Section 2.5

(3/19/08)

Derivatives as functions and estimating derivatives

OVERVIEW: *In this section we discuss more examples of derivatives as rates of change and show how approximate derivatives can be obtained from graphs and tables. We also look at the concept of* MARGINAL CHANGE *from economics, and discuss acceleration and other higher-order derivatives.*

Topics:

- ***The derivative as a function***
- ***Finding approximate derivatives from graphs***
- ***Using a graph to find where a derivative has a particular value***
- ***Estimating a derivative from a table***
- ***The term "marginal" in economics***
- ***Differentiablity and continuity***
- ***Higher-order derivatives***
- ***Acceleration***

The derivative as a function

Imagine that a tank contains $V(t) = 12t - t^2$ gallons of water at time t (minutes) for $0 \le t \le 12$. As can be seen from the graph of $V = V(t)$ in the tV-plane of Figure 1, the tank is empty at $t = 0$, water is added for six minutes and then drained out, and the tank is empty again at $t = 12$.

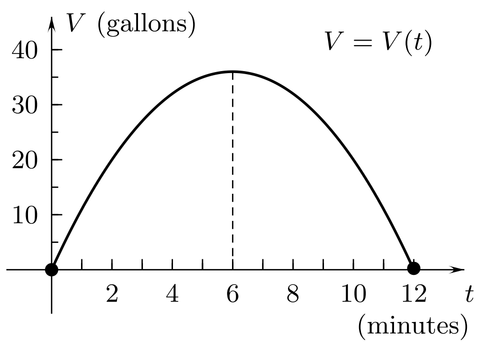


FIGURE 1

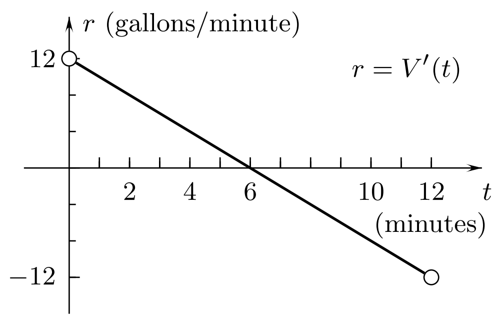


FIGURE 2

Example 1 **(a)** Find a formula for the rate of change with respect to time of the volume of water in the tank and draw its graph. **(b)** What do the rates of change of the volume at $t = 2, t = 6$, and $t = 10$ indicate about the water in the tank?

SOLUTION **(a)** The rate of change of the volume is the derivative $V'(t) = \dfrac{d}{dt}(12t - t^2) = 12 - 2t$. It is defined for $0 < t < 12$ because $V(t)$ is defined for $0 \le t \le 12$. The graph of $r = V'(t)$ is the line segment in Figure 2.

(b) Because $V'(2) = 12 - 2(2) = 8$, water is flowing into the tank at the rate of 8 gallons per minute at $t = 2$. Because $V'(6) = 12 - 2(6) = 0$, water is neither flowing into or out of the tank at $t = 6$. Since $V'(10) = 12 - 2(10) = -8$, water is flowing out of the tank at the rate of 8 gallons per minute at $t = 10$. □

The values of $r = V'(t)$ at $t = 2, 6$, and 10 are plotted in Figure 3, and the corresponding tangent lines to the graph of V are shown in Figure 4. The tangent line at $t = 2$ has slope 8 because the value of the derivative at $t = 2$ is 8. The tangent line at $t = 6$ is horizontal (has zero slope) because the value of the derivative at $t = 6$ is 0. The tangent line at $t = 2$ has slope -8 because the value of the derivative at

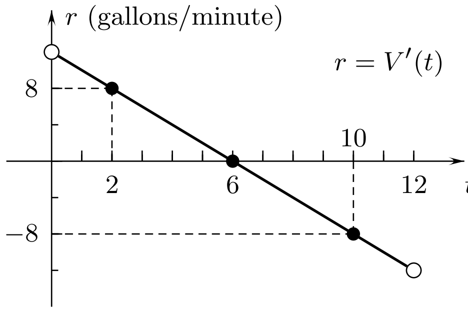


FIGURE 3

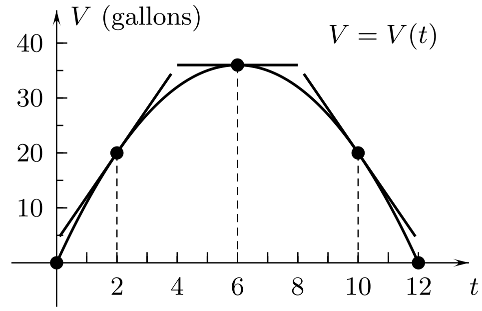


FIGURE 4

$t = 10$ is -8.

Finding approximate derivatives from graphs

If you are given a function by a drawing of its graph rather than by an exact formula, you cannot find exact values of its derivative. You can, however, find approximate values of the derivative, as in the next two examples, by drawing plausible tangent lines to the graph and determining their approximate slopes by estimating the coordinates of pairs of points on the lines.

Example 2 Figure 5 shows the graph of an object's position $s = s(t)$ on an s-axis as a function of the time t. What is the object's approximate velocity in the positive s-direction at $t = 2$?

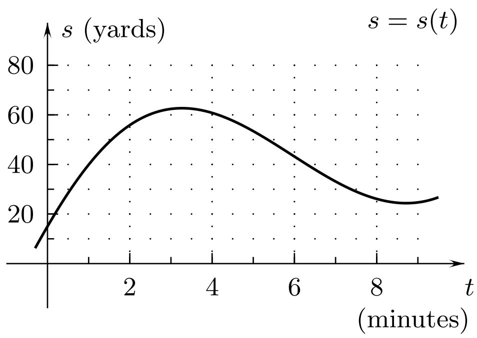


FIGURE 5

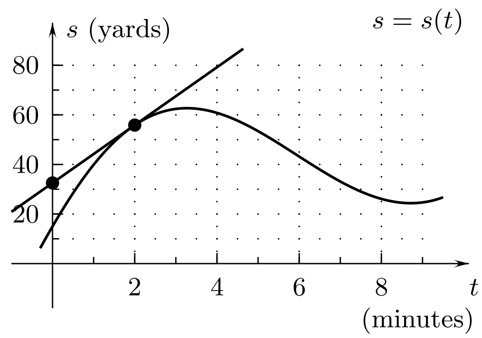


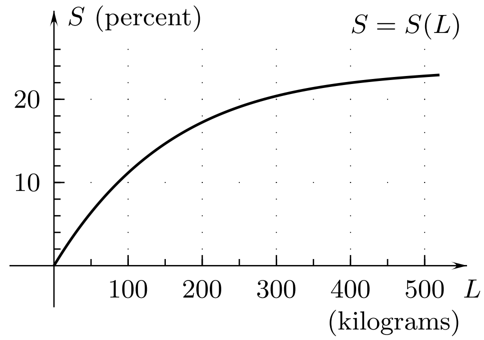
FIGURE 6

SOLUTION We draw a plausible tangent line at $t = 2$, as in Figure 6. Its slope is the approximate velocity at $t = 2$. The points shown by dots on the tangent line have approximate coordinates $(0, 32)$ and $(2, 55)$, so that

$$[\text{Velocity at } t = 2] = s'(2) \approx \frac{55 - 32}{2 - 0} = 11.5 \text{ yards per minute. } \square$$

Vertebrae in the human spine are separated by fibrous elastic disks that compress when the load on them increases. The load on the disk is called the STRESS, and the percent of compression is called the STRAIN. The function $S = S(L)$ in Figure 7 gives the strain S of lower back (lumbar) vertebral disks as a function of the stress L on them.[1]

FIGURE 7



Example 3 Sketch the graph of the derivative $r = S'(L)$ of the stress function $S = S(L)$ of Figure 7.

