Section 6.8

(3/20/08)

Integration by substitution

OVERVIEW: *With the Fundamental Theorem of Calculus every differentiation formula translates into integration formula. In this section we discuss the technique of* INTEGRATION BY SUBSTITUTION *which comes from the Chain Rule for derivatives. Then we use it with integration formulas from earlier sections. We end the section with a discussion of some of the highlights in the history of the integral.*

Topics:

- ***Integration by substitution***
- ***Substitution in definite integrals***
- ***Special linear substitutions***
- ***Integrals of tangents, cotangents, secants, and cosecants***
- ***History of the integral***

Integration by substitution

We begin with the following result.

Theorem 1 (Integration by substitution in indefinite integrals) *If $y = g(u)$ is continuous on an open interval and $u = u(x)$ is a differentiable function whose values are in the interval, then*

$$\int g(u)\,\frac{du}{dx}\,dx = \int g(u)\,du. \tag{1}$$

Equation **(1)** states that an x-antiderivative of $g(u)\,\dfrac{du}{dx}$ is a u-antiderivative of $g(u)$. Formula **(1)** is called INTEGRATION BY SUBSTITUTION because the variable x in the integral on the left of **(1)** is replaced by the substitute variable u in the integral on the right.

Proof of Theorem 1: Suppose that $y = G(u)$ is a u-antiderivative of $y = g(u)^{\dagger}$, so that $G'(u) = g(u)$ and

$$\int g(u)\,du = G(u) + C. \tag{2}$$

The Chain Rule gives, with $u = u(x)$,

$$\frac{d}{dx}[G(u)] = G'(u)\,\frac{du}{dx} = g(u)\,\frac{du}{dx}.$$

The last equation shows that $y = G(u)$ is an x-antiderivative of $g(u)\,\dfrac{du}{dx}$, so that

$$\int g(u)\,\frac{du}{dx}\,dx = G(u) + C. \tag{3}$$

Equations **(2)** and **(3)** give **(1)**. There is no constant C of integration in equation **(1)** because there are indefinite integrals on both sides. **QED**

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†Part II of the Fundamental Theorem in Section 6.4 implies that every continuous function has antiderivatives.

We use Leibniz notation for the derivative du/dx in the integrals on the left of **(1)** so that we can use the formula

$$du = \frac{du}{dx}\, dx$$

in our calculations. Here du/dx denotes the derivative of the function $u = u(x)$; dx is the symbol that is used in integrals with x as variable; and du is the symbol in integrals with u as variable.

When we apply substitution to the integration formulas from earlier sections, we obtain the following list.

$$\begin{aligned}
\int u^n \frac{du}{dx}\, dx &= \int u^n\, du = \frac{1}{n+1} u^{n+1} + C \text{ for } n \neq -1 \\
\int \frac{1}{u} \frac{du}{dx}\, dx &= \int \frac{1}{u}\, du = \ln|u| + C \\
\int e^u \frac{du}{dx}\, dx &= \int e^u\, du = e^u + C \\
\int b^u \frac{du}{dx}\, dx &= \int b^u\, du = \frac{1}{\ln(b)} b^u + C \text{ for positive } b \neq 1 \\
\int \cos u \frac{du}{dx}\, dx &= \int \cos u\, du = \sin u + C \\
\int \sin u \frac{du}{dx}\, dx &= \int \sin u\, du = -\cos u + C \\
\int \sec^2 u \frac{du}{dx}\, dx &= \int \sec^2 u\, du = \tan u + C \\
\int \csc^2 u \frac{du}{dx}\, dx &= \int \csc^2 u\, du = -\cot u + C \\
\int \sec u \tan u \frac{du}{dx}\, dx &= \int \sec u \tan u\, du = \sec u + C \\
\int \csc u \cot u \frac{du}{dx}\, dx &= \int \csc u \cot u\, du = -\csc u + C \\
\int \frac{1}{\sqrt{a^2 - u^2}} \frac{du}{dx}\, dx &= \int \frac{1}{\sqrt{a^2 - u^2}}\, du = \sin^{-1}\left(\frac{u}{a}\right) + C \text{ for } a > 0 \\
\int \frac{1}{a^2 + u^2} \frac{du}{dx}\, dx &= \int \frac{1}{a^2 + u^2}\, du = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C \text{ for } a > 0 \\
\int \cosh u \frac{du}{dx}\, dx &= \int \cosh u\, du = \sinh u + C \\
\int \sinh u \frac{du}{dx}\, dx &= \int \sinh u\, du = \cosh u + C
\end{aligned}$$

As will be shown in examples, we choose substitutions that put integrals we want to find in the forms of the first integrals in the above equations. The second integrals in these equations are then used in the actual calculations.

In many cases the appropriate substitution for a particular integral is one that simplifies the most complicated part of the integrand.

