Fractals

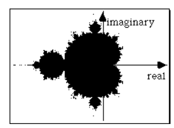
from Wikipedia: list of fractals by Hausdoff dimension

Sierpinski Triangle 

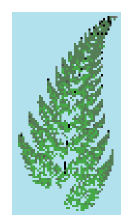
3D Cantor Dust 

Lorenz attractor 

Coastline of Great Britain 

Mandelbrot Set 

What makes a fractal?



I'm using 2 references:
Fractal Geometry by Kenneth Falconer
Encounters with Chaos by Denny Gulick

1) A fractal is a subset of P^n with non integer dimension.
Of course this make no sense without a definition of dimension.

2) Fractal contains copies of itself at many scales.

3) It is too irregular to be described by traditional language.

Mandelbrot coined the term in 1977 though many examples were known.
"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."(Mandelbrot, 1983).

Other references: the web, books by Mandelbrot
M. Barnsley, *Fractals Everywhere*

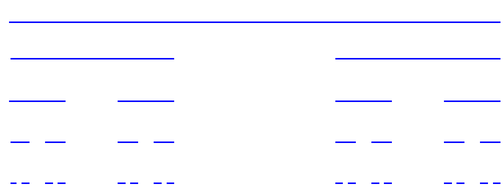
1) Cantor Dust.

From the interval [0,1] remove the middle third. Then remove the middle third from the remaining 2 intervals [0,1/3] and [2/3,1].

Keep going with this removal of middle thirds forever …..

You end up with the Cantor dust. Impossible to draw it. We give the first 5 steps. Later we will see the Cantor Dust has box dimension $\ln 2/\ln 3 \cong .63$.

Higher dimension than a point but smaller than an interval. There are many interesting facts about the Cantor dust. For example, the set is uncountable, but if you integrate the function that is 1 on the Cantor set and 0 off the set (using the Lebesgue integral), you get 0. The Riemann integral cannot deal with this.



Defn. Start with $C_0=[0,1]$. Let $C_1=[0,1/3]\cup[2/3,1]$. Continue in this way, defining C_k as a union of 2^k subintervals, each of length 3^{-k} obtained by removing middle thirds of the intervals in C_{k-1}. The **Cantor set** C is the intersection of all the C_k, for k running over all integers ≥ 0.

Problem 1. Show that the Cantor set can be identified with all the real numbers in [0,1] that can be represented in the form

$$x=\sum_{n=1}^{\infty}\frac{d_n}{3^n}, \quad with \quad d_n\in\{0,2\}.$$

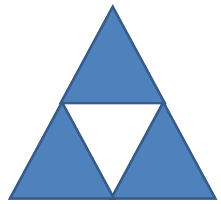
This includes, for example, $20/27 = 2/3 + 2/27 = 0.202000\ldots$. Here, if a number has 2 expansions, we are saying 1 and only 1 of these expansions has no d_n taking the value 1. Thus $1/3\in C$ since even though $1/3=.10000\ldots$, since we also have $1/3=.0222\ldots$.

This implies there are as many points in the Cantor set as there are real numbers. The Cantor set is uncountable.

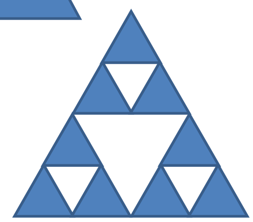
Hint. The numbers requiring a 1 in the 1st place of their ternary expansion lie in the interval (1/3,2/3). The numbers requiring a 1 in the 2nd place of their ternary expansion lie in the union of the intervals (1/9,2/9) and (7/9,8,9).

2) Sierpinski Triangle.

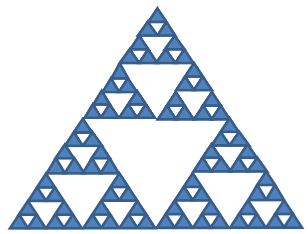
Start with an equilateral triangle
and remove the center triangle.



Remove the center triangles from each of the
3 remaining triangles.



Keep going forever.



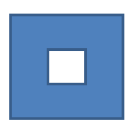
Fractal dimensions give a way of comparing fractals. Fractal dimensions can be defined in connection with real world data, such as the coastline of Great Britain. It turns out to have fractal dimension approximately 1.2.
Here we will only look at the box dimension. It is only one of a wide variety of notions of fractal dimension. (The oldest and perhaps most important is the Hausdoff dimension. It is harder to calculate and you will need to look at the references such as Falconer to find out what that is.)

Definition of Box Dimension.

Defn. Suppose S is a subset of P^n, n=1,2,3. By an **n-box**, we mean a closed interval if n=1; a square if n=2; a cube if n=3.

Defn. $\forall\varepsilon>0$ let **$N(\varepsilon)$** be the smallest number of n-boxes of side length ε needed to cover S.

Example.
It takes 8 boxes of length 1/3 to cover the unit square with center square of side 1/3 removed.



