Section 1.2

More on finite limits

Overview: *In this section we find limits where the function is not defined at the limiting value of the variable because it involves a denominator which is zero at that point. This is the most important type of finite limit in calculus because derivatives are defined as such limits. We will find the limits by rewriting the formulas for the functions.*

Topics:

- ***Limits of quotients of polynomials that tend to zero***
- ***Rationalizing differences of square roots***

Limits of quotients of polynomials that tend to zero

At the beginning of the last section we considered a ball that falls $h = 16t^2$ feet in t seconds (Figure 1) , so that the graph of the distance it falls after $t = 0$ is the parabola in Figure 2.

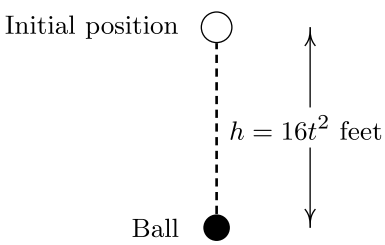

FIGURE 1

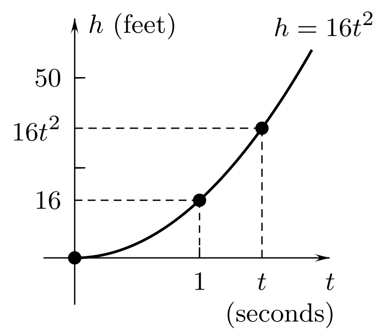

FIGURE 2

We saw that because the ball falls $16t^1 - 16$ feet in the $t - 1$ seconds between time 1 to a later time t, its average velocity in that time interval is

$$[\text{Average velocity}] = \frac{16t^2 - 16}{t - 1} \text{ feet per second.}$$

By calculating values of the average velocity from times t near 1, we predicted that the average velocity would approach 32 feet per second as t approaches 1. We can now verify that prediction.

Example 1 What is the limit of $\lim_{t \to 1} \dfrac{16t^2 - 16}{t - 1}$ as $t \to 1$?

Solution We cannot apply Theorem 1 or Theorem 2 in Section 1.1 on limits of quotients of functions and limits of rational functions because the denominator of $\dfrac{16t^2 - 16}{t - 1}$ is zero at $t = 1$. Instead, we factor the numerator to obtain

$$\frac{16t^2 - 16}{t - 1} = \frac{16(t^2 - 1)}{t - 1} = \frac{16(t + 1)(t - 1)}{t - 1}.$$

Then we cancel the factor $t - 1$ in the numerator with the denominator to have for $t \neq 1$,

$$\frac{16t^2 - 16}{t - 1} = 16(t + 1).$$

This shows that the average velocity **(1)**, which is not defined at $t = 1$, equals $16(t+1)$ for all other values of t. Its graph in Figure 3 consists of the line $A = 16(t+1)$ with the point at $t = 1$ removed. Because $y = 16(t + 1)$ is a polynomial, its limit as t tends to 1 is its value at $t = 1$ and

$$\lim_{t\to 1} \frac{16t^2 - 16}{t - 1} = \lim_{t\to 1} [16(t + 1)]$$

$$= \Big[16(t + 1)\Big]_{t=1} = 16(1 + 1) = 32.$$

The limit of the average velocity is 32 feet per second. □

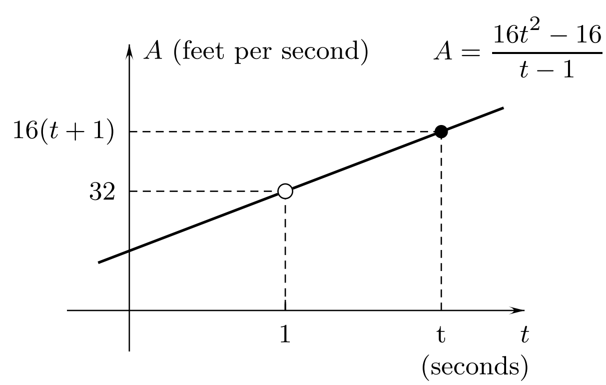

FIGURE 3

The limits in the next two examples are also found by factoring and canceling.

Example 2 Find the limit $\lim_{x\to 2} \frac{x^2 - 4}{x^2 - x - 2}$.

SOLUTION We cannot find the limit directly because $y = \frac{x^2 - 4}{x^2 - x - 2}$ is not defined at $x = 2$. We use the factorizations $x^2 - 4 = (x - 2)(x + 1)$ and $x^2 - x - 2 = (x - 2)(x + 1)$ to write for $x \neq -1, 2$,

$$\frac{x^2 - 4}{x^2 - x - 2} = \frac{(x - 2)(x + 2)}{(x - 2)(x + 1)} = \frac{x + 2}{x + 1}.$$

Then because the rational function $y = \frac{x + 2}{x + 1}$ is defined at $x = 2$, its limit as $x \to 2$ is its value at 2 and we obtain

$$\lim_{x\to 2} \frac{x^2 - 4}{x^2 - x - 2} = \lim_{x\to 2} \frac{x + 2}{x + 1} = \left[\frac{x + 2}{x + 1}\right]_{x=2} = \frac{2 + 2}{2 + 1} = \tfrac{4}{3}. \ \square$$

Example 3 Find the limit of $R(x) = \dfrac{(x+2)^2 - 4}{(x+1)^2 - 1}$ as $x \to 0$.