Example 1 Find the antiderivative $\int (x^2+1)^5(2x)\,dx$.

SOLUTION Because the most complicated part of the integrand in this example is $(x^2+1)^5$, we try the substitution $u = x^2+1$ which would convert $(x^2+1)^5$ into u^5. Then we calculate

$$du = \frac{du}{dx}\,dx = \frac{d}{dx}(x^2+1)\,dx = 2x\,dx.$$

With this u and du, the given integral takes the form,

$$\int (x^2+1)^5(2x)\,dx = \int u^5\,du.$$

The u-integration can be carried out using the formula,

$$\int u^5\,du = \tfrac{1}{6}u^6 + C.$$

Then we substitute the formula $u = x^2+1$ for the final answer:

$$\int (x^2+1)^5(2x)\,dx = \int u^5\,du = \tfrac{1}{6}u^6 + C = \tfrac{1}{6}(x^2+1)^6 + C.\ \square$$

As with any indefinite integral, we can check Example 1 by differentiating the result. This requires the Chain Rule because the technique of substitution is derived from the Chain Rule. We obtain

$$\frac{d}{dx}\left(\tfrac{1}{6}(x^2+1)^6\right) = \tfrac{1}{6}[6(x^2+1)^5]\frac{d}{dx}(x^2+1) = (x^2+2)^5(2x).$$

The formula for the indefinite integral in Example 1 is correct because its derivative is the original integrand.

Usually when we carry out an integration by substitution, we have to adjust a constant in the integrand to construct du. This procedure is illustrated in the next example.

Example 2 Perform the integration $\int x^3\sqrt{x^4+16}\,dx$.

SOLUTION Because the most complicated part of the integrand is the square root of x^4+16 in the integral, we use the substitution $u = x^4+16$, for which $\dfrac{du}{dx} = \dfrac{d}{dx}(x^4+16) = 4x^3$ and

$$du = \frac{du}{dx}dx = 4x^3\,dx.$$

To make this substitution, we construct du from the dx and other elements of the integral. First, we move the x^3 next to the dx to have

$$\int x^3\sqrt{x^4+16}\,dx = \int \sqrt{x^4+16}\,(x^3\,dx).$$

The integral on the right of the last equation would equal $\int \sqrt{u}\,du$ if it contained $4x^3\,dx$ instead of $x^3\,dx$. Since the missing factor 4 is a constant, we can insert it if we compensate by dividing the entire integral by 4. We write

$$\begin{aligned}\int x^3\sqrt{x^4+16}\,dx &= \int \sqrt{x^4+16}\,(x^3\,dx)\\ &= \tfrac{1}{4}\int \sqrt{x^4+16}\,(4x^3\,dx)\\ &= \tfrac{1}{4}\int \sqrt{u}\,du = \tfrac{1}{4}\int u^{1/2}\,du\end{aligned} \tag{4}$$

To finish the example, we carry out the u-integration and put the result in terms of the original variable x by setting $u = x^4 + 16$. We obtain

$$\begin{aligned}\int x^3\sqrt{x^4+16}\,dx &= \tfrac{1}{4}\int u^{1/2}\,du = \tfrac{1}{4}\left[\frac{1}{3/2}u^{3/2}\right] + C\\ &= \tfrac{1}{4}\left(\tfrac{2}{3}\right)u^{3/2} + C = \tfrac{1}{6}u^{3/2} + C = \tfrac{1}{6}(x^4+16)^{3/2} + C. \;\square\end{aligned}$$

Notice the transition from the first integral to the third integral in **(4)**. Most calculations in mathematics involve simplifying expressions. Here, instead, we put the integral $\int x^3\sqrt{x^4+16}\;dx$ in the more complicated form $\frac{1}{4}\int \sqrt{x^4+16}\;(4x^3\;dx)$ in order to construct $du = 4x^3\;dx$ from the terms $x^3\;dx$ in the original integral.

Example 3 Find the antiderivatives $\displaystyle\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\;dx$.

SOLUTION The key here is to recognize that the derivative of $u = \sqrt{x} = x^{1/2}$ is $\frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$, so that $du = \dfrac{1}{2\sqrt{x}}\,dx$. To make this substitution, we divide and multiply by 2 and obtain

$$\begin{aligned}\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\;dx &= 2\int \cos(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\;dx\right) = 2\int \cos u\;du\\ &= 2\sin u + C = 2\sin\left(\sqrt{x}\right) + C. \;\square\end{aligned}$$

Substitution in definite integrals

Often the easiest way to evaluate a definite integral by substitution is to make the substitution in the corresponding indefinite integral.

Example 4 Figure 1 shows the region between $y = \dfrac{10x}{(x^2+1)^2}$ and the x-axis for $0 \le x \le 3$. Find its area.

