{\Delta s} = \dfrac{-1}{200} = -0.005$ degrees per mile. □

[1] Data adapted from *Zoogeography of the Sea* by S. Elkman, London: Sidgwich and Jackson, 1953, p. 144.

The gradient vector

The formula

$$D_{\mathbf{u}}f(x_0,y_0) = f_x(x_0,y_0)\,u_1 + f_y(x_0,y_0)\,u_2 \tag{5}$$

from Theorem 1 for the derivative of f at (x_0,y_0) in the direction of the unit vector $\mathbf{u} = \langle u_1,u_2\rangle$ has the form of the dot product of $\mathbf{u}$ with the vector $\langle f_x, f_y\rangle$ at (x_0,y_0). This leads us to define the latter to be the GRADIENT VECTOR of f, which is denoted ∇f.[†]

Definition 2 *The gradient vector of $f(x,y)$ at (x_0,y_0) is*

$$\nabla f(x_0,y_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle. \tag{6}$$

The gradient vector **(6)** is drawn as an arrow with its base at (x_0,y_0). Because its length is a derivative (a rate of change) rather than a distance, its length can be measured with any convenient scale. We will, however, use the scales on the coordinate axes whenever possible.

Example 5 Draw $\nabla f(1,1), \nabla f(-1,2)$, and $\nabla f(-2,-1)$ for $f(x,y) = x^2y$. Use the scale on the x- and y-axes to measure the lengths of the arrows.

SOLUTION We calculate $\nabla f(x,y) = \left\langle \dfrac{\partial}{\partial x}(x^2y), \dfrac{\partial}{\partial y}(x^2y)\right\rangle = \langle 2xy, x^2\rangle$, and then

$$\begin{aligned}\nabla f(1,1) &= \langle 2(1)(1), 1^2\rangle = \langle 2,1\rangle \\ \nabla f(-1,2) &= \langle 2(-1)(2), (-1)^2\rangle = \langle -4,1\rangle \\ \nabla f(-2,-1) &= \langle 2(-2)(-1), (-2)^2\rangle = \langle 4,4\rangle.\end{aligned}$$

These vectors are drawn in Figure 10. □

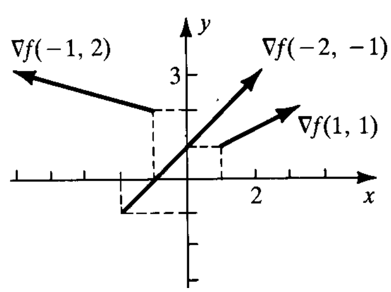


FIGURE 10

With Definition 2, formula **(3)** for the directional derivative becomes

$$D_{\mathbf{u}}f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\mathbf{u}. \tag{7}$$

This representation is useful because we know from Theorem 1 of Section 13.2 that the dot product $\mathbf{A}\cdot\mathbf{B}$ of two nonzero vectors equals the product $|\mathbf{A}||\mathbf{B}|\cos\theta$ of their lengths and the cosine of an angle θ between them. Because $\mathbf{u}$ is a unit vector, its length $|\mathbf{u}|$ is 1 and we obtain the following theorem.

[†]The symbol ∇ is called "nabla" or "del."

Theorem 2 *If $\nabla f(x_0, y_0)$ is not the zero vector, then for any unit vector* $\mathbf{u}$,

$$D_{\mathbf{u}} f(x_0, y_0) = |\nabla f(x_0, y_0)| \cos\theta \tag{8}$$

where θ is an angle between ∇f and $\mathbf{u}$ (Figure 11). If $\nabla f(x_0, y_0)$ is the zero vector, then $D_{\mathbf{u}}(x_0, y_0) = 0$ for all unit vectors $\mathbf{u}$.

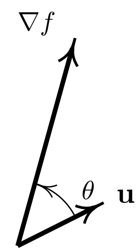


FIGURE 11

Look closely at formula **(8)**. If the point (x_0, y_0) is fixed, then $|\nabla f(x_0, y_0)|$ is a positive constant and as θ varies, $\cos\theta$ varies between 1 and -1: $\cos\theta$ equals 1 when $\nabla f(x_0, y_0)$ and $\mathbf{u}$ have the same direction, equals -1 when $\nabla f(x_0, y_0)$ and $\mathbf{u}$ have opposite directions and θ is a straight angle, and is zero when $\nabla f(x_0, y_0)$ and $\mathbf{u}$ are perpendicular so that θ is a right angle. This establishes the next result.

