The Principle of Inclusion - Exclusion
• Fundamental Problem in Combinatorics: given a (finite) set $S$, determine $|S|$.

• Different strategies, depending on how the set $S$ is "given."

• If $S$ is contained in a larger set $\overline{S}$, you can try "count the complement" strategy.
• At the level of sets,

\[ \bar{S} = S \cup \bar{S} \setminus S. \]

• At the level of numbers,

\[ |\bar{S}| = |S| + |\bar{S} \setminus S|. \]

• If you know how to compute \( |\bar{S}| \) and \( |\bar{S} \setminus S| \), you can solve for \( |S| \) ("Subtraction Principle"): \n
\[ |S| = |\bar{S}| - |\bar{S} \setminus S|. \]
Problem: How many n-bit strings contain at least one 1?
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Solution: \( S = \{ n \text{-bit strings with at least one 1} \} \).

\( \overline{S} = \{ n \text{-bit strings} \} \). Clearly, \( |\overline{S}| = 2^n \).

\( \overline{S} \setminus S = \{ n \text{-bit strings with no 1} \} = \{ 000...0 \} \). Clearly, \( |\overline{S} \setminus S| = 1 \).

Subtraction Principle:
\[ |S| = |\overline{S}| - |\overline{S} \setminus S| = 2^n - 1. \]
• Subtraction Principle really just an instance of "Partition Principle"

\[ A = A_1 \cup A_2 \implies |A_1 \cup A_2| = |A_1| + |A_2|. \]

• Notation: \( A_1 \cup A_2 \) means the union of two disjoint sets, \( A_1 \cap A_2 = \emptyset \),

\[ A_1 \quad \text{and} \quad A_2 \]
What happens when $A = A_1 \cup A_2$ is the union of two non-disjoint sets?

What is the relationship between the numbers $|A_1 \cup A_2|, |A_1|, |A_2|$?

Think of this as an "area" or "paving" computation.
• First, pave oval $A_1$:

• Then pave oval $A_2$:
• The overlapping area was paved twice:

• To get a smooth paving, you have to strip one layer of pavement off the overlap:
• To get a smooth paving, you have to strip one layer of pavement off the overlap:

\[ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|. \]
• In formulas,

\[ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| . \]

• Observe that this degenerates to the partitioning principle if \( A_1 \cap A_2 = \emptyset \),

\[ |A_1 \cup A_2| = |A_1| + |A_2| - 0. \]
Consider now the case of three sets:

\[
\begin{align*}
A_1 & \quad A_2 \\
A_3 &
\end{align*}
\]

We want to evenly pave the region delineated by these three circles.
• Start by paving $A_1$:

- $A_1$
- $A_2$
- $A_3$

• No problem so far.
• Now pave $A_2$:

• Problem: the region $A_1 \cap A_2$ has been paved twice.
• Now pave $A_3$.

• Multiple issues:
  • The region $A_1 \cap A_2$ has been paved twice;
  • The region $A_1 \cap A_3$ has been paved twice;
  • The region $A_2 \cap A_3$ has been paved twice;
  • The region $A_1 \cap A_2 \cap A_3$ has been paved three times.
• Strip a layer of pavement off $A_1 \cap A_3$: 

![Venn Diagram]
• Strip a layer of pavement off \( A_2 \cap A_3 \):
• Strip a layer of pavement off $A_1 \cap A_2$: 

\[ A_1 \cap A_2 = \{ \_ , \_ \} \]
Re-pave $A_1 \cap A_2 \cap A_3$. 

![Venn diagram with three overlapping circles labeled A_1, A_2, and A_3]
• Repave \( A_1 \cap A_2 \cap A_3 \),

\[ |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \]
**Extrapolation:**

\[ |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \]

\[ |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \]

\[ |A_1 \cup A_2 \cup A_3 \cup A_4| = ? \]
Extrapolation:

\[ |A_1 \cup A_2| = \sum \text{cardinality} \]

\[ - \sum \text{cardinality of intersection} \]

\[ |A_1 \cup A_2 \cup A_3| = \sum \text{cardinality} \]

\[ - \sum \text{cardinality of intersection} \]

\[ + \sum \text{cardinality of triple intersection} \]