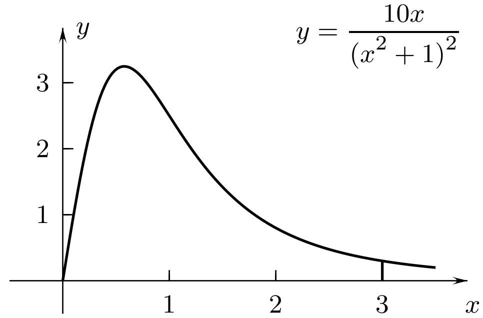


FIGURE 1

SOLUTION Because $\dfrac{10x}{(x^2+1)^2}$ is nonnegative for $0 \le x \le 3$, the area is given by the integral,

$$[\text{Area}] = \int_0^3 \frac{10x}{(x^2+1)^2}\,dx.$$

We evaluate this definite integral by making a change of variables in the corresponding indefinite integral. We use the substitution $u = x^2+1$, for which $du = \dfrac{d}{dx}(x^2+1)\,dx = 2x\,dx$. To construct du from $x\,dx$, we write $10 = 5(2)$ and put the 2 with the x and dx to obtain

$$\begin{aligned}\int \frac{10x}{(x^2+1)^2}\,dx &= 5\int \frac{1}{(x^2+1)^2}\,(2x\,dx) = 5\int \frac{1}{u^2}\,du\\ &= 5\int u^{-2}\,du = -5u^{-1} + C\\ &= -5(x^2+1)^{-1} + C = -\frac{5}{x^2+1} + C.\end{aligned}$$

Then we have

$$\begin{aligned}[\text{Area}] &= \int_0^3 \frac{10x}{(x^2+1)^2}\,dx = \left[\int \frac{10x}{(x^2+2)^2}\,dx\right]_0^3\\ &= \left[-\frac{5}{x^2+1}\right]_0^3 = \left[-\frac{5}{3^2+1}\right] - \left[-\frac{5}{0^2+1}\right] = -\tfrac{5}{10} + 5 = \tfrac{9}{2}. \ \square\end{aligned}$$

Example 5 Find the value of $\displaystyle\int_0^1 \sin(\pi x)\,dx$.

SOLUTION We start with the indefinite integral. We use the change of variables $u = \pi x$, for which

$$du = \frac{d}{dx}(\pi x)\,dx = \pi\,dx.$$

To construct du from the dx in the integrand we multiply and divide by π:

$$\int \sin(\pi x)\,dx = \frac{1}{\pi}\int \sin(\pi x)\,(\pi\,dx) = \frac{1}{\pi}\int \sin u\,du.$$

We perform the u-integration and substitute the formula for u in terms of x:

$$\int \sin(\pi x)\,dx = \frac{1}{\pi}\int \sin u\,du - \frac{1}{\pi}\cos u + C = -\frac{1}{\pi}\cos(\pi x) + C.$$

Then we evaluate the definite integral:

$$\begin{aligned}\int_0^1 \sin(\pi x)\,dx &= \left[\int \sin(\pi x)\,dx\right]_0^1 - \left[-\frac{1}{\pi}\cos(\pi x)\right]_0^1\\ &= [-\frac{1}{\pi}\cos(\pi)] - [-\frac{1}{\pi}\cos(0)] = -\frac{1}{\pi}(-1) + \frac{1}{\pi}(1) = \frac{2}{\pi}.\end{aligned}$$

We used the values $\cos(\pi) = -1$ and $\cos(0) = 1$ to simplify the result. $\square$

We can also make substitutions directly in definite integrals by switching the limits of integration to values of the new variable. We use the following result.

Theorem 2 (Integration by substitution in definite integrals) *If $y = g(u)$ is continuous on an open interval and $u = u(x)$ is a differentiable function whose values are in the interval, then for a and b in the domain of u,*

$$\int_{x=a}^{x=b} g(u)\,\frac{du}{dx}\,dx = \int_{u=u(a)}^{u=u(b)} g(u)\,du. \tag{5}$$

Proof: Suppose that $G(u)$ is an antiderivative of $g(u)$. Then

$$\int_{x=a}^{x=b} g(u)\,\frac{du}{dx}\,dx = \int_{x=a}^{x=b} \frac{d}{dx}[G\left(u(x)\right)]\,dx = G\left(u(b)\right) - G\left(u(a)\right) = \int_{u=u(a)}^{u=u(b)} g(u)\,du. \text{ qed}$$

Example 6 Evaluate the integral $\displaystyle\int_0^1 e^{-2x}\,dx$ by making a change of variables in the definite integral.