Theorem 3 *Suppose that $\nabla f(x_0, y_0)$ is not the zero vector. Then* **(a)** *the maximum directional derivative of f at (x_0, y_0) is $|\nabla f(x_0, y_0)|$ and occurs for $\mathbf{u}$ with the same direction as $\nabla f(x_0, y_0)$,* **(b)** *the minimum directional derivative of f at (x_0, y_0) is $-|\nabla f(x_0, y_0)|$ and occurs for $\mathbf{u}$ with the opposite direction as $\nabla f(x_0, y_0)$, and* **(c)** *the directional derivative of f at (x_0, y_0) is zero for $\mathbf{u}$ with either of the two directions perpendicular to $\nabla f(x_0, y_0)$.*

Example 6 **(a)** What is the maximum directional derivative of $g(x, y) = y^2 e^{2x}$ at $(2, -1)$ and in the direction of what unit vector does it occur? **(b)** What is the minimum directional derivative of g at $(2, -1)$ and in the direction of what unit vector does it occur?

SOLUTION **(a)** We find the gradient vector:

$$\nabla g(x, y) = \left\langle \frac{\partial}{\partial x}(y^2 e^{2x}), \frac{\partial}{\partial y}(y^2 e^{2x}) \right\rangle = \langle y^2 e^{2x} \frac{\partial}{\partial x}(2x), 2ye^{2x} \rangle = \langle 2y^2 e^{2x}, 2ye^{2x} \rangle.$$

This formula yields $\nabla g(2, -1) = \langle 2e^4, -2e^4 \rangle$. By Theorem 3, the maximum directional derivative is $|\nabla g(2, -1)| = |\langle 2e^4, -2e^4 \rangle| = \sqrt{(2e^4)^2 + (-2e^4)} = \sqrt{8}\, e^4$. It occurs in the direction of the unit vector,

$$\mathbf{u} = \frac{\nabla g(2, -1)}{|\nabla g(2, -1)|} = \frac{\langle 2e^4, -2e^4 \rangle}{|\langle 2e^4, -2e^4 \rangle|} = \frac{\langle 1, -1 \rangle}{\sqrt{2}}.$$

(b) The minimum directional derivative is $-|\nabla g(2, -1)| = -\sqrt{8}\, e^4$ and occurs in the direction of the unit vector $\mathbf{u} = -\langle 1, -1 \rangle / \sqrt{2} = \langle -1, 1 \rangle / \sqrt{2}$. □

Example 7 Give the two unit vectors $\mathbf{u}$ such that the function $z = g(x, y)$ of Example 6 has zero derivatives at $(2, -1)$ in the direction of $\mathbf{u}$.

SOLUTION The derivative is zero in the two directions perpendicular to the unit vector $\dfrac{\langle 1, -1\rangle}{\sqrt{2}}$ that has the direction of the gradient. Interchanging the components and multiplying one or the other of the components by -1 gives the perpendicular unit vectors. The directional derivative is zero in the directions of $\mathbf{u} = \langle -1, -1\rangle/\sqrt{2}$ and $\mathbf{u} = \langle 1, 1\rangle/\sqrt{2}$. □

Gradient vectors and level curves

If the gradient vector of $z = f(x, y)$ is zero at a point, then the level curve of f may not be what we would normally call a "curve" or, if it is a curve it might not have a tangent line at the point. The gradient of $f = x^2 + y^2$, for example, is $\nabla f = \langle 2x, 2y\rangle$. It is the zero vector at the origin and the level curve $x^2 + y^2 = 0$ at the origin in Figure 12 consists of the single point $(0, 0)$. The function $g = x^2 - y^3$, on the other hand, has the gradient vector $\nabla g = \langle 2x, -3y^2\rangle$, which is also the zero vector at the origin. Its level curve $x^2 - y^3 = 0$ through the origin is the curve $y = x^{2/3}$ in Figure 13, but it has a cusp and no tangent line at the origin.