SOLUTION We cannot find the limit directly because $R(x)$ is not defined at $x = 0$. Instead, we expand the squares in the numerator and denominator and factor the results to obtain for nonzero x close to 0,

$$R(x) = \frac{(x+2)^2 - 4}{(x+1)^2 - 1} = \frac{(x^2 + 4x + 4) - 4}{(x^2 + 2x + 1) - 1} = \frac{x^2 + 4x}{x^2 + 2x} = \frac{x(x+4)}{x(x+2)}.$$

We can cancel the x's in the last formula to have for nonzero x close to 0

$$R(x) = \frac{x+4}{x+2}.$$

Since the rational function $y = (x+4)/(x+2)$ is defined at $x = 0$, its limit as x tends to 0 is its value at $x = 0$ and

$$\lim_{x \to 0} R(x) = \lim_{x \to 0} \left[\frac{x+4}{x+2}\right] = \left[\frac{x+4}{x+2}\right]_{x=0} = \tfrac{4}{2} = 2. \ \square$$

Rationalizing differences of square roots

Limits of some functions involving square roots can be found by a procedure called RATIONALIZATION to differences of square roots. In this procedure a difference $\sqrt{a} - \sqrt{b}$ of square roots is multiplied and divided by the sum $\sqrt{a} + \sqrt{b}$ to obtain

$$\sqrt{a} - \sqrt{b} = \frac{(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b})}{\sqrt{a} + \sqrt{b}} = \frac{(\sqrt{a})^2 - (\sqrt{b})^2}{\sqrt{a} + \sqrt{b}} = \frac{a - b}{\sqrt{a} + \sqrt{b}}.$$

The examples and exercises in this section that involve the square root function use the fact[†] that for any positive number a,

$$\lim_{x \to a} \sqrt{x} = \sqrt{a}.$$

Example 4 Find the limit of $\dfrac{\sqrt{x} - 1}{x^2 - 1}$ as $x \to 1$.

SOLUTION We cannot apply Theorem 1 of Section 1.1 on limits of quotients because the denominator $x^2 - 16$ of the given expression is zero at $x = 1$. Instead, we write the numerator $\sqrt{x} - 1$ as a difference of square roots, $\sqrt{x} - \sqrt{1}$, and rationalize it by multiplying and dividing by the sum $\sqrt{x} + \sqrt{1}$ of the square roots: for positive $x \neq 1$,

$$\begin{aligned}\frac{\sqrt{x} - 1}{x^2 - 1} &= \frac{\sqrt{x} - \sqrt{1}}{x^2 - 1} = \frac{(\sqrt{x} - \sqrt{1})(\sqrt{x} + \sqrt{1})}{(x^2 - 1)(\sqrt{x} + \sqrt{1})} \\ &= \frac{(\sqrt{x})^2 - (\sqrt{1})^2}{(x^2 - 1)(\sqrt{x} + \sqrt{1})} = \frac{x - 1}{(x^2 - 1)(\sqrt{x} + 1)}\end{aligned}$$

[†]See Theorem 2 of Section 1.3.

Then we factor $x^2 - 1$ in the denominator and cancel one factor with the numerator to obtain for positive $x \neq 1$,

$$\frac{\sqrt{x}-1}{x^2-1} = \frac{x-1}{(x^2-1)(\sqrt{x}+1)} = \frac{x-1}{(x-1)(x+1)(\sqrt{x}+1)} = \frac{1}{(x+1)(\sqrt{x}+1)}. \quad \textbf{(1)}$$

The denominator $(x+1)(\sqrt{x}+1)$ on the right of **(1)** tends to the nonzero number $(1+1)(\sqrt{1}+1) = 2(2) = 4$ as $x \to 1$ and the numerator is constant. By Theorem 1 of Section 1.1 on limits of quotients of functions,

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x^2-11} = \lim_{x\to 1} \frac{1}{(x+1)(\sqrt{x}+1)} = \frac{1}{\lim_{x\to 1}\left[(x+1)(\sqrt{x}+1)\right]} = \tfrac{1}{4}. \ \square$$

Interactive Examples 1.2

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.†

1. Predict the limit of $y = \dfrac{x^2-9}{x-3}$ as $x \to 3$ by calculating values of the function.

Find the limits in Examples 2 through 6.