SOLUTION We use the change of variables $u = -2x$, for which $\dfrac{du}{dx} = \dfrac{d}{dx}(-2x) = -2$ and consequently $du = -2\,dx$. We also note that $u(0) = -2(0) = 0$ and $u(1) = -2(1) = -2$, so that u goes from 0 to -2 as x goes from 0 to 1. We display this information by writing

$$0 \underset{u}{\longrightarrow} -2 \quad \text{as} \quad 0 \underset{x}{\longrightarrow} 1. \tag{6}$$

We construct $du = -2\,dx$ from dx by multiplying and dividing by -2, change the limits of integration as indicated in **(6)**, and then carry out the u-integration:

$$\begin{aligned}\int_0^1 e^{-2x}\,dx &= -\tfrac{1}{2}\int_{x=0}^{x=1} e^{-2x}\,(-2\,dx) = -\tfrac{1}{2}\int_{u=0}^{u=-2} e^{u}\,du \\ &= -\tfrac{1}{2}\Big[e^{u}\Big]_{u=0}^{u=-2} = -\tfrac{1}{2}(e^{-2} - e^{0}) = \tfrac{1}{2}(1 - e^{-2})\end{aligned} \tag{7}$$

We used equations, $x = 0, x = 1, u = 0$, and $u = -2$ as limits of integration in **(7)** to help keep track of the values. □

Special linear substitutions

An integral of the product $y = x^n(ax+b)^r$ of a small, positive integer power of x and any power of a linear function $ax + b$ can be found by using the following procedure, where we substitute formulas for x and dx in terms of u and du instead of constructing du from dx as in the examples above.

Rule 1 (Linear substitutions) *To perform the integration $\displaystyle\int x^n(ax+b)^r\,dx$, where n is a small positive integer and $r, a \neq 0$, and $b \neq 0$ are any constants, set $u = ax + b$. Solve this equation for x and use the result to express dx in terms of du. Then replace $ax + b$ in the integral by u and replace x and dx by their formulas in terms of u and du.*

This rule is often applied when r is either a positive integer greater than the integer n, or r is a fraction, a negative number, or an irrational number. The effect of the calculations described in the rule is to express the integrand as a linear combination of powers of u.

Example 7 Perform the integration $\int x(2x+1)^6\,dx$.

SOLUTION Rule 1 applies with $u = 2x+1$, $n = 1$, and $r = 6$. We solve the equation $u = 2x+1$ for x by writing $2x = u-1$ and then $x = \frac{1}{2}(u-1)$. This gives $dx = \frac{1}{2}\,du$, and making these substitutions in the integral yields

$$\begin{aligned}\int x(2x+1)^6\,dx &= \int \tfrac{1}{2}(u-1)(u^6)\left(\tfrac{1}{2}\,du\right)\\ &= \tfrac{1}{4}\int (u^7-u^6)\,du\\ &= \tfrac{1}{4}\left(\tfrac{1}{8}u^8\right)-\tfrac{1}{4}\left(\tfrac{1}{7}u^7\right)+C\\ &= \tfrac{1}{32}(2x+1)^8-\tfrac{1}{28}(2x+1)^7+C.\ \Box\end{aligned}$$

Notice how the procedure in this example differs from that used in earlier examples. In the earlier examples we constructed du's from the dx's and other elements of the integrals. In applications of Rule 2, as in Example 7, we solve for x in terms of u. We use this formula to find du and then substitute the formulas for u and du in the integrals.

Example 8 Evaluate $\int_0^1 x\sqrt{1-x}\,dx$.

SOLUTION We will make the substitution in the definite integral. We set $u = 1-x$ and then solve for $x = 1-u$. This gives $dx = -du$. We make these substitutions and change the limits of integration to values of u. Since $1 \underset{u}{\longrightarrow} 0$ as $0 \underset{x}{\longrightarrow} 1$, we obtain

$$\begin{aligned}\int_{x=0}^{x=1} x\sqrt{1-x}\,dx &= \int_{u=1}^{u=0}(1-u)\sqrt{u}\,(-du) = -\int_{u=1}^{u=0}(u^{1/2}-u^{3/2})\,du\\ &= -\left[\tfrac{2}{3}u^{3/2}-\tfrac{2}{5}u^{5/2}\right]_{u=1}^{u=0}\\ &= -[\tfrac{2}{3}(0^{3/2})-\tfrac{2}{5}(0^{5/2})]+[\tfrac{2}{3}(1^{3/2})-\tfrac{2}{5}(1^{5/2})] = \tfrac{4}{15}\ \Box\end{aligned}$$

Integrals of tangents, cotangents, secants, and cosecants

Integrals of $y = \tan x, y = \cot x, y = \sec x$, and $y = \csc x$ can be found with the substitutions described in the following rule.

Rule 2 (Integrals of $\mathbf{y} = \tan \mathbf{x}, \mathbf{y} = \cot \mathbf{x}, \mathbf{y} = \sec \mathbf{x}$, and $\mathbf{y} = \csc \mathbf{x}$)

(a) *To perform the integration* $\int \tan x\,dx$, *write* $\tan x = \dfrac{\sin x}{\cos x}$ *and use the substitution* $u = \cos x, du = -\sin x\,dx$.