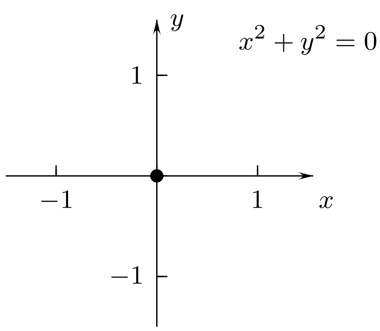


FIGURE 12

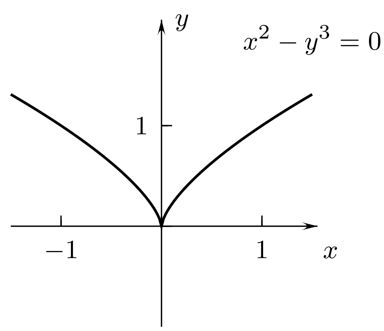


FIGURE 13

If, on the other hand, the gradient vector of a function is not zero at a point, then its level curve through that point is a curve with a tangent line at the point, as is established in the next theorem.

Theorem 4 (The Implicit Function Theorem) *If $\nabla f(x_0, y_0)$ is not the zero vector, then a portion of the level curve of $z = f(x, y)$ through (x_0, y_0) is a parameterized curve with a nonzero velocity vector and therefore a tangent line at (x_0, y_0).*

This theorem is proved in advanced courses. We use it to establish the next result.

Theorem 5 *If $\nabla f(x_0, y_0)$ is not zero, then $\nabla f(x_0, y_0)$ is perpendicular to the level curve of f through (x_0, y_0) (Figure 14).*

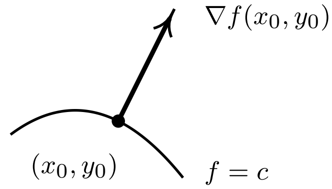


FIGURE 14

Proof: Suppose that $\nabla f(x_0, y_0)$ is not the zero vector. By Theorem 4, a portion of the level curve of f through (x_0, y_0) has parametric equations $x = x(t), y = y(t)$ with $x(t_0) = x_0$ and $y(t_0) = y_0$ and a nonzero velocity vector $\mathbf{v}(t_0) = \langle x'(t_0), y'(t_0)\rangle$. Then for t near t_0, the composite function $z = f(x(t), y(t))$ has the constant value c, and its derivative

$$\frac{d}{dt}[f(x(t), y(t))] = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t)$$

is zero. Setting $t = t_0$ gives

$$0 = f_x(x_0, y_0)\, x'(t_0) + f_y(x_0, y_0)\, y'(t_0) = \nabla f(x_0, y_0) \cdot \mathbf{v}(t_0).$$

Since $\mathbf{v}(t_0)$ is a nonzero velocity vector tangent to the level curve, the last equation shows that $\nabla f(x_0, y_0)$ is perpendicular to the level curve, as is stated in the theorem. **QED**

Example 8 **(a)** Draw the gradient vector of $f(x, y) = xy$ at $(1, 2)$ and the level curve of f through that point. **(b)** Draw $\nabla f(-3, 1)$ and the level curve of f through $(-3, 1)$.

SOLUTION **(a)** Because $\nabla f = \nabla(xy) = \left\langle \frac{\partial}{\partial x}(xy), \frac{\partial}{\partial y}(xy) \right\rangle = \langle y, x\rangle$, the gradient at $(1, 2)$ is $\langle 2, 1\rangle$. Also, because $xy = 2$ at $(1, 2)$, the level curve is $xy = 2$ or $y = 2/x$. The curve and vector are drawn in Figure 15. Notice that the gradient vector is perpendicular to the level curve.