2. $\displaystyle\lim_{x\to 3} \frac{x^2-9}{x-3}$

3. $\displaystyle\lim_{x\to 0} \frac{x^3+4x^2}{x^5+2x^2}$

4. $\displaystyle\lim_{x\to 2} \frac{x^2-4}{x+2}$

5. $\displaystyle\lim_{x\to 2} \frac{x^2-x-2}{x^2+3x+2}$

6. $\displaystyle\lim_{x\to 5} \frac{x-5}{\sqrt{x}-\sqrt{5}}$

Exercises 1.2

AAnswer provided. **O**Outline of solution provided. **C**Graphing calculator or computer required.

CONCEPTS:

1. It was shown in the solution of Example 1 that the functions $y = \dfrac{16t^2-16}{t-1}$ and $y = 16(t+1)$ have the same values for $t \neq 1$. How do the graphs of these functions differ?

2. Find the limit $\displaystyle\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$ by making the substitution $z = \sqrt{x}$ instead of rationalizing the numerator.

---

†In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

BASICS:

In Exercises 3 through 6 predict the limits of the functions as $x \to 1$ by calculating values for x near 1 on a calculator or computer.

3.[O] $y = \dfrac{x^3 - 1}{1 - x^4}$

4.[A] $y = \dfrac{x^2 + x - 2}{\sqrt{x} - 1}$

5.[A] $y = \dfrac{x^6 - 1}{x^2 - 1}$

6. $y = \dfrac{x^5 - 1}{\sqrt{2x + 7} - 3}$

In Exercises 7 through 10 **(a)** find the limits and then [C]**(b)** generate the graphs of the functions in the given windows to illustrate the results.

7.[O] $\lim\limits_{x\to 2} \dfrac{1 - 2/x}{x - 2}$ $(0 \le x \le 4, 0 \le y \le 2)$

8.[O] $\lim\limits_{x\to 0} \dfrac{(x-1)^2 - 1}{x}$ $(-2 \le x \le 4, -4 \le y \le 3)$

9.[A] $\lim\limits_{x\to 2} \dfrac{1 - 2/x}{x - 2}$ $(-1 \le x \le 4, -0.5 \le x \le 1.5)$

10. $\lim\limits_{x\to \pm 1} \dfrac{x^2 - 1}{x^4 - x^2}$ $(-2.5 \le x \le 2.5, -0.5 \le y \le 3)$

Find the limits in Exercises 11 through 24.

11.[O] $\lim\limits_{x\to 3} \dfrac{x^2 - 9}{x - 3}$

12.[O] $\lim\limits_{x\to 3} \dfrac{(9/x^2) - 1}{12 - 4x}$

13.[A] $\lim\limits_{x\to 1} \dfrac{(x-1)^2}{x^2 - 1}$

14.[A] $\lim\limits_{x\to -2} \dfrac{2 + x}{4 - x^2}$

15. $\lim\limits_{x\to 1} \dfrac{x^2 - 1}{3x - 3}$

16.[O] $\lim\limits_{x\to 2} \dfrac{x^2 - 4}{x^2 + 4}$

17.[A] $\lim\limits_{x\to 2} \dfrac{(1/x) - (1/2)}{x^2 - 4}$

18.[A] $\lim\limits_{x\to 0} \dfrac{(3+x)^2 - 9}{x}$

19. $\lim\limits_{x\to 0} \dfrac{x^2 + 4x}{6x - x^3}$

20. $\lim\limits_{x\to 2} \dfrac{1 - (2/x)}{x^2 - 4}$

21. $\lim\limits_{x\to -2} \dfrac{x^2 + 4}{x^3 - 8}$

22.[O] $\lim\limits_{x\to 2} \dfrac{x^2 - 4x + 4}{x^2 + 3x - 10}$

23.[A] $\lim\limits_{x\to 1} \dfrac{x^2 + x - 2}{x^2 + 3x - 4}$

23.[A] $\lim\limits_{x\to 0} \dfrac{4 - (x+2)^2}{9 - (x+3)^2}$

24. $\lim\limits_{x\to -1} \dfrac{x^2 + 3x + 2}{x + 1}$

EXPLORATION:

Find the limits in Exercises 25 through 33.

25.[O] $\lim_{x\to 2} \dfrac{x^4-16}{x-2}$

26. $\lim_{x\to 1} \dfrac{x^4+x^2-2}{x-1}$

27.[O] $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{x^2-1}$

28.[A] $\lim_{x\to 4} \dfrac{\dfrac{2}{\sqrt{x}}-4}{x-4}$

29. $\lim_{x\to 8} \dfrac{\sqrt{x+1}-3}{8-x}$

30.[A] $\lim_{x\to 5} \dfrac{\sqrt{5}-\sqrt{x}}{\sqrt{5}+\sqrt{x}}$

31. $\lim_{x\to 1} \dfrac{x^2-1}{x^3-1}$

32. $\lim_{x\to 3} \sqrt{\dfrac{x^2-9}{x-3}}$

33. $\lim_{x\to 1} \dfrac{\sqrt{x+3}-2}{\sqrt{x}-1}$

(End of Section 1.2)