(b) *To perform the integration* $\int \cot x\,dx$, *write* $\cot x = \dfrac{\cos x}{\sin x}$ *and use the substitution* $u = \sin x, du = \cos x\,dx$.

(c) *To perform the integration* $\int \sec x\,dx$, *use the substitution* $u = \sec x + \tan x$, $du = (\sec x \tan x + \sec^2 x)\,dx$ *after multiplying and dividing* $\sec x$ *by* u.

(d) *To perform the integration* $\int \csc x\,dx$, *use the substitution* $u = \csc x + \cot x$, $du = (-\csc x \cot x - \csc^2 x)\,dx$ *after multiplying and dividing* $\csc x$ *by* u.

Example 9 Evaluate $\int_0^{\pi/3} \tan x \, dx$.

SOLUTION We use $u = \cos x, du = -\sin x \, dx$ in the indefinite integral after writing $\tan x$ as $\frac{\sin x}{\cos x}$. We obtain

$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx = -\int \frac{1}{\cos x} \, (-\sin x \, dx) \\ &= -\int \frac{1}{u} \, du = -\ln|u| + C = -\ln|\cos x| + C\end{aligned}$$

Then we evaluate the definite integral:

$$\begin{aligned}\int_0^{\pi/3} \tan x \, dx &= \left[\int \tan x \, dx\right]_0^{\pi/3} = \Big[-\ln|\cos x|\Big]_0^{\pi/3} \\ &= [-\ln|\cos(\tfrac{1}{3}\pi)|] - [-\ln|\cos(0)|] \\ &= [-\ln(\tfrac{1}{2})] - [-\ln(1)] = -\ln(\tfrac{1}{2}) + 0 = \ln(2).\end{aligned}$$

We used here the value $\cos(\frac{1}{3}\pi) = \frac{1}{2}$ and the fact that $\ln(\frac{1}{2}) = -\ln(2)$. □

Example 10 Perform the integration $\int \sec x \, dx$.

SOLUTION We multiply and divide by $u = \sec x + \tan x$ and then use this substitution, for which $du = (\sec x \tan x + \sec^2 x) \, dx$. We obtain

$$\begin{aligned}\int \sec x \, dx &= \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx = \int \frac{1}{u} \, du \\ &= \ln|u| + C = \ln|\sec x + \tan x| + C. \ \square\end{aligned}$$

Example 11 Imagine your investments at the beginning of 2000 were worth \$50,000 and that their value was increasing at the rate of $R = 800e^{t/5}$ dollars per year t years after the beginning of 2000. What would have been the value of your investments at the beginnings of 2001 and 2002? Give the answers rounded to the nearest cent.

SOLUTION Let $I(t)$ be the value of the investments t years after the beginning of 2000. Then $I(0) =$ 50,000 and $I'(t) = 800e^{t/5}$ for $t \geq 0$. With the substitution $u = \frac{1}{5}t, du = \frac{1}{5} \, dt$, we have

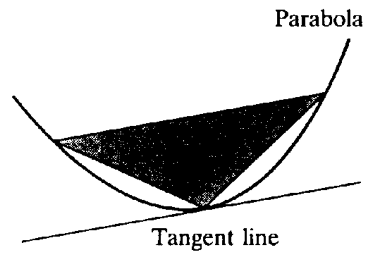
$$\begin{aligned}I(t) &= \int 800e^{t/5} \, dt = 5\int 800e^{t/5} \, (\tfrac{1}{5} \, dt) \\ &= 4000\int e^u \, du = 4000e^u + C = 4000e^{t/5} + C.\end{aligned}$$

Setting $t = 0$ and $I(0) =$ 50,000 gives 50,000 $= 4000e^0 + C$ and then $C =$ 50,000 - 4,000 = 46,000. Therefore, $I(t) = 4000e^{t/5} +$ 46,000. Your investments would have been worth $I(1) = 4000e^{1/5} +$ 46,000 $\doteq$ \$50,885.61 at the beginning of 2001 and $I(2) = 4000e^{2/5} +$ 46,000 $\doteq$ \$51,967.30 at the beginning of 2002. □

History of the integral

Archimedes (ca. 287–212 B.C.) was the first known mathematician to determine the exact area of a region with a curved boundary by an approximation procedure. He demonstrated, in particular, that the area of a region bounded by a parabola and by a straight line as in Figure 2 has area equal to four thirds of the area of the inscribed triangle whose vertex is at the point where the tangent line is parallel to the top of the region. (The Greeks did not think of areas as numbers; they only compared areas of related regions.) We describe two ways that Archimedes obtained this result in the exercise sets of Sections 11.3 and 17.2.