(b) ∇f equals $\langle 1, -3\rangle$ at $(-3, 1)$. Moreover, $f(-3, 1)$ equals $(-3)(1) = -3$, so the level curve is $xy = -3$, which has the equivalent formula $y = -3/x$. This curve and gradient vector are shown in Figure 16. □

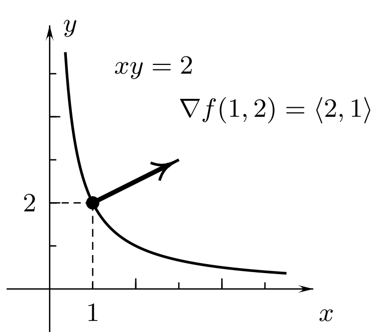


FIGURE 15

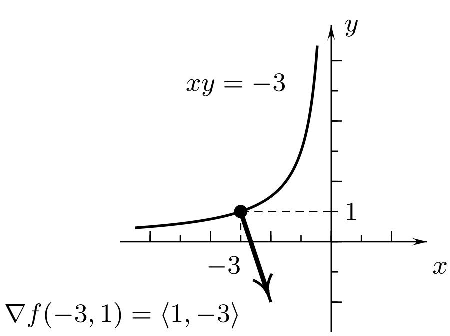


FIGURE 16

Estimating gradient vectors from level curves

To estimate the gradient of a function from its level curves, we could estimate the x- and y-derivatives. We want, however, to emphasize the gradient vector's geometric properties, so in the next example we will instead estimate the length and direction of the gradient vector directly from the level curves.

Example 9 Level curves of a function $z = f(x, y)$ are shown in Figure 17. Find the approximate length and direction of $\nabla f(3, 2)$ and then draw it with the level curves, using the scales on the axes to measure the length of the arrow.

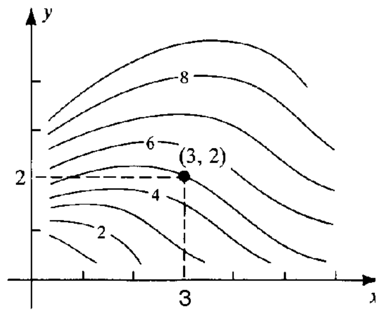


FIGURE 17

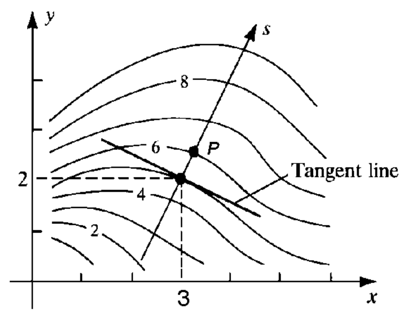


FIGURE 18

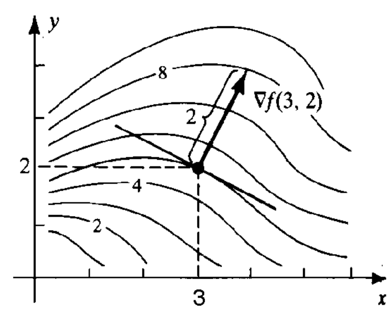


FIGURE 19

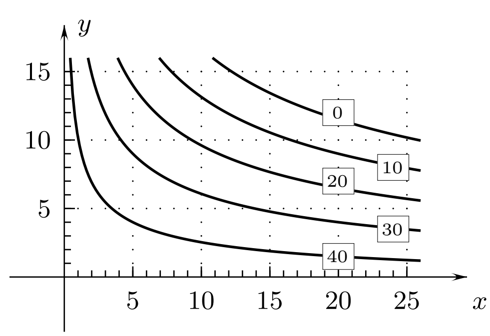
SOLUTION We draw an approximate tangent line at $(3, 2)$ to the level curve of f through that point and a perpendicular s-axis with its positive side in the direction in which f increases, as in Figure 18. By Theorem 5, $\nabla f(3, 2)$ points in the direction of the positive s-axis and by Theorem 3, its length is the rate of change of f at $(3, 2)$ in that direction. The change in f on the s-axis from the level curve $f = 5$ at $(3, 2)$ to point P on the level curve $f = 6$ above it is $\Delta f = 6 - 5 = 1$, and the distance between the level curves along the s-axis is $\Delta s \approx \frac{1}{2}$. Therefore, for $\mathbf{u}$ in the positive s-direction, $D_{\mathbf{u}}(3, 2) \approx \dfrac{\Delta f}{\Delta s} = \dfrac{1}{\frac{1}{2}} = 2$. Since $D_{\mathbf{u}} f(3, 2) = |\nabla f(3, 2)|$ for this vector $\mathbf{u}$, we draw $\nabla f(3, 2)$ as an arrow of length 2 pointing in the direction of the positive s-axis, as in Figure 19. □