SOLUTION We draw approximate tangent lines near the left and the right ends of the graph of S as in Figure 8. The tangent line at $L = 50$ on the left contains points with approximate coordinates $(50, 6.5)$ and $(100, 12)$, so its slope is approximately $\dfrac{12 - 6.5}{100 - 50} = 0.11$ and $S'(50) \approx 0.11$ percent per kilogram. The tangent line at $L = 400$ on the right passes through points with approximate coordinates $(200, 20)$ and $(400, 22)$, so its slope is approximately $\dfrac{22 - 20}{400 - 300} = 0.02$ and $S'(400) \approx 0.02$ percent per kilogram. We plot these values in an Lr-plane as in Figure 9. Then, because the slope of the tangent line decreases as L increases, we draw a curve that starts above $r = 0.11$ at $L = 0$ and decreases through the two plotted points. □

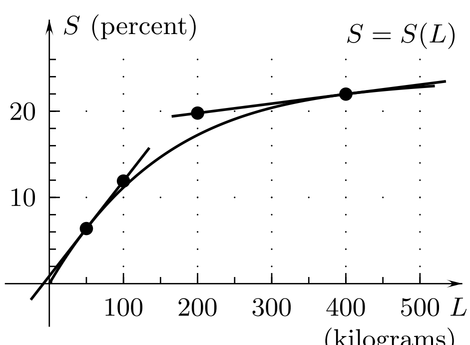


FIGURE 8

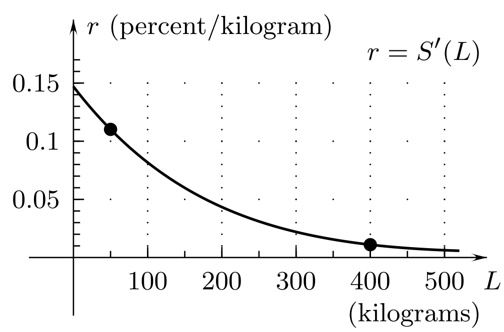


FIGURE 9

[1] Data adapted from *Physics with Illustrative Examples from Medicine and Biology*, Vol. 1, G. Benedeck and F. Villars, Reading, Massachusetts: Addison-Wesley, 1973, p.343.

Using a graph to find where a derivative has a particular value

If we are given the graph of a function and we want to determine where its derivative has a specified value, we can look for places on the graph where the tangent line has the given slope.

Example 4 The graph of the height $s = s(t)$ (feet) of a balloon as a function of time is shown in Figure 10. Find the approximate values of two times when the balloon's upward velocity is 20 feet per minute.

SOLUTION The upward velocity is the rate of change $v = ds/dt$ (feet per minute) of the balloon's height, and its value at each t is the slope of the tangent line to the graph of the height at that value of t. To estimate where the velocity is 20, we draw the line at the lower right in Figure 11, which runs from $(4, 0)$ to $(10, 120)$ and has slope $\dfrac{120-0}{10-4} = 20$. Then we determine approximately where the tangent line to the graph $s = s(t)$ is parallel to that line. (A lined, transparent ruler is useful in this step.) As can be seen in Figure 11, the velocity is 20 feet per minute at $t \approx 2.4$ and $t \approx 7.6$. □

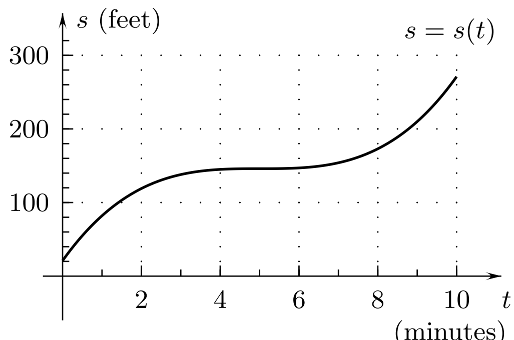


FIGURE 10

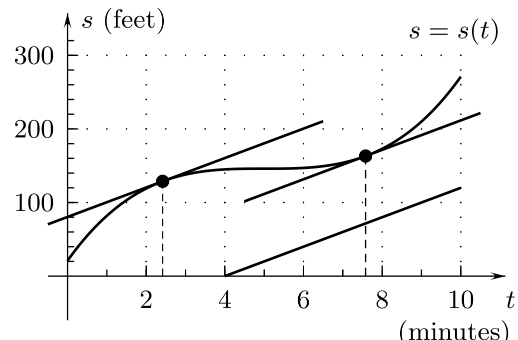


FIGURE 11

Estimating a derivative from a table

The next table lists the wind speed at three-hour intervals one day in Dodge City, Kansas.[2] The time t is measured in hours with $t = 0$ at midnight. We want to use this data to estimate the rate of change $W'(t)$ of the wind's velocity with respect to time at 9:00 AM ($t = 9$).

WIND VELOCITY $W(t)$ (MILES PER HOUR) IN DODGE CITY, KANSAS

t	0	3	6	9	12	15	18	21	24
$W(t)$	12.55	12.40	12.57	15.32	16.00	15.66	14.92	11.30	12.91

Figures 12 through 14, which show the data from the table, illustrate three techniques for estimating the derivative $W'(9)$. We could use the secant line in Figure 12, whose slope is the average rate of change of $W = W(t)$ with respect to t for $9 \le t \le 12$; we could use the secant line in Figure 13, whose slope is the average rate of change for $6 \le t \le 9$; or we could use the secant line in Figure 14, whose slope is the average rate of change for $6 \le t \le 12$.

[2] Data from *Wind Energy Systems*, G. L. Johnson, Englewood Cliffs, N. J: Prentice Hall International, Inc., 1985, p. 47.

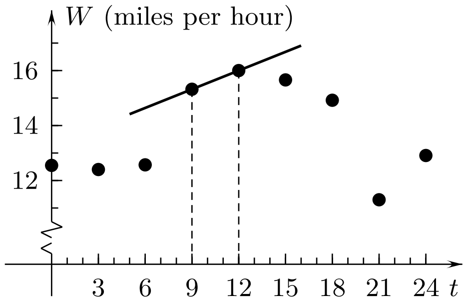


FIGURE 12

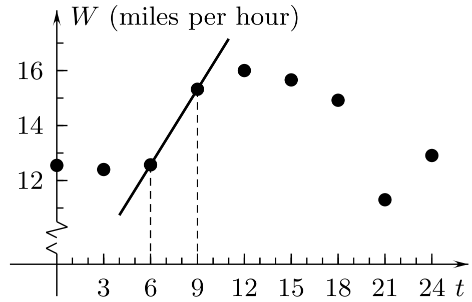


FIGURE 13

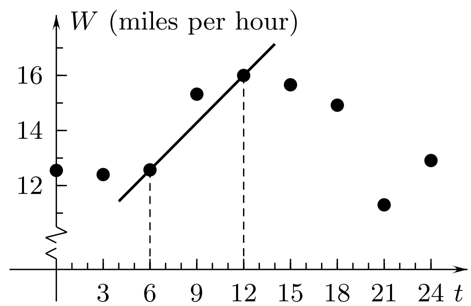


FIGURE 14

Example 5 Give three estimates of $W'(9)$ by finding the slopes of the secant lines **(a)** in Figure 12, **(b)** in Figure 13, and **(c)** in Figure 14.