FIGURE 2



The problem of determining the areas of regions bounded by the curves $y = x^n$ for general rational constants n did not arise until the study of analytic geometry in the seventeenth century led to the consideration of these curves. Results amounting to the integration formula,

$$\int_a^b x^n\,dx = \frac{1}{n+1}\,b^{n+1} - \frac{1}{n+1}\,a^{n+1} \quad (0 < a < b, n \neq -1) \tag{8}$$

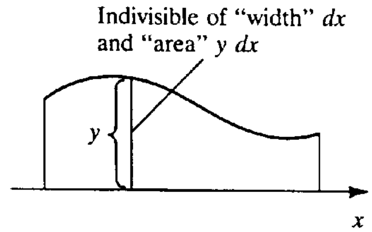
for particular positive integers n appeared in various forms during the years 1635–1660 in the work of two Italian students of Galileo — Bonaventura Cavalieri and Evangelista Torricelli — a Frenchman, Gilles Personne de Roberval, and the philosopher Blaise Pascal. The French lawyer Pierre Fermat (1601–1665) is usually given credit for first establishing integration formula **(8)** for arbitrary rational numbers $n \neq -1$. One of his procedures for doing this is described in the exercise set for Section 11.3. Pascal and others also derived results which, with modern definitions and notation, would read

$$\int_a^b \cos x\,dx = \sin(b) - \sin(a)$$
$$\int_a^b \sin x\,dx = -\cos(b) + \cos(a).$$

Many early derivations of integration formulas employed the concept of INDIVISIBLES. In the context of area, indivisibles are infintely narrow rectangles of infinitesimal area. They are called “indivisibles” because they are assumed to be so small that they cannot be divided into narrower rectangles. A region as in Figure 3 was considered to consist of an infinite number of parallel indivisibles such as is represented by the line in the sketch. The area of each indivisible was considered to be its length multiplied by its infinitesimal width, and the area of the region was considered to be the sum of the areas of the infinite number of indivisibles in it.

Leibniz used the symbol dx to denote the width of an indivisible, so that the area of an indivisible of length y was given by the product $y\,dx$. He then introduced the symbol $\int y\,dx$ for the sum or “integral” of the areas of the indivisibles that gives the area of the region.

FIGURE 3



The use of indivisibles in integration, like the use of infinitesimals in early treatments of the derivative, has always been subject to criticism. The French writer Voltaire, for example, once complained that calculus was "the Art of numbering and measuring exactly a Thing whose Existence cannot be conceived."[1] Yet even those who argued that there were logical difficulties with the use of infinitesimals and indivisibles could not deny the importance of the mathematical tools to which these concepts led. By the second half of the seventeenth century, indivisibles had been used extensively in determining the areas, volumes, and centers of gravity of a large number of geometric figures.

Later, mathematicians who were schooled in the classical Greek mathematics of Euclid and Archimedes supplied rigorous proofs, based on approximation procedures, of the integration formulas they derived using indivisibles. But, eventually, as more results were obtained, many felt that supplying the rigorous proof to meet the standards of classical mathematics was a waste of time. The German astronomer Johann Kepler (1571-1630) took this point of view in his popular book *Nova Stereometria Doliorum Binariorum (New Solid Geometry of Wine Barrels)*, in which he computed volumes of a large number of solids of revolution. Referring to the absence of classical proofs in the book, he wrote in its preface, "We could obtain absolute and in all respects perfect demonstrations from the books of Archimedes themselves, were we not repelled by the thorny reading thereof."[2]

Special cases of the Fundamental Theorem of Calculus were discovered long before its full significance and generality were recognized. The fourteenth-century French bishop Nicole Oresme (ca. 1323–1382) showed in special cases that if a time interval were represented on a horizontal line and if the corresponding variable velocity were represented by vertical line segments along it, then the distance traveled would be equal to the area of the resulting region. Galileo used the same idea in his study of falling bodies. Fermat used a primitive version of the derivative to solve problems that we solve today with integrals, and he solved certain tangent line problems by using areas.

It was the English theologian Isaac Barrow (1630-1677), however, who was the first to clearly recognize that the finding of tangent lines and the computing of areas are inverse operations. This insight contributed to the work of his protégé Newton, and perhaps also that of Leibniz, who purchased a copy of Barrow's *Lectures* on a trip to London in 1673. The Fundamental Theorem played a central role in the rules and procedures of calculus as developed, independently, by Newton and Leibniz.

The point of view that integrals are obtained geometrically through the use of indivisibles prevailed until the early nineteenth century. The definition of the integral of a continuous function as a limit of finite sums was given by Augustin Cauchy (1789-1857) in the 1820's. Thirty years later, G.F.B. Riemann (1826-1866) realized that Cauchy's definition could be extended to apply to certain discontinuous functions. He gave the definition of the definite integral that we use today.