Interactive Examples 14.5

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.[†]

1. Find the derivative of $g(x, y) = x^2 y^3$ at $(1, -1)$ in the direction toward the point $(2, 2)$.

2. Level curves of $z = K(x, y)$ are shown in Figure 20. Find the approximate derivative of K at $(5, 5)$ in the direction toward the origin.



Level curves of $z = K(x, y)$
FIGURE 20

[†]In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

3. What is the gradient vector of $f(x,y) = 2x - \ln y$ at $(0,3)$?

4. **(a)** What is the maximum directional derivative of $f = x \sin y$ at $(5, \frac{1}{3}\pi)$? **(b)** What is the unit vector in the direction of the maximum directional derivative?

5. The directional derivative of $g = x^2 - 3y^3$ at $(3,2)$ is zero in two directions. Give the unit vectors in those directions.

6. Figure 21 shows the level curves of $f(x,y) = y - \frac{1}{4}x^2$ through the points $(0,0), (-2,-1)$, and (2,3). Draw ∇f at those points, using the scales on the axes to measure its components.

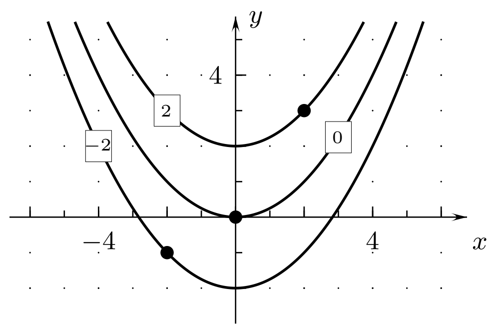


FIGURE 21

7. Figure 22 shows level curves of a function $z = G(x,y)$. Add the approximate gradient vector $\nabla G(7,4)$ to the drawing, using the scales on the axes to measure its components.

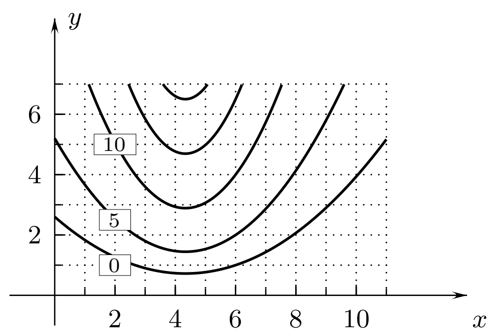


FIGURE 22

Exercises 14.5

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

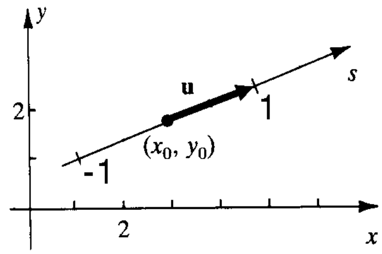
CONCEPTS:

1. What are the maximum and minimum directional derivatives of $z = f(x,y)$ at (x_0, y_0) if the gradient $\nabla f(x_0, y_0)$ is the zero vector?

2. How are the directional derivatives of a function **(a)** in the positive x-direction, **(b)** in the positive y-direction, **(c)** in the negative x-direction, and **(d)** in the negative y-direction related to the x- and y-derivatives?

3. Unit vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ are required in the formula for the derivative of $z = f(x, y)$ at (x_0, y_0) in the direction of $\mathbf{u}$. What would you get if a vector $\mathbf{u}$ of length 2 were used instead, as in Figure 23?

FIGURE 23



4. How does the gradient vector of $g(x, y) = 2xy$ at $(1, 2)$ differ from the gradient vector of $f(x, y) = xy$ at $(1, 2)$ in Figure 15?