SOLUTION **(a)** $W'(9) \approx \dfrac{W(12) - W(9)}{12 - 9} = \dfrac{16.00 - 15.32}{3} \doteq 0.23$ miles per hour per hour

(b) $W'(9) \approx \dfrac{W(9) - W(6)}{9 - 6} = \dfrac{15.32 - 12.57}{3} \doteq 0.92$ miles per hour per hour

(c) $W'(9) \approx \dfrac{W(12) - W(6)}{12 - 6} = \dfrac{16.00 - 12.57}{6} \doteq 0.57$ miles per hour per hour □

The slopes of the secant lines in Figures 12 through 14 are called a RIGHT-DIFFERENCE QUOTIENT, a LEFT-DIFFERENCE QUOTIENT, and a CENTERED-DIFFERENCE QUOTIENT, respectively. These can be used to approximate a derivative of any function: if $f(x)$ has a derivative at a, then for positive Δx,

$$f'(a) \approx \frac{f(a + \Delta x) - f(a)}{\Delta x} \quad \text{(a right-difference quotient)} \tag{1}$$

$$f'(a) \approx \frac{f(a) - f(a - \Delta x)}{\Delta x} \quad \text{(a left-difference quotient)} \tag{2}$$

$$f'(a) \approx \frac{f(a + \Delta x) - f(a - \Delta x)}{2\Delta x} \quad \text{(a centered-difference quotient).} \tag{3}$$

The centered-difference quotient in **(3)** is also the slope of the tangent line at $x = a$ to the approximation of the graph of $y = f(x)$ that is obtained by using the parabola that passes through the points at $x = a - \Delta x$, $x = a$, and $x = a + \Delta x$ on it. This is the case because the tangent line to the parabola at $x = a$ is parallel to the secant line through the points at $x = a - \Delta x$ and $x = a + \Delta x$, as can be seen in Figure 15 for the function W of Example 5.[†]

[†]A short calculation shows that for a function $g(x) = Ax^2 + Bx + C$ whose graph is a parabola, the centered-difference quotient $\dfrac{g(a + \Delta x) - g(a - \Delta x)}{2\Delta x}$ is $2Aa + B$, which equals $g'(a)$.

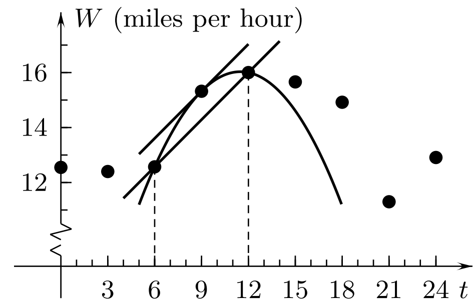


FIGURE 15

If we want to estimate the derivative at a value of the variable between the values listed in a table, we can use the slope of the secant line through points on both sides. For example, we can estimate the rate of change of the wind velocity at 10:00 ($t = 10$) from the table above by using the slope 0.23 miles per hour per hour of the secant line in Figure 12 through the points at $t = 9$ and $t = 12$ that we calculated in Example 5a.

The term "marginal" in economics

Economists often use the word "marginal" for "derivative of," as in the statement

$$[\text{Marginal } f(x)] = f'(x). \tag{4}$$

They also use "marginal" for the change in a function when its variable is increased by 1:

$$[\text{Marginal } f(x)] = [f(x+1) - f(x)]. \tag{5}$$

These uses are closely related because the quantity in **(5)** is the average rate of change of $f(x)$ from x to $x + 1$

$$f(x+1) - f(x) = \frac{f(x+1) - f(x)}{1}$$

which in many cases is close to the derivative $f'(x)$.

Example 6 A company makes $R(x) = 2.50x + 0.04x^2$ dollars revenue if it sells x units of a product. **(a)** Use **(4)** to calculate the marginal revenue on 100 units. **(b)** Calculate the marginal revenue on 100 units using **(5)**, and compare the result with that from part (a).

SOLUTION **(a)** The derivative of the revenue with respect to x is $R'(x) = \frac{d}{dx}(2.50x + 0.04x^2)$ $= 2.50 + 0.08x$, so that by formula **(4)** the marginal revenue at $x = 100$ is $R'(100) = 2.50 + 0.08(100) = 10.50$ dollars per unit.
(b) By formula **(5)** the marginal revenue at $x = 100$ is $R(101) - R(100)$ $= 660.54 - 650 = 10.54$ dollars per unit. The marginal revenues given by the two formulas differ by 4 cents per unit. □

The income tax for individuals in the United Kingdom for the 2001–2002 tax year can be calculated approximately as follows:[3] there is a 22% tax on the first £30,000 of taxable income and a 40% tax on any additional taxable income. With this formula, the tax $T = T(x)$ on taxable income of x pounds is

$$T(x) = \begin{cases} 0.22x & \text{for} \quad 0 \le x \le 30{,}000 \\ 6600 + 0.40(x - 30{,}000) & \text{for} \quad x > 30{,}000. \end{cases} \tag{6}$$

The graph of $T = T(x)$ is in Figure 16. This function is called PIECEWISE LINEAR because its domain consists of intervals on each of which it is linear. Graphs of piecewise-linear functions are formed from line segments and possibly isolated points.

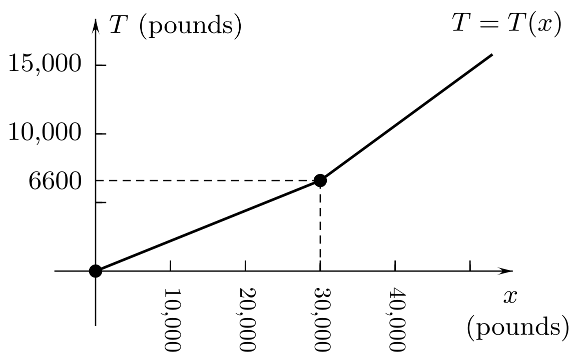


FIGURE 16

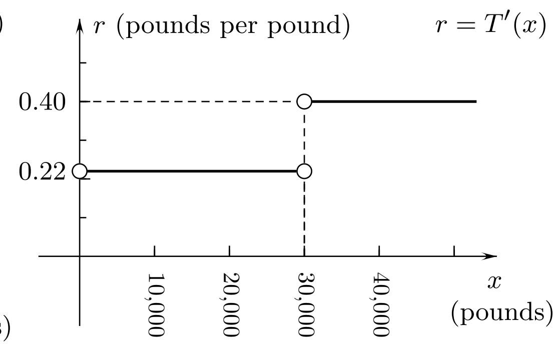


FIGURE 17

Example 7 Find formulas for the derivative $r = T'(x)$ of the tax **(6)** and sketch its graph.