Defn. **Box dimension** of $S\subset P^n$ is defined to be the following limit if it exists:

$$\dim_B S = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{\ln N(\varepsilon)}{\ln 1/\varepsilon}.$$

here we use ln(x) rather than log(x)

Note. The box dimension is often called capacity. It sometimes differs from the Hausdorff dimension which we do not define here. But both definitions of dimension agree on most of the simple examples. For an m-dimensional surface in P^n, n=1,2,3, the box dimension can be shown to be m. Thus, for example, the box dimension of the surface of the sphere is 2. See Falconer, p. 44.

Problem 2. Show that the box dimension of the unit square is 2.

Theorems to aid us in computing the box dimension.

Theorem 1. Let S be a subset of $\mathbb{R}^n$, n=1,2,3. If 0<r<1, then the limit defining $\dim_B S$ exists iff the following limit exists and then they are equal

$$\dim_B S = \lim_{k\to\infty} \frac{\ln N(r^k)}{\ln \frac{1}{r^k}}.$$

Proof Sketch. Given $\varepsilon>0$ and r with 0<r<1, you just have to find a positive integer k such that

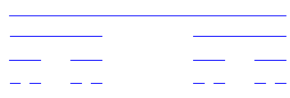
$$r^{k+1} < \varepsilon \le r^k. \quad \textit{Then} \quad N(r^k) \le N(\varepsilon) < N(r^{k+1}).$$

$$\textit{As } \ln(x) \textit{ monotone} \uparrow, \quad \ln\left(\frac{1}{r^k}\right) \le \ln\left(\frac{1}{\varepsilon}\right) < \ln\left(\frac{1}{r^{k+1}}\right),$$

$$\textit{and} \qquad \ln N(r^k) \le \ln N(\varepsilon) \le \ln N(r^{k+1}).$$

$$\textit{So} \qquad \frac{\ln N(r^k)}{\ln\left(1/r^{k+1}\right)} \le \frac{\ln N(\varepsilon)}{\ln\left(1/\varepsilon\right)} \le \frac{\ln N(r^{k+1})}{\ln\left(1/r^k\right)}.$$

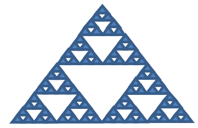
Example.
The Box dim of the Cantor set C.



Recall that *C* is obtained from the interval [0,1] by continually removing middle thirds. At each stage there are twice as many intervals as the preceding stage. And each interval has length 1/3 that of the preceding stage. Since N(1/3) =2, we find that, by induction, $N(1/3^k)=2^k$, for each $k\ge 1$.
Let r=1/3 in the preceding theorem and find

$$\dim_B C = \lim_{k\to\infty} \frac{\ln N\left(\frac{1}{3^k}\right)}{\ln\left(\frac{1}{1/3^k}\right)} = \lim_{k\to\infty} \frac{\ln 2^k}{\ln 3^k}$$

$$= \lim_{k\to\infty} \frac{k\ln 2}{k\ln 3} = \frac{\ln 2}{\ln 3} \cong 0.63.$$



Problem 3. Show that the box dimension of the set of rational numbers in [0,1] is 1.

Defn. Suppose our set S is a subset of P^n, n=1,2,3.
The distance between 2 points x,y is denoted Πx-y Π.
Suppose that a function f:S→S has the property that for some constant r with 0<r<1, Πf(x)-f(y)Π = r Πx-y Π, for all x,y in S. Then we call f a similarity of S. The constant r is called the similarity constant.

Example. Let S=[0,1] & f(x)=2/3 + x/3. Then f is a similarity with constant 1/3. It maps [0,1] onto [2/3,1].

Defn. If there are m similarities $f_1,\ldots,f_m$ of S such that $S=f_1(S) \cup \ldots \cup f_m(S)$, and the images are non-overlapping except possibly for boundaries, we say that S is a self-similar set. It is composed of m (shrunk) copies of itself.

Example. The Canter set C. Define g(x)=x/3 and f(x)= 2/3 + x/3. Then C=f(C) ∪ g(C). Both f and g are similarity functions with similarity constant 1/3.

Problem 4. Show that the Sierpinski triangle is a self-similar set. Use this to see that the box dimension of the Sierpinski triangle is ln3/ln2≅1.58 using the following theorem. You need to define functions of vectors in the plane. First put the origin at the left hand base point of the big triangle. Then figure out what function shrinks the big triangle to the small one at the origin. Next what vector must you add to that function to shift the left small triangle over to the right one at the base?

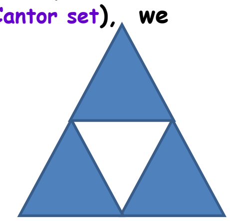
Another Theorem making it easy to compute the box dimension.