BASICS:

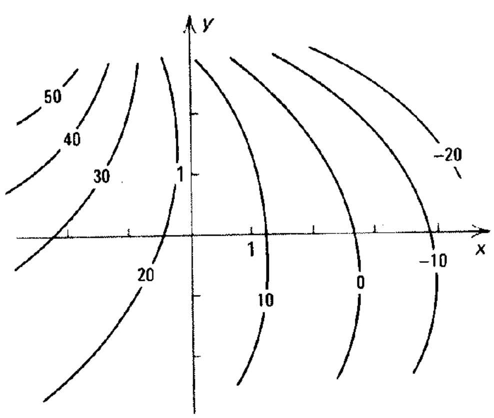
5.$^{\text{O}}$ What is the derivative of $f = x^2y - xy^3$ at $(3, -2)$ in the direction toward $(5, 6)$?

6.$^{\text{O}}$ What is the derivative of $g = \sin(xy)$ at $(\frac{3}{4}, \pi)$ in the direction of the unit vector $\mathbf{u} = (-\mathbf{i}+2\mathbf{j})/\sqrt{5}$?

7. What is the derivative of $h = x^2e^{2y}$ at $(4, 3)$ in the direction of the (nonunit) vector $2\mathbf{i} - 3\mathbf{j}$?

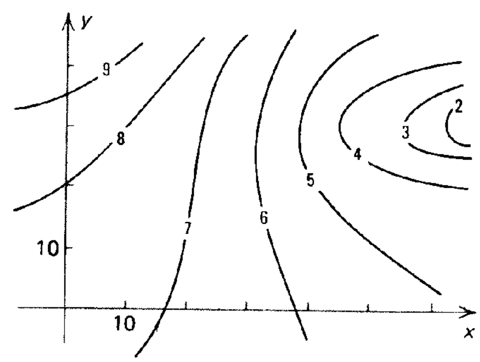
8. What is the derivative of $k = \ln(x^2 - y^2)$ at $(4, 1)$ in the direction toward $(4, -5)$?

9.$^{\text{O}}$ Figure 24 shows level curves of $z = g(x, y)$. Find the approximate derivative of g at $(-1, 1)$ in the direction toward $(0, -2)$.



Level curves of $z = g(x, y)$

FIGURE 24



Level curves of $z = h(x, y)$

FIGURE 25

10. Figure 25 shows level curves of $z = h(x, y)$. What is the approximate derivative of h at $(50, 30)$ in the direction of the vector $\langle -5, -2 \rangle$?

In Exercises 11 through 14 find the gradients of the given functions at the given points.

11.[O] The gradient of $f(x, y) = \ln(xy)$ at $(5, 10)$

12.[A] The gradient of $g(x, y) = x^5y^{20}$ at $(-1, 1)$

13. The gradient of $h(x, y) = x^3y^2 - y^3x^2$ at $(2, -3)$

14. The gradient of $k(x, y) = (x^2 - y^2)^{3/2}$ at $(5, 4)$

15.[O] What is the maximum directional derivative of $z = x^5 + y^3$ at $(1, 5)$? Give the unit vector in the direction of the maximum derivative.

16.[A] What is the minimum directional derivative of $z = x^5e^{4y}$ at $(1, 0)$? What is the unit vector in the direction of the minimum derivative?

17.[O] Give unit vectors in the directions in which the directional derivative of $f(x, y) = x + \sin(5y)$ at $(2, 0)$ are zero.

18. Give unit vectors in the directions in which the directional derivative of $z = e^{x+3y}$ at $(2, 5)$ are zero.

19.[A] **(a)** Draw and label the level curves of $g(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ through the points $(1, 1), (1, -2)$, and $(-3, -1)$. **(b)** draw ∇g at those points, using the scales on the axes to measure its components.

20. Draw the level curve $k(x, y) = 2$ of $k(x, y) = y + \sin x$ and $\nabla k(x, y)$ at four points on it. Describe how ∇k varies along the curve.