SOLUTION Since $T'(x) = \dfrac{d}{dx}(0.22x) = 0.22$ for $0 < x < 30{,}000$ and $T'(x)$ $= \dfrac{d}{dx}[6600 + 0.40(x - 30{,}000)] = 0.40$ for $x > 30{,}000$, we have

$$T'(x) = \begin{cases} 0.22 & \text{for} \quad 0 < x < 30{,}000 \\ 0.40 & \text{for} \quad x > 30{,}000. \end{cases}$$

The graph $r = T'(x)$ of the derivative is shown in Figure 17. □

The derivative $r = T'(x)$ is called a STEP FUNCTION because it is piecewise constant and its graph in Figure 17 consists of horizontal line segments that look like steps. The derivative $T'(x)$ is the marginal tax rate for a taxable income of x pounds and is the highest rate paid on that taxable income.

[3] Data adapted from *Inland Revenue Tax Calculation Guide for the Year Ended 5 April, 2001*, http://www.inlandrevenue.gov.uk/rates/it.htm, 2001, p. 2. (The actual tax is 10% on the first £1520 of taxable income, 22% on the next £26,880, and 40% on any additional amount.)

Differentiabililty and continuity

A function $y = f(x)$ that has a derivative at a value $x = a$ of its variable is said to be DIFFERENTIABLE at that point. This implies—as is shown in the next theorem—that it is continuous at $x = a$.

Theorem 1 *If $y = f(x)$ has a derivative at $x = a$, then $y = f(x)$ is continuous at $x = a$.*

Proof: Suppose that $y = f(x)$ has a derivative at a. Then the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. We write for $x \neq a$,

$$f(x) - f(a) = (x - a)\left[\frac{f(x) - f(a)}{x - a}\right].$$

As $x \to a$, the first factor on the right of this equation tends to 0 and the second tends to the number $f'(a)$. Since the limit of the product is the product of the limits, $f(x) - f(a) \to 0$ as $x \to a$. This implies that $f(x) \to f(a)$ as $x \to a$, so that f is continuous at a. **QED**

The converse of this theorem is not valid. A function may or may not have a derivative at a point where it is continuous. This is illustrated by the function $T = T(t)$ of Figure 16, which is continuous at all positive t and has a derivative at all positive $t \neq 30{,}000$ but does not have a derivative at $t = 30{,}000$.

Higher-order derivatives

The SECOND DERIVATIVE of a function $y = f(x)$ is the derivative of its first derivative and is denoted f'' or d^2f/dx^2. The THIRD DERIVATIVE is the derivative of the second derivative and is denoted $f^{(3)}$ or d^3f/dx^3. The nTH DERIVATIVE is obtained by differentiating n times. It is also called the derivative of ORDER n and is denoted $f^{(n)}$ or d^nf/dx^n.

Example 8 What are the first, second, and third derivatives of $f(x) = x^5$?

SOLUTION $f'(x) = \frac{d}{dx}(x^5) = 5x^4, f''(x) = \frac{d}{dx}(5x^4) = 20x^3$, and $f^{(3)}(x) = \frac{d}{dx}(20x^3) = 60x^2$ □

Acceleration

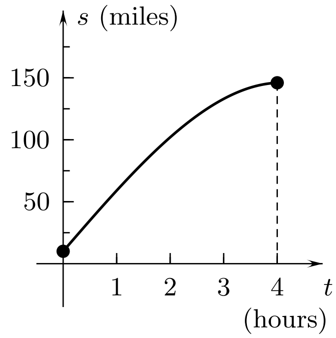
As we have seen before, if an object's position on an s-axis at time t is given by the function $s(t)$, then the derivative $v = s'(t)$ is the object's velocity in the positive s-direction at time t. The derivative of the velocity, which is the second derivative of the position function, is the object's ACCELERATION in the positive s-direction:

$$a(t) = v'(t) = s''(t).$$

Example 9 A car is $s = 10 + 50t - t^3$ miles past a truck stop at the time t (hours) for $0 \le t \le 4$. **(a)** Where is it, **(b)** how fast is it going, and **(c)** how rapidly is it accelerating or decelerating at $t = 2$?

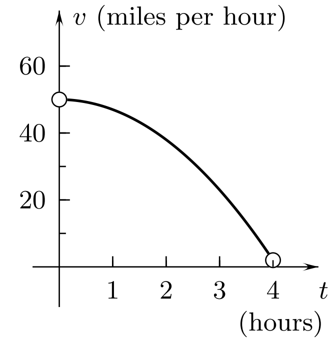
SOLUTION **(a)** The car is $s(2) = 10 + 50(2) - 2^3 = 102$ miles past the truck stop at $t = 2$.
(b) The car's velocity in the positive s-direction is $v(t) = s'(t)$ $= \frac{d}{dt}(10 + 50t - t^3) = 50 - 3t^2$ at time t and $v(2) = 50 - 3(2^2) = 38$ at $t = 2$. The car is going 38 miles per hour away from the truck stop at that time.
(c) The car's acceleration in the positive s-direction is $a(t) = v'(t)$ $= \frac{d}{dt}(50 - 3t^2) = -6t$ at time t and $a(2) = -6(2) = -12$ at $t = 2$. Since the car's velocity is positive at $t = 2$, it is decelerating 12 miles per hour at that time. □

You can see from the graph of $s = 10 + 50t - t^3$ in Figure 18 that the car in Example 9 has positive velocity for $0 < t < 4$ but slows down as t increases from 0 to 4. This can also be seen from the graph of the velocity $v = 50 - 3t^2$ in Figure 19: the velocity is positive for $0 < t < 4$ and decreases as t increases from 0 to 4. Figure 20 shows the graph of the acceleration $a(t) = -6t$. The car is slowing down for $0 < t < 4$ because the velocity is positive and the acceleration is negative during that time.



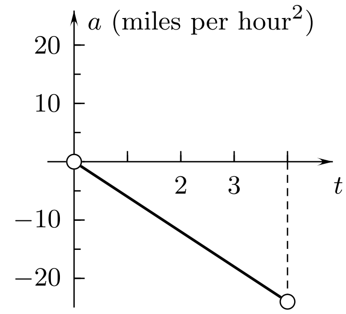
$s = 10 + 50t - t^3$

FIGURE 18



$v = 50 - 3t^2$

FIGURE 19



$a = -6t$

FIGURE 20

Interactive Examples 2.5

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.†

1. Use the graph of the function P in Figure 21 to estimate $P'(3)$.

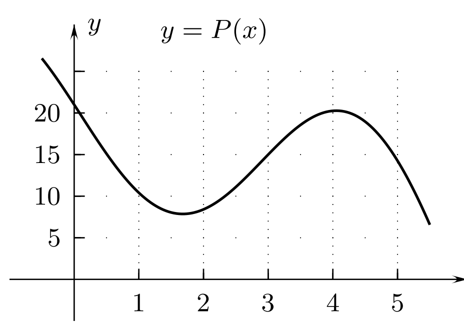


FIGURE 21

2. Use the graph of the function P in Figure 21 to find the approximate values of x with $0 \le x \le 5$ such that $P'(x) = 0$.

3. Use the graph of the function P in Figure 21 to find an approximate value of x such that $P'(x) = 5$.

†In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

4. The graph in Figure 22 gives the price $p = p(t)$ of newsprint in the U.S. as a function of the year.[4] **(a)** During what decade was dp/dt approximately zero? **(b)** Approximately how fast was the price of newsprint increasing when it was 400 dollars per ton?

