Problem 11.39. The first graph in problem 12 is connected and cubic and doesn’t have a hamiltonian cycle (which isn’t easy to see). Another example is suggested by the book: Construct a graph with 5 vertices, 4 vertices of which have degree 3 and 1 has degree 2. Take three copies of this graph $G_1, G_2, G_3$. Introduce another vertex $w$, and connect it to the three vertices with degree 2 in $G_1, G_2, G_3$. Then this new graph (16 vertices) has no Hamiltonian cycle since any closed walk visiting all of the vertices must visit $w$ at least twice.

Problem 11.40. Since $G$ has $(n - 1)(n - 2)/2 + 2$ edges, this means that there are $n - 3$ non-edges (missing edges). For sake of contradiction, suppose a pair $x$ and $y$ that are not adjacent in $G$ satisfies $d(x) + d(y) < n$. Then there are at least $2(n - 2) - n + 1 = n - 3$ missing edges between $\{x, y\}$ and $V - \{x, y\}$. Plus, $(x, y)$ is not an edge so there are at least $n - 4$ missing edges. This contradicts the above observation.

Problem 11.42. Construct a new graph $G'$ on $n + 1$ vertices by adding a vertex $w$ and connect $w$ to every vertex in $G$. Then this new graph has the Ore property. Indeed, for any two nonadjacent vertices $x, y \in V(G')$, both $x, y \in V(G)$ (i.e. neither of them can be $w$ since $w$ is adjacent to everyone). Then

$$d(x) + d(y) \geq n - 1 + 2 = n + 1 = |V(G')|.$$ 

Hence there is a Hamiltonian cycle in $G'$. By deleting $w$ in this Hamiltonian cycle, we get a Hamiltonian path in $G$.

Problem 11.47. Any bipartite graph with a Hamiltonian cycle must have the same number of vertices in each part since the cycle must alternate across the parts at each step. If $G$ has an odd number of vertices and is bipartite, then the parts do not have equal size.

Problem 11.53. $G$ is a tree if and only if it does not contain any cycles is shown in Theorem 11.5.4. We also know that it has exactly $n - 1$ edges. By adding an edge, $G$ is no longer a tree which means that there is an edge that is not a bridge. By Lemma 11.5.3, there is a cycle containing this edge.

Problem 11.54. Since trees have exactly $n - 1$ edges, only the path $P_n$ has the Eulerian path.

Problem 11.53. Since trees have exactly $n - 1$ edges, only the path $P_n$ has a hamiltonian path.
**Problem 12.4.** A graph is two colorable if and only if it is bipartite. Since $C_n$ is an odd cycle, we know the chromatic number is at least 3. All we need to prove now is that it is 3 colorable (i.e. the chromatic number is at most 3). We proceed by induction on $n$ for $n$ odd. The base case is when $n = 3$, and clearly we need three colors. For the inductive step, assume the statement holds for all odd integers $n' < n$. Let’s prove that $C_n$ is 3-colorable for $n$ odd. Consider two consecutive vertices $x$ and $y$ on $C_n$, and their neighbors $x'$ and $y'$. If we remove them and connect their neighbors $x'$ and $y'$, we have an odd cycle of length $n - 2$. By the induction hypothesis, we can color $C_{n-2}$ with three colors.

**Problem 12.5.** a) $\chi(G) = 2$. It is at least 2 since there are edges. Moreover, by inspection, one can properly 2-color $G$. b) $\chi(G) = 3$. It is at least three since there are triangles. Moreover, by inspection, one can color $G$ with three colors. c) $\chi(G) = 4$. It is at least 4 since $G$ contain $K_4$. Moreover, it is at most 4 by inspection.

**Problem 12.6.** If $\chi(G) = k$, then we have a vertex partition $V = V_1 \cup \cdots \cup V_k$, where $V_i$ have no edges inside. Then between any two parts $V_i, V_j, i \neq j$, we must have at least one edge. Indeed, otherwise we can combine $V_i \cup V_j$ to be one color class and color $G$ with $k - 1$ colors. Hence there must be at least $\binom{k}{2}$ edges in $G$. 