21. Draw the level curve of $L(x, y) = \ln(2y - x)$ through $(2, 2)$ and $\nabla L(x, y)$ at four points on it.

22.[O] Figure 26 shows level curves of the depth (feet) of the ocean in the Monterey Canyon off the coast of California.[2] What is the approximate rate of change of the depth with respect to distance at P in the direction of the s-axis?

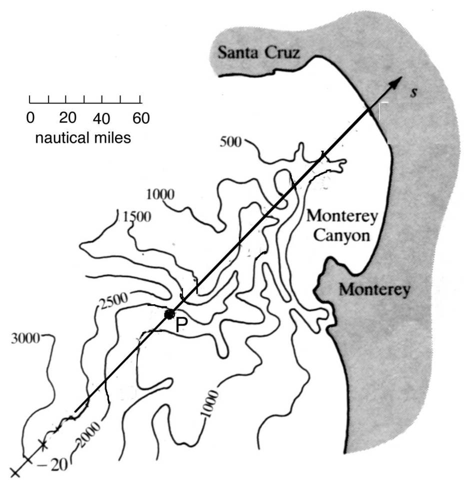


FIGURE 26

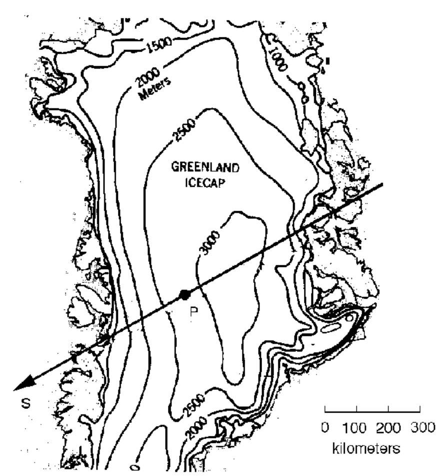


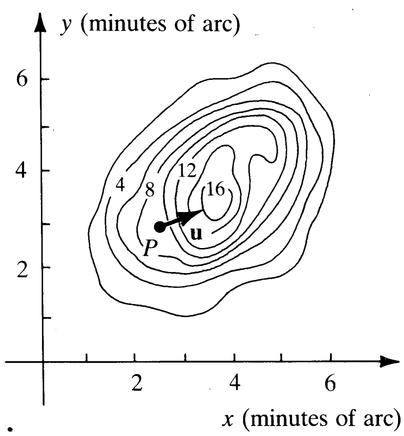
FIGURE 27

23. Figure 27 shows level curves of the elevation (meters) of the Greenland icecap above sea level.[3] What is the approximate rate of change of the elevation with respect to distance at the point P in the direction of the positive s-axis?

[2] Adapted from *Submarine Canyons and Other Sea Valleys* by F. Shepard and R. Dill, Skokie, IL: Rand McNally, 1966, p. 82

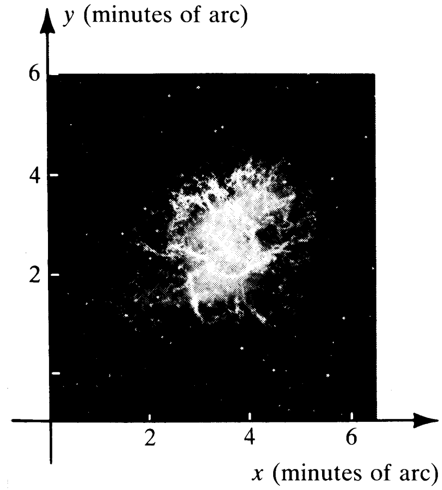
[3] Adapted from *Introduction to Physical Geography* by A. Strahler, New York, NY: John Wiley & Sons, 1970, p. 343

24. Figure 28 shows level curves of the intensity $I = I(x, y)$ (thousand degrees Kelvin) of radio signals from the portion of the sky near the Crab Nebula shown in Figure 29.[4] What is the approximate directional derivative of I at the point P in the direction of the vector $\mathbf{u}$?



Intensity of radio signals
with wavelength 21.3 cm.