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[1] *History of the Calculus* by C. Boyer, New York, NY: Dover Publications, Inc., 1959, p.238

[2] *A Source Book of Mathematics* by D. Struik, Cambridge, MA: Harvard University Press, 1969, p. 188.

Interactive Examples 6.8

Interactive solutions are on the web page http//www.math.ucsd.edu/~ashenk/.[†]

1. Find the antiderivative $\displaystyle\int \frac{x}{x^2+3}\,dx$. Check the result by differentiation.

2. Evaluate the integral $\displaystyle\int_{-1}^{0} \frac{1}{6x-1}\,dx$.

3. Give a formula for the indefinite integral. $\displaystyle\int \sin x\, e^{\cos x}\,dx$. Check the result by differentiation.

4. Evaluate $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$ by making a substitution in the indefinite integral.

5. Evaluate $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$ by making a substitution in the definite integral.

Exercises 6.8

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

1. Show that $\displaystyle\int x^4\,dx = \int x\cdot x^3\,dx$ is not equal to $\displaystyle x\int x^3\,dx$.

2. When the substitution $u = 2x, du = 2\,dx$ is made in the definite integral $\displaystyle\int_{x=0}^{x=1} 2x\,dx$ we obtain

$$\int_{x=0}^{x=1} 2x\,dx = \tfrac{1}{2}\int_{u=0}^{u=2} u\,du$$

since $0 \underset{u}{\longrightarrow} 2$ when $0 \underset{x}{\longrightarrow} 1$ and $dx = \frac{1}{2}du$. Illustrate this equation geometrically by drawing the regions whose areas equal the two integrals.

BASICS:

3.[O] Find the antiderivatives $\displaystyle\int x^2\sqrt{x^3-1}\,dx$ and check the result by differentiation.

4.[O] Perform the integration $\displaystyle\int \frac{1}{(3x+4)^2}\,dx$ and check the result by differentiating it.

[†]In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

Perform the integrations in Exercises 5 through 29.

5.$^{\mathsf{O}}$ $\displaystyle\int_0^1 \frac{1}{(3-x)^3}\,dx$

6.$^{\mathsf{O}}$ $\displaystyle\int \frac{x}{x^2+1}\,dx$

7.$^{\mathsf{O}}$ $\displaystyle\int \frac{\sqrt{\ln x}}{x}\,dx$

8.$^{\mathsf{A}}$ $\displaystyle\int e^x\sqrt{10+e^x}\,dx$

9. $\displaystyle\int \frac{x}{\sqrt{x+3}}\,dx$

10.$^{\mathsf{O}}$ $\displaystyle\int \sinh(4x)\,dx$

11.$^{\mathsf{O}}$ $\displaystyle\int \frac{x}{(x^2+4)^3}\,dx$

12.$^{\mathsf{A}}$ $\displaystyle\int x^8(2+x^9)^3\,dx$

13. $\displaystyle\int \frac{x^3}{(5-x^4)^2}\,dx$

14.$^{\mathsf{O}}$ $\displaystyle\int \frac{x^{1/2}}{x^{3/2}+3}\,dx$

15.$^{\mathsf{A}}$ $\displaystyle\int_0^1 \sin(5x)\,dx$

16.$^{\mathsf{A}}$ $\displaystyle\int_2^3 \frac{x}{1-x^2}\,dx$

17.$^{\mathsf{O}}$ $\displaystyle\int_1^2 \frac{4x^3+1}{1-x-x^4}\,dx$

18.$^{\mathsf{A}}$ $\displaystyle\int_0^{\pi} \frac{\sin x}{4-\cos x}\,dx$

19.$^{\mathsf{O}}$ $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx$

20.$^{\mathsf{A}}$ $\displaystyle\int xe^{-x^2}\,dx$

21. $\displaystyle\int e^{-5x}\,dx$

22.$^{\mathsf{O}}$ $\displaystyle\int_0^2 e^{-3t}\,dt$

23.$^{\mathsf{A}}$ $\displaystyle\int_1^9 \frac{1}{\sqrt{x}}\,e^{\sqrt{x}}\,dx$

24. $\displaystyle\int_{-1}^1 x^2e^{x^3}\,dx$

25.$^{\mathsf{A}}$ $\displaystyle\int \frac{1}{x\sqrt{\ln x}}\,dx$

26. $\displaystyle\int e^x\sqrt{2+e^x}\,dx$

27. $\displaystyle\int \frac{1}{7x+1}\,dx$

28.$^{\mathsf{O}}$ $\displaystyle\int_0^4 (2x+1)^{-1/2}\,dx$

29.$^{\mathsf{A}}$ $\displaystyle\int \frac{\ln x}{x}\,dx$

30.$^{\mathsf{O}}$ Evaluate $\displaystyle\int_0^1 [4(1+x)^3-5(1-x)^4]\,dx$ by writing it as the difference of two integrals and using one substitution in one and another substitution in the other.