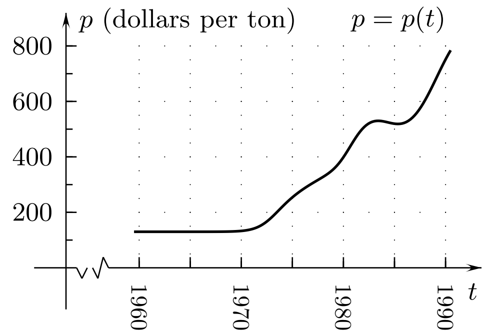


FIGURE 22

5. The table below gives the temperature $T = T(t)^\circ$F of the radiator in an automobile t minutes after its engine is started.[5] **(a)** Approximately how fast is the temperature of the radiator increasing after 15 minutes? **(b)** After 22 minutes?

t (minutes)	0	5	10	15	20	25
$T(t)^\circ$F	87	110	113	116.5	119	121

6. Find the acceleration of an object with position function $s = t^{10} + 9\sqrt{t}$ kilometers at time t (hours).

[4] Data adapted from *Newsprint, Canadian Supply and American Demand* by T. Roach, Durham, NC: Forest History Society, 1994, p. 43.

[5] Data adapted from "A Flow Network Approach to Vehicle Underhood Heat Transfer Problem" by P. Li, G. Chui, J. Glidewell, T. Chue, and M. Lai, *1993 Vehicle Thermal Management Systems Proceedings*, Warrendale, PA: Society of Automotive Engineers, 1993, p. 46.

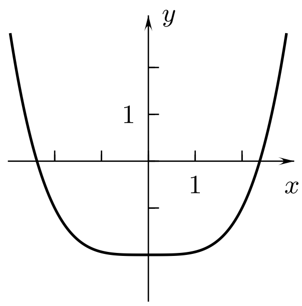
Exercises 2.5

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

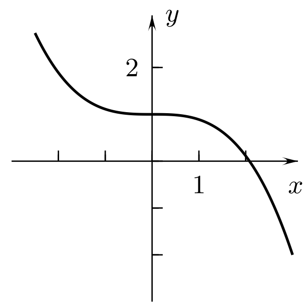
1. How are the estimates (a) $\dfrac{W(12)-W(9)}{12-9}$, (b) $\dfrac{W(9)-W(6)}{9-6}$, and (c) $\dfrac{W(12)-W(6)}{12-6}$ of the rate of change of the wind speed in Example 4 related?

2. Match **(a)** $y = P(x)$, **(b)** $y = Q(x)$, and **(c)** $y = R(x)$ of Figures 23 through 25 to the graphs of their first derivatives in Figures 26 through 28. Explain your choices.



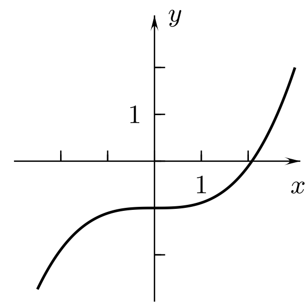
$y = P(x)$

FIGURE 23



$y = Q(x)$

FIGURE 24



$y = R(x)$

FIGURE 25

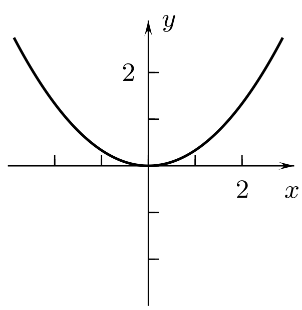


FIGURE 26

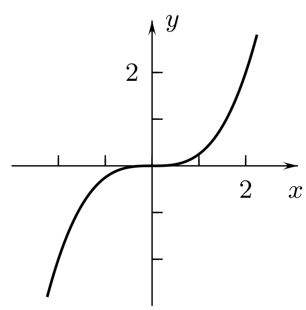


FIGURE 27

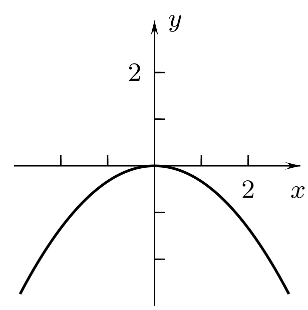


FIGURE 28

3. The graph of M in Figure 29 is symmetric about the line $x = 4$ and $M'(2) = -2$. What is $M'(6)$?

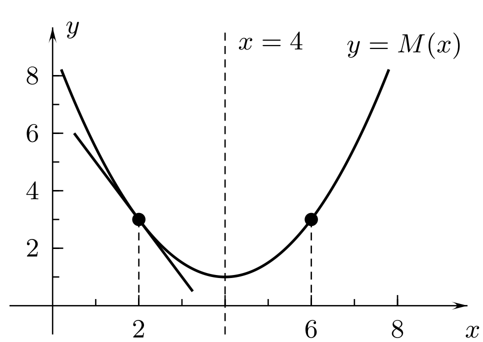


FIGURE 29

4. Figure 30 shows the graph of the function $F(x) = \begin{cases} x^2 & \text{for} \quad x < 1 \\ 2 & \text{for} \quad x \geq 1 \end{cases}$ which is discontinuous at $x = 1$ because it has different limits from the left and from the right at that point. Figure 31 shows the secant line to the graph through the point at $x = 1$ and a point with $x < 1$. Use this drawing to explain, without using Theorem 1, why F does not have a derivative at $x = 1$.

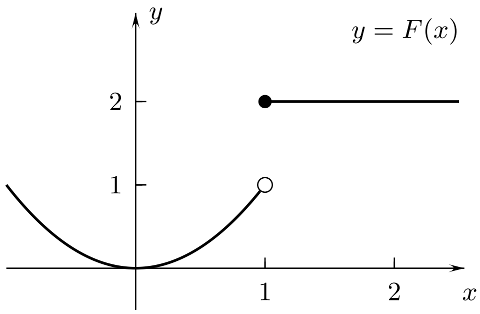


FIGURE 30

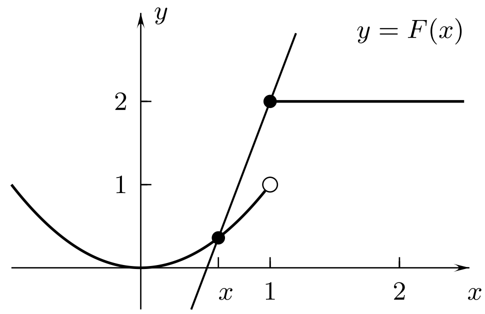


FIGURE 31

BASICS:

5.$^{\mathbf{O}}$ The piecewise linear function H of Figure 32 is defined for $-2 \leq x \leq 2$. Draw the graph of its derivative.

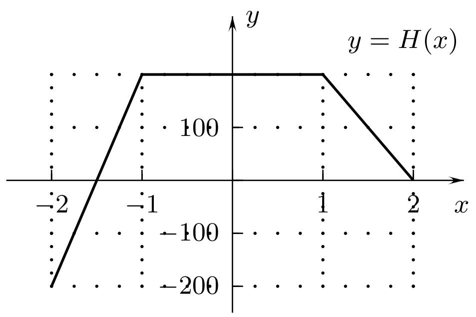


FIGURE 32

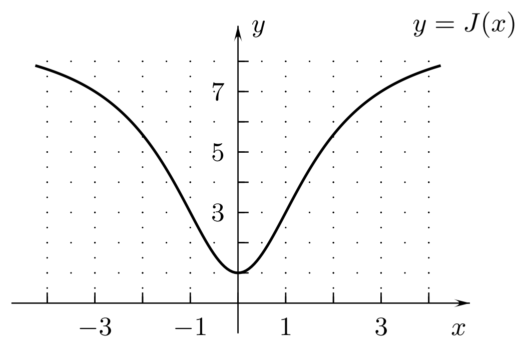


FIGURE 33

6.$^{\mathbf{O}}$ Find the approximate greatest and least values of $J'(x)$ for $y = J(x)$ of Figure 33.