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\[ - \sum \text{cardinality of intersection} \]

\[ + \sum \text{cardinality of triple intersection} \]

\[ - \sum \text{cardinality of quadruple intersection}. \]
Theorem (Principle of Inclusion-Exclusion)

For any positive integer \( n \), and any finite sets \( A_1, \ldots, A_n \),

\[
\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \atop |I| = i}} \left| \bigcap_{j \in I} A_j \right|
\]

Equivalently,

\[
\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\substack{I \subseteq \{1,\ldots,n\} \atop I \neq \emptyset}} (-1)^{|I|-1} \left| \bigcap_{i \in I} A_i \right|
\]
Example: $|A_1 \cup A_2 \cup A_3 \cup A_4|$ equals

\{1\} \ {2,3} \ {3,4} \ {4,3}

\{1,2,3\} \ {1,3,4} \ {2,3,4} \ {3,4,3}

\{1,2,3,4\}
Example: \(|A_1 \cup A_2 \cup A_3 \cup A_4|\) equals

\[
\begin{align*}
\{1\} & \quad \{2\} & \quad \{3\} & \quad \{4\} \\
+ |A_1| & + |A_2| & + |A_3| & + |A_4| \\
\{1,2\} & \quad \{1,3\} & \quad \{1,4\} & \quad \{2,3\} & \quad \{2,4\} & \quad \{3,4\} \\
- |A_1 \cap A_2| & - |A_1 \cap A_3| & - |A_1 \cap A_4| & - |A_2 \cap A_3| & - |A_2 \cap A_4| & - |A_3 \cap A_4| \\
\{1,2,3\} & \quad \{1,2,4\} & \quad \{1,3,4\} & \quad \{2,3,4\} \\
+ |A_1 \cap A_2 \cap A_3| & + |A_1 \cap A_2 \cap A_4| & + |A_1 \cap A_3 \cap A_4| & + |A_2 \cap A_3 \cap A_4| \\
\{1,2,3,4\} \\
- |A_1 \cap A_2 \cap A_3 \cap A_4| 
\end{align*}
\]
Proof: • We want to prove that

\[
|A_1 \cup \ldots \cup A_n| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{\substack{S \subseteq \{1, \ldots, n\} \ 1 \leq |S| = i}} |\bigcap_{j \in S} A_j|.
\]  

\text{(⋆)}

• Choose an arbitrary element \( x \in A_1 \cup \ldots \cup A_n \).

• The LHS of (⋆) counts \( x \) exactly once.

• How many times does the RHS of (⋆) count \( x \)?
• Let \( r \) be the number of sets on the list \( A_1, \ldots, A_n \) which contain \( x \).

• Since \( x \in A_1 \cup \ldots \cup A_n \), it must belong to at least one \( A_i \), so \( r \geq 1 \). But \( r \) could be as large as \( n \) (if \( x \in A_1, \ldots, x \in A_n \)).

• Without loss in generality, suppose \( x \in A_1, \ldots, x \in A_r \). For example, if \( r \) were 3 and \( x \in A_1, x \in A_2, x \in A_n \), we could rename \( A_n \) "\( A_3 \)" and rename \( A_3 \) "\( A_n \)."

• This means that, when considering how many times (*) counts \( x \), we need only consider the partial sum

\[
\sum_{i=1}^{r} (-1)^{i-1} \sum_{\substack{S \subseteq \{1, \ldots, r\} \\ |S| = i}} |\bigcap_{j \in S} A_j| \quad (**)
\]
• Consider the $i=1$ term in (**). The internal sum is

$$\sum_{S \subseteq \{1, \ldots, r\}, |S|=1} |\cap_{j \in S} A_j| = |A_1| + |A_2| + \ldots + |A_r|,$$

which counts $x \cdot r$ times.

• Consider the $i=2$ term in (**). The internal sum is

$$\sum_{S \subseteq \{1, \ldots, r\}, |S|=2} |\cap_{j \in S} A_j| = |A_1 \cap A_2| + |A_1 \cap A_3| + \ldots + |A_1 \cap A_r|$$

$$+ |A_2 \cap A_3| + \ldots + |A_2 \cap A_r| + \ldots + |A_{r-1} \cap A_r|,$$

which counts $x \cdot \binom{r}{2}$ times.
In general, the $i^{th}$ term of

$$\sum_{i=1}^{r} (-1)^{i-1} \sum_{\substack{S \subseteq \{1, \ldots, r\} \\mid |S| = i}} |\bigcap_{j \in S} A_j| \quad (**)$$

counts a total of \((r)\) times, once for each term of the internal sum, and with a sign of \((-1)^{i-1}\).