FIGURE 28

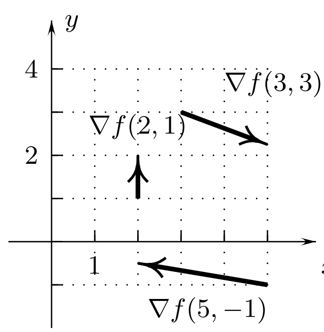


The Crab Nebula in
the constellation Taurus

FIGURE 29

25. Figure 30 shows the gradient of $z = f(x, y)$ at three points with the lengths of the arrows measured by the scales on the axes. Give the values of **(a^O)** $f_x(5, -1)$, **(b^A)** $f_y(3, 3)$, **(c^A)** $f_x(2, 1)$, **(d)** $f_y(2, 1)$, **(e^O)** $D_{\mathbf{u}}f(2, 1)$ with $\mathbf{u} = \langle 1, 1\rangle/\sqrt{2}$, and **(f)** $D_{\mathbf{u}}f(3, 3)$ with $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5}\rangle$.

FIGURE 30



[4] Adapted from *The Radio Universe*, 3rd Edition by J. Hey, ???: Pergamon Books, Ltd, 1970, p. 157

EXPLORATION:

26.[O] Find the approximate x- and y-components of $\nabla f(2,1)$ for the function $z = f(x,y)$ whose level curves are shown in Figure 31.

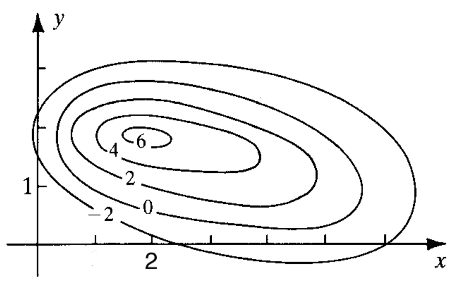


FIGURE 31

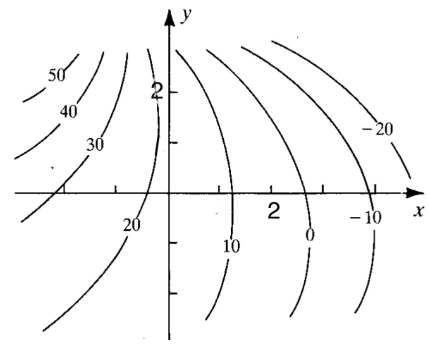
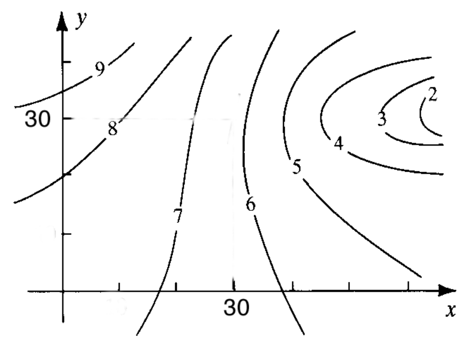


FIGURE 32

27.[A] Find the approximate x- and y-components of $\nabla g(2,2)$ for the function $z = g(x,y)$ whose level curves are shown in Figure 31.

28. Find the approximate x- and y-components of $\nabla h(30,30)$ for the function $z = h(x,y)$ whose level curves are shown in Figure 33.

FIGURE 33



In each of Problems 29 through 31: **(a)** Find a formula for the cross section $F(s) = f(x_0 + su_1, y_0 + su_2)$ of the function $z = f(x,y)$ through a point (x_0, y_0) in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$. **(b)** Use a directional derivative to calculate the slope of the tangent line to the cross section at $s = 0$. [C]**(c)** Generate the graph of the cross section and its tangent line on a calculator or computer and copy them on your paper.

29.[A] The cross section of $f(x,y) = x + \sin y + 2$ through $(0,0)$ in the direction of $\mathbf{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$.

30. The cross section of $f(x,y) = xe^y$ through $(0,0)$ in the direction of $\mathbf{u} = \dfrac{\langle 3,1 \rangle}{\sqrt{10}}$.

31. The cross section of $f(x,y) = xy^2$ through $(2,2)$ in the direction of $\mathbf{u} = \dfrac{\langle 1,1 \rangle}{\sqrt{2}}$.

(End of Section 14.5)