7.$^{\mathbf{O}}$ Use the values of $y = Q(x)$ in the next table to estimate **(a)** $Q'(10)$ and **(b)** $Q'(10.35)$.

x	9.6	9.8	10.0	10.2	10.4
$Q(x)$	410.9	414.1	417.0	420.2	422.7

8.$^{\mathbf{O}}$ Find the second derivatives of **(a)** $w(x) = 5x^3 - 3x^5$ and **(b)** $y(x) = x^{1/4} + x^{1/3}$.

9.$^{\mathbf{O}}$ Sketch the graphs of the function M and of its derivative, where

$$M(x) = \begin{cases} x & \text{for} \quad 0 \leq x < 2 \\ 4 - x & \text{for} \quad 2 \leq x < 3 \\ x - 2 & \text{for} \quad 3 \leq x < 4 \\ 6 - x & \text{for} \quad 4 \leq x \leq 6. \end{cases}$$

10.^O Figure 34 shows the graph of the temperature $T = T(h)$ (degrees Celsius) as a function of the altitude h (feet) above sea level one summer day in Los Angeles.[6] The temperature distribution is called an "inversion" because the air is warmer at 3000 feet than at 1000 feet, while normally the air is cooler at higher altitudes. The inversion is caused by warm air from the deserts that blows over the cooler air from the ocean and traps air pollution by preventing the upward circulation of warm air. What are the approximate temperature and the approximate rate of change of the temperature with respect to altitude at altitudes of **(a)** 500 and **(b)** 7000 feet?

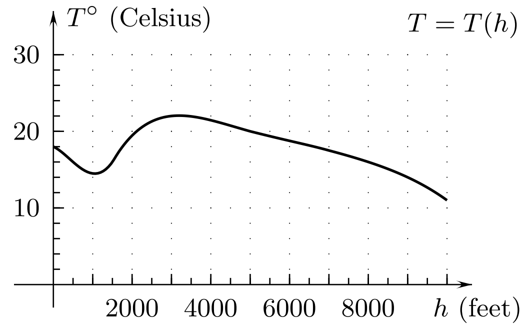


FIGURE 34

11. Figure 35 shows the graph of a bird's height $h = h(t)$ above the ground as a function of time. **(a)** Approximately how fast is the bird's height decreasing at $t = 2$? **(b)** Approximately how fast is the bird's height increasing at $t = 6$?

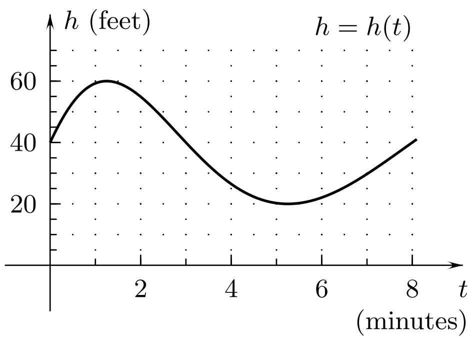


FIGURE 35

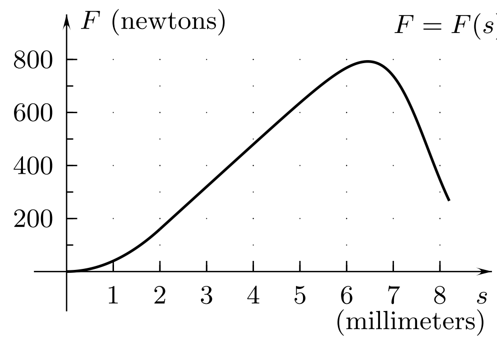


FIGURE 36

12. Figure 36 is the graph of the force $F = F(s)$ exerted by a monkey ligament as a function of how much it is stretched.[7] The ligament can easily be stretched very small amounts. This gives the portion of the graph for small s the "toe" shape shown in the figure. The portion of the curve that looks like a line is the "linear region," and the rest of the curve on the right is the "fatigue region", where the ligament weakens because increasing numbers of microfibers in it are torn. What is the approximate rate of change of the force with respect to the length in the linear region?

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[6] Data adapted from "The role of meteorology in the study and control of air pollution" by M. Neiburger, *Bulletin of the American Meteorological Society*, Dec., 1969, Boston, MA: American Meteorological Society, 1969, p. 960.

[7] Data adapted from *Human Adaptation and Accommodation* by A. Frisancho, Ann Arbor, MI: The University of Michigan Press, 1993, p. 119.

13.[A] Based on the next table of the number of farms $N = N(t)$ (millions) and total farm acreage $A = A(t)$ (million acres) in the United States,[8] what were the approximate rates of change with respect to time of **(a)** the number of farms at the beginning of 1979 and **(b)** the total farm acreage at the beginning of 1987? **(c)** What was the average size of an American farm at the beginning of 1976? **(d)** At the beginning of 1991?

t (year)	1976	1979	1982	1985	1988	1991
$N(t)$ (millions)	2.497	2.437	2.407	2.293	2.197	2.105
$A(t)$ (million acres)	1054	1042	1028	1012	995	983

14. The following table gives the velocity of sound in water $v = v(T)$ (feet per second) as a function of the temperature $T°$F of the water.[9] **(a)** Does the velocity of sound increase or decrease as the temperature increases? **(b)** Based on this data, does the velocity decrease more rapidly when the temperature is 210°F or when it is 410°F?

$T°$F	200	240	280	320	360	400	440
$v(T)$ (feet per second)	5079	5000	4879	4724	4537	4331	4081

15. The table below lists the percentage of roses sold in the U.S. that were imported in the odd-numbered years from 1981 through 1993.[10] Based on this data, when between 1981 and 1993 was the percentage of imported roses increasing the most rapidly, and how fast was it increasing then?

Year	1981	1983	1985	1987	1989	1991	1993
% of roses imported	14	22	26	31	34	48	58

16.[O] What is the fourth derivative of $y = \frac{1}{90}x^{10} - \frac{1}{56}x^8$?

17.[A] Find the third derivative of $y = x^{10} - x^8$.

18. What is $F''(8)$ for $F(x) = 4x^{2/3}$?

19.[A] Find the first and second derivatives of $y = 6ax^3 - 4bx^5$, where a and b are constant parameters.

In Exercises 20 through 22 find formulas for the acceleration of objects with the given position functions.

20.[O] $s = t^4 + 2t - 1$ meters at time t (seconds)

21.[A] $s = \left(\sqrt{t}\right)^5$ kilometers at time t (hours)

22. $s = t^{11/3}$ kilometers at time t (hours)

23. A sailboat that is coasting with its sails down has a velocity of $v = 200t^{-1}$ feet per second for $t \geq 20$. **(a)** Give a formula for its acceleration as a function of t. **(b)** When is it decelerating 0.08 feet per second^2?