Thus

$$|A_1 \cup \ldots \cup A_n| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{\substack{S \subseteq \{1, \ldots, n\} \\mid |S| = i}} |\bigcap_{j \in S} A_j| \quad (*)$$

is true if and only if

$$1 = \sum_{i=1}^{r} (-1)^{i-1} (r) \quad (***)$$

is true.
- Let us verify that (***), is correct.

- Clearly, (***), is equivalent to the identity

\[ \left| 1 - \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} \right| = 0. \] (***)

- But the LHS of (****) is

\[ \left| 1 - \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} \right| = \left| 1 + \sum_{i=1}^{r} (-1)^{i} \binom{r}{i} \right| = \sum_{i=0}^{r} (-1)^{i} \binom{r}{i} = (-1)^{r+1} = 0. \]
Problem: You are mailing letters to your friends 1, 2,..., n. You decide to close your eyes and randomly put the letters in the addressed envelopes. What is the probability that everyone gets the wrong letter?
Solution: • Each way of placing the letters in the envelopes defines a permutation 
\[ \pi: \{1, \ldots, n\} \to \{1, \ldots, n\}. \]

• The formula \( \pi(i) = j \) means “letter \( i \) goes in envelope \( j \).”

• The probability we want to calculate is thus
\[
\frac{|\{\pi \in S_n : \pi(i) \neq i \text{ for all } 1 \leq i \leq n\}|}{|S_n|},
\]
where \( S_n \) is the set of all permutations \( \pi: \{1, \ldots, n\} \to \{1, \ldots, n\} \).

• The denominator of \( p_n \) is \( |S_n| = n! \); it remains to calculate the numerator.
• By the Subtraction Principle,

\[ |\{\pi \in S_n : \pi(i) \neq i \text{ for all } 1 \leq i \leq n\} | = n! - |\{\pi \in S_n : \pi(i) = i \text{ for some } 1 \leq i \leq n\} |. \]

• So, we need to compute the number \( |\{\pi \in S_n : \pi(i) = i \text{ for some } 1 \leq i \leq n\} | \) of permutations that have at least one fixed point.

• This number is \( |A_1 \cup A_2 \cup \ldots \cup A_n| \), where \( A_i \) is the set of permutations which map \( i \) to itself:

\[ A_i = \{\pi \in S_n : \pi(i) = i\}. \]
• By PIE,

\[ |A_1 \cup \ldots \cup A_n| = \sum_{i=1}^{n} (-1)^{i+1} \sum_{|S|=i, \ S \subseteq \{1, \ldots, n\}} |\bigcap_{j \in S} A_j| \]

• Now, for any nonempty \( S \subseteq \{1, \ldots, n\} \), we have

\[ \bigcap_{j \in S} A_j = \{ \pi \in S_n : \pi(j) = j \ \text{for each} \ j \in S \} \]

• The cardinality of this intersection is thus

\[ |\bigcap_{j \in S} A_j| = (n - |S|)! \]
• Returning to the PIE formula, we thus have

\[ |A_1 \cup \ldots \cup A_n| = \sum_{i=1}^{n} (-1)^{i-1} \sum_{S \subseteq \{1, \ldots, n\} \atop |S| = i} (n-|S|)! \]

\[ = \sum_{i=1}^{n} (-1)^{i-1} (n-i)! \sum_{S \subseteq \{1, \ldots, n\} \atop |S| = i} 1 \]

\[ = \sum_{i=1}^{n} (-1)^{i-1} (n-i)! \binom{n}{i} \]

\[ = \sum_{i=1}^{n} (-1)^{i-1} \frac{n!}{i!}. \]
Returning to the Subtraction Principle formula, we thus find that the number of permutations with no fixed point is

\[ n! - |A_1 \cup \ldots \cup A_n| = n! - \sum_{i=1}^{n} (-1)^{i-1} \frac{n!}{i!} \]

\[ = n! \left( 1 + \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i!} \right) \]

\[ = n! \sum_{i=0}^{n} \frac{(-1)^{i}}{i!}. \]