Find the areas of the regions in Exercises 31 through 36.

31.$^{\mathsf{O}}$ The region bounded by the curve $y=x^2(1-x^3)^{1/3}$ and the x-axis

32.$^{\mathsf{A}}$ The region between $y=4e^{-x}$ and the x-axis for $0\le x\le 2$

33.$^{\mathsf{A}}$ The region bounded by $y=x^2/(x^3+2)$, the x-axis, and the line $x=2$

34.$^{\mathsf{A}}$ The region bounded by the curve $y=e^{2x}$, the x-axis, and the lines $x=-1$ and $x=1$

35. The region between the x-axis and the curve $y=xe^{-x^2}$ for $0\le x\le 4$

36. The region between $y=\dfrac{\ln x}{x}$ and the x-axis for $1\le x\le e$

37.$^{\mathsf{O}}$ An object's velocity in the positive s-direction on an s-axis is $v=t^2(1+t^3)^3$ feet per minute for $0\le t\le 1$. It is at $s=3$ (feet) at $t=0$ (minutes). Where is it at $t=1$?

38.$^{\mathsf{A}}$ An object that is at $s=25$ (meters) on an s-axis at $t=0$ (minutes) has veocity $v=100/(t+2)$ meters per minute in the positive direction at time $t>0$. Give its position as a function of $t\ge 0$.

39. An object that is at $s = 10$ (feet) on an s-axis at $t = 0$ (seconds) has veocity $v = t/(1+t^2)$ feet per second in the positive direction at time $t > 0$. Give its position as a function of $t \geq 0$.

EXPLORATION:

Perform the integration in Exercises 40 through 56.

40.[O] $\displaystyle\int \frac{x}{\sqrt{1-x}}\,dx$

41.[A] $\displaystyle\int \frac{x}{(x-4)^3}\,dx$

42. $\displaystyle\int \frac{t}{\sqrt{t+10}}\,dt$

43.[O] $\displaystyle\int \frac{(\sqrt{x}+9)^9}{\sqrt{x}}\,dx$

44.[A] $\displaystyle\int \frac{x+1}{\sqrt{x^2+2x+5}}\,dx$

45. $\displaystyle\int_0^1 x^{1/2}(1-x^{3/2})\,dx$

46. $\displaystyle\int_0^1 x(x-1)^{3/2}\,dx$

47. $\displaystyle\int \frac{x}{(2x+1)^3}\,dx$

48. $\displaystyle\int \cosh(x^{1/2})\,x^{-1/2}\,dx$

49.[A] $\displaystyle\int \sinh x\cosh x\,dx$

50.[A] $\displaystyle\int (e^{3x}+1)^2\,dx$

51.[O] $\displaystyle\int x(10^{x^2})\,dx$

52.[A] $\displaystyle\int 5^{x^{1/2}}x^{-1/2}\,dx$

53. $\displaystyle\int_0^{2\pi} 2^{\cos x}\sin x\,dx$

54.[A] $\displaystyle\int \frac{e^{\ln x}}{x}\,dx$

55. $\displaystyle\int_0^e (x^e + e^x)\,dx$

56. $\displaystyle\int \cos(2^x)\,2^x\,dx$

57. What is the area of the region between $y = 1 + x(x-1)^4$ and the x-axis for $0 \leq x \leq 1$?

58. Give a formula for $\displaystyle\int \frac{x}{\sqrt{x+a}}\,dx$ in terms of the parameter a.

59. Evaluate $\displaystyle\int_0^1 (1+\sqrt{x})^3\,dx$ by using the change of variables $u = 1+\sqrt{x}$.

60. Find $b > 0$ such that the region between $y = \dfrac{x}{(1+x^2)^2}$ and the x-axis for $0 \leq x \leq b$ has area $\frac{1}{4}$.

61. **(a)** Find the positive number k such that $\displaystyle\int_3^k x\sqrt{x^2-9}\,dx = \frac{64}{3}$. **(b)** Why is there no negative k that satisfies the equation in part (a)?

62. What is the area of the region between $y = (e^x+1)^2$ and the x-axis for $0 \leq x \leq 5$?

63. Find the value of one positive number and one negative number a such that the region bounded by the x- and y-axes, by the curve $y = e^{-x}$, and by the line $x = a$ has area 0.99.

65. Find the antiderivative $y = f(x)$ of $y = xe^{-x^2}$ such that $\displaystyle\lim_{x\to\pm\infty} f(x) = 10$.

66. Find the value of $k > 0$ such that $\displaystyle\int_0^1 \frac{1}{x+k}\,dx = 0.9$.

67. What is $\displaystyle\lim_{x\to-3} F(x)$ if $F(x)$ is an antiderivative of $f(x) = \dfrac{1}{2x+6}$?

(End of Section 6.8)