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(8) Data from *1993 Statistical Abstract of the United States*, Washington, D.C: U.S. Bureau of the Census, 1993, p. 655.
(9) Data from *CRC Handbook of Chemistry and Physics*, Boca Raton, FL: CRC Press, 1981, p. E-46.
(10) Data adapted from *Los Angeles Times*, January 28, 1995, p. D1. Source: Department of Agriculture.

EXPLORATION:

24. The functions G and H of Figures 37 and 38 have the same derivative K. Give formulas for K.

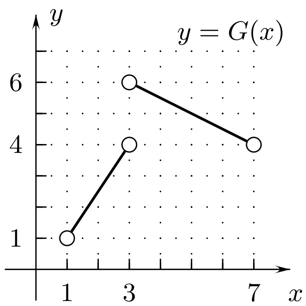


FIGURE 37

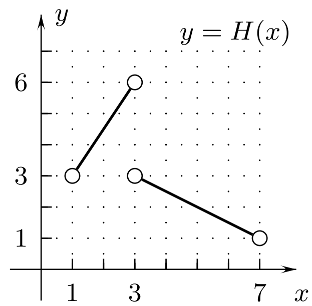


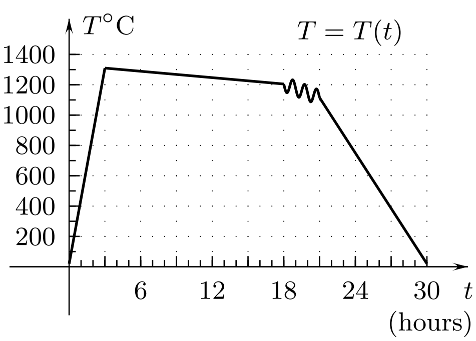
FIGURE 38

25.[A] A car is $s = 60t + \frac{1}{10}t^5$ (kilometers) past a service station at time t (hours). Where is it and how fast is it traveling when it is accelerating 16 kilometers per hour per hour?

26. A watermelon is dropped from a dormitory window 20 meters above the ground. If there were no air resistance it would fall $s = 4.9t^2$ meters in t seconds. What are its downward velocity and downward acceleration when it is 10.2 meters from the ground?

27.[O] Figure 39 gives a mathematical model of the temperature $T = T(t)$ as a function of time in the manufacture of glass.[(11)] Sand and other materials are heated to the melting point and formed into the desired shapes. The glass is cooled slightly over a long period of time, to allow bubbles of air to escape. Then the glass is repeatedly heated and cooled in a process called annealing, which removes internal stress. Finally, the glass is cooled. Give approximate formulas for dT/dt, disregarding the annealing period.

FIGURE 39



[(11)]Data adapted from "Ancient Glazes" by P. Vandiver, *Scientific American*, April, 1990, New York, NY: Scientific American, Inc. 1990, p. 108.

28.$^{\mathsf{A}}$ The percent $P = P(T)$ of water in ice cream that is frozen is given in Figure 40 as a function of the temperature at which the ice cream is stored.(12) **(a)** Why does $P(T)$ decrease as T increases for $-20 < T < 32$? **(b)** Why is $P(T)$ zero for $T \geq 32$? **(c)** Find the approximate value of $P'(10)$ and use it determine approximately how much more water is frozen at 9.99°F than at 10°F.

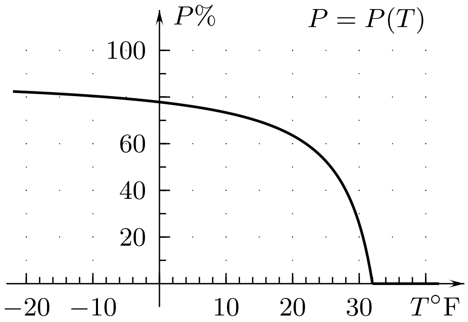


FIGURE 40

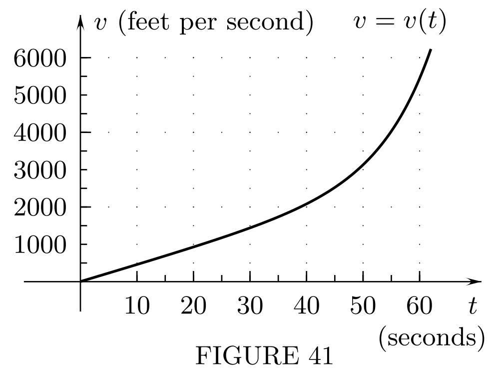


FIGURE 41

29. The function $v = v(t)$ of Figure 41 is the upward velocity of a World War II V-2 rocket t seconds after it is fired.(13) The rocket's engine exerts a constant force but the rocket becomes lighter as it uses fuel and the air resistance is less at higher altitudes. Explain how this affects the shape of the graph.

30.$^{\mathsf{A}}$ A bucket sitting outside contains water one inch deep when it begins to rain. After half an hour of constant rainfall, the water is 1.2 inches deep in the bucket. It then rains at a greater, constant rate for one hour and then stops, adding 0.8 inches of water in the bucket. Draw the graphs of the depth $y = h(t)$ of the water and of its derivative $r = h'(t)$ as functions of time in the two hours after the rain starts.

31. A trust fund's investments are worth 10 thousand dollars on January 1 and increase at the constant rate of \$100 per month for three months. The value decreases at the constant rate of \$50 per month for the next six months, and then increases at the constant rate of \$75 per month for three months. Draw the graphs of the value $B = B(t)$ of the investments t months after January 1 and of its derivative $r = B'(t)$ for $0 \leq t \leq 12$.

32.$^{\mathsf{A}}$ The percent of water measured by weight, in a piece of wood is given as a function of the relative humidity h of the environment is $W = W(h)$, where $W'(70) = 0.55$.(14) **(a)** Why is it plausible that $W'(70)$ is positive? **(b)** What is the approximate percent of water in the wood at $h = 69$ if $W(70) = 11\%$?

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(12) Data adapted from *McGraw-Hill Encyclopedia of Science and Technology*, Vol. vii, New York: NY: McGraw-Hill, Inc., 1992, p. 251.

(13) Data adapted from *Introduction to Rocket Technology* by V. Feodosiev and G. Siaiarev, New York, NY: Academic Press, 1959, p. 253.

(14) Data adapted from *Handbook of Engineering Materials*, D. Miner and J. Seastone, New York, NY: John Wiley & Sons, 1955, pp. 3-22.

33.[A] Figure 42 shows the upward force (lift) $L = L(x)$ (pounds) of an airplane wing as a function of the angle x (degrees) it makes with the horizontal.[15] **(a)** What is the approximate rate of change of the lift with respect to the angle at $x = 0$? **(b)** At $x = 13$? **(c)** What is the approximate maximum lift of the wing and at approximately what angle does it occur? **(d)** What is the rate of change of the lift with respect to the angle when the lift is a maximum?

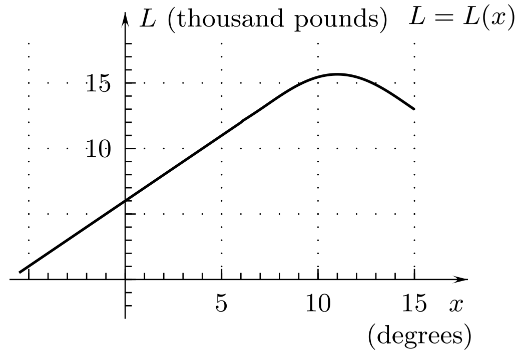


FIGURE 42

34. The percent $P = r(t)$ of the surface of a steel sheet that has rusted t years after it was placed outside is given by the graph in Figure 43.[16] Let $P = s(t)$ denote the percent that has not rusted. **(a)** How are $P = s(t)$ and $P = r(t)$ related? **(b)** Draw the graph of $P = s(t)$. **(c)** How are $y = s'(t)$ and $y = r'(t)$ related? Match them to their graphs in Figures 44 and 45.

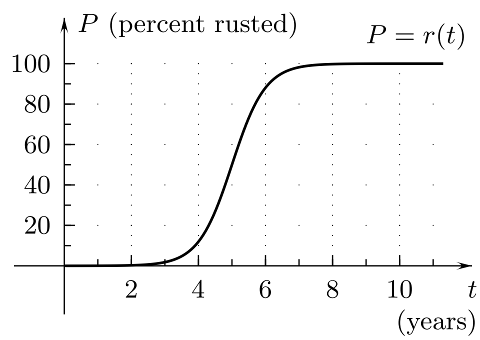
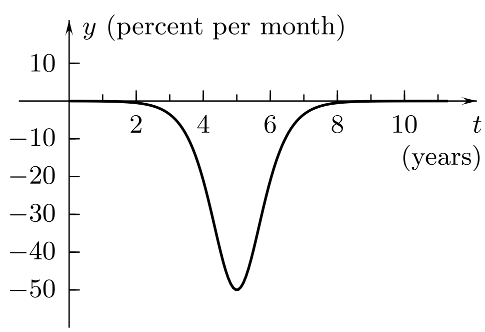


FIGURE 43



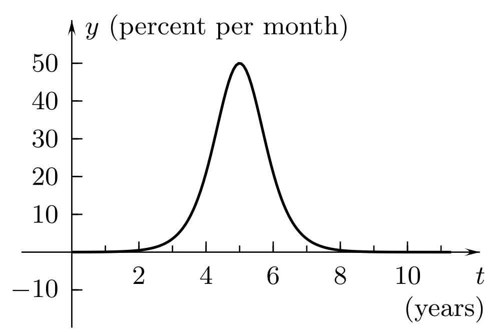


FIGURE 44

FIGURE 45

[15] Data adapted from *Boundary Layer Theore* by H. Schlichting, New York, NY: McGraw-Hill, Inc, 1968, p. 21.

[16] Data adapted from *Handbook of Engineering Materials* by D. Miner and J. Seastone, New York, NY: John Wiley & Sons, 1955, pp. 2-48.

35. Large amounts of organic pollutants in a river can consume so much dissolved oxygen that the water does not contain enough oxygen to support life. This occurred during the 1960's in parts of the Rhine River in Germany. Figure 46 gives the graph of the average concentration $C = C(t)$ of dissolved oxygen in the Rhine from 1955 to 1990.[17] **(a)** What was the approximate relative change in the concentration of dissolved oxygen from 1955 to its minimum value? **(b)** Approximately when was the oxygen level in the Rhine decreasing 0.3 milligrams per liter per year? **(c)** When was it increasing 0.2 milligrams per liter per year?

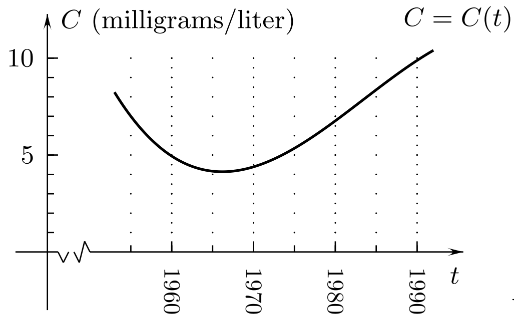


FIGURE 46

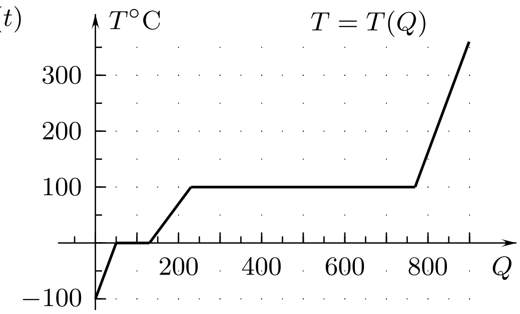


FIGURE 47

36.$^{\mathsf{A}}$ A gram of H_2O is initially ice at $-100°$C and it is heated to $200°$C. Figure 44 shows the graph of its temperature $T = T(Q)$ as a function of the number Q of calories of heat that have been added to it. The ice's temperature increases $2°$C for every calorie of heat until the ice begins to melt at $0°$C. It takes 80 calories to melt the ice into water at $0°$C. The temperature of the gram of water increases 1 degree for every calorie of heat until the water starts to boil at $100°$C. 539 calories are required to boil the water into steam at $100°$C, and then the steam's temperature increases 2 degrees for every calorie of heat.[18] **(a)** The three values of dT/dQ are the SPECIFIC HEAT of ice, water, and steam. What are they? **(b)** Draw the graph of $r = T'(Q)$.

37. According to the February 14, 1994 issue of *Newsweek* magazine, "The rate of increase of health costs nudged downward in the last year, a welcome sign. But costs are still increasing." Rephrase these statements in terms of derivatives.

In Exercises 38 through 42 use definition (4) of "marginal $f(x)$" as the derivative $f'(x)$.

38.$^{\mathsf{A}}$ It costs a company $C(x) = 2500 + 10x - 30x^{2/3}$ dollars to manufacture x shirts $(100 \le x \le 2000)$. What is the marginal cost at $x = 1000$?

39.$^{\mathsf{A}}$ A company can sell $x(p) = 21000p^{-1/2} - 3000$ jackets if it charges p dollars per jacket $(8 \le p \le 49)$. **(a)** How many will it sell and what revenue will it receive from the jackets if it charges 16 dollars per jacket? **(b)** What revenue will it receive from the jackets if it charges p dollars per jacket? **(c)** Give a formula for the marginal revenue with respect to price as a function of the price.

40. It costs a company $C(x) = 600 + 325x + x^2$ dollars to manufacture x radios in a week and it can sell them for \$500 each. **(a)** What are the revenue and profit of x radios? **(b)** What are the marginal cost, marginal revenue, and marginal profit with respect to x?

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[17] Data adapted from "Cleaning up the river Rhine" by K-G. Malle, *Scientific American*, January, 1996, Washington, DC: Scientific American, Inc., 1996, p. 70.

[18] Data adapted from *CRC Handbok of Chemistry and Physics*, 62nd Edition, R. Weast, editor, Boca Raton FL: CRC Press, 1981, pp. D149–D150.

41. It costs \$25 per bookcase plus \$300 overhead to build bookcases. **(a)** What is the average total cost per bookcase if x bookcases are made? **(b)** What is the marginal total cost with respect to x if x bookcases are made?

42.[A] A company can sell 1000 units of a product if it charges \$30 per unit. For every dollar it raises the price, up to a price of \$50 per unit, it will sell 20 fewer units. **(a)** Give the demand and revenue as functions of price. **(b)** What is the marginal revenue with respect to price at $p = 30, p = 40$, and $p = 48$?

(End of Section 2.5)