Theorem 2. Suppose that S is a self similar set in P^n, n=1,2,3; i.e., $S=f_1(S) \cup \ldots \cup f_m(S)$, non-overlapping, and such that each similarity function f_i has the same similarity constant r. Then

$$\dim_B S = \ln m/\ln(1/r).$$

Proof. We use Theorem 1. Since $N(r^k) = cm^k$, for some positive constant c (Why? is Problem 5. Hint. Think about the Cantor set), we have

$$\lim_{k\to\infty}\frac{\ln N(r^k)}{\ln(1/r)^k} = \lim_{k\to\infty}\frac{\ln m^k}{\ln(1/r)^k} = \frac{\ln m}{\ln(1/r)}.$$



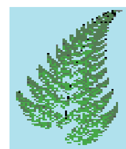
Defn. A set M of real numbers is said to have **Lebesgue measure 0** iff for every $\varepsilon > 0$ there is a sequence of open intervals E_n such that

$$M \subset \bigcup_{n\geq 1} E_n \;\; and \;\; \sum_{n\geq 1} length(E_n) < \varepsilon.$$

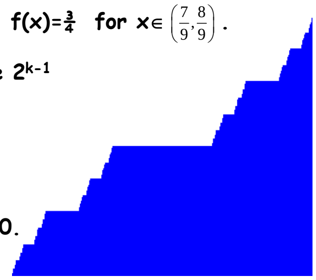
Sets of Lebesgue measure 0 are considered negligible in the theory of the Lebesgue integral. One can ignore what a function does on such sets when computing Lebesgue integrals. And one identifies functions that are equal except on a set of Lebesgue measure 0.

Problem 6. Show that any countable set M of real numbers has Lebesgue measure 0.

Problem 7. Show that the Cantor set has Lebesgue measure 0.
Hint. To do this recall that setting $C_0=[0,1]$. Let $C_2=[0,1/3]\cup[2/3,1]$. Continue in this way, defining C_k as a union of 2^k subintervals, each of length 3^{-k} obtained by removing middle thirds of the intervals in C_{k-1}. The Cantor set C is the intersection of all the C_k, for k running over all integers ≥ 0.
Show that C_k is has length $(2/3)^k$.



Defn. The Devil's Staircase or the Almost Perfect Sneak.
Define a function on the interval [0,1] as follows. Write the complement of the Cantor set in [0,1] as a union of intervals. First define f on the complement of the Cantor set.
Define $f(x)=\frac{1}{2}$ for $x\in\left(\frac{1}{3},\frac{2}{3}\right)$; $f(x)=\frac{1}{4}$ for $x\in\left(\frac{1}{9},\frac{2}{9}\right)$, $f(x)=\frac{3}{4}$ for $x\in\left(\frac{7}{9},\frac{8}{9}\right)$.



Then in the kth step, going from left to right on the 2^{k-1} subintervals left out of the Cantor set define

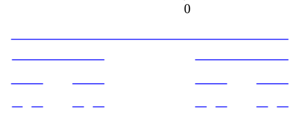
$$f(x) = \frac{1}{2^k},\frac{3}{2^k},\ldots,\frac{2^k-1}{2^k}$$

Define f(x) for $x\in C$=the Cantor set by
$f(x)$ = l.u.b.$\{ f(t) \mid t < x, t\in[0,1]-C\}$ and set $f(0)=0$.

Problem 8. Prove that f(x) is increasing, continuous and has derivative $f'(x)=0$ except on the Cantor set C, which has Lebesgue measure 0. But $f(0)=0$ and $f(1)=1$. So the fundamental theorem of calculus fails for this function:

$$1 = f(1) - f(0) = \int_0^1 f'(t)dt = 0.$$

Hint. If f were not continuous, f would have a jump. Then there would have to be an open subinterval of [0,1] containing no value of f. But the range of f contains all numbers of the form $(2n+1)/2^k$ in [0,1].



A person moving toward you according to the Devil's staircase law y=f(x) would cover a unit distance in a unit of time, but you might never see him or her move even if you were watching all the time. Thus Korevaar, *Mathematical Methods*, p. 404, calls this function "the almost perfect sneak."

In 1872 Weierstrass found functions that were continuous everywhere but nowhere differentiable. This shocked many famous mathematicians who had thought such a function impossible.

Hermite described these functions as a "dreadful plague."

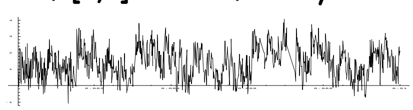
Poincaré wrote: "Yesterday, if a new function was invented it was to serve some practical end; today they are specially invented only to show up the arguments of our fathers, and they will never have any other use." Even as late as the 1960's, before "everyone" had a computer fast enough to graph these things, such examples were viewed as pathological monsters. Now there are thousands of websites with pictures of approximations of them.

The Weierstrass Nowhere Differentiable Function.

Weierstrass published this construction in 1872. It too is a fractal.

Defn. The Weierstrass function f. The definition involves parameters $\lambda>1$ and $1<s<2$. Then $f:[0,1]\to P$ is defined by

$$f(t)=\sum_{k=1}^{\infty}\lambda^{(s-2)k}\sin(\lambda^k t).$$

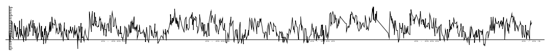


Theorem 3. If λ is large enough, the box dimension of the graph of f(t) in P^2 is s. By the graph of f, we mean the set of points (t,f(t)), for all t in [0,1].

There are lots of pictures of this graph on the web. For example
http://en.wikipedia.org/wiki/Weierstrass_function.
http://planetmath.org/encyclopedia/WeierstrassFunction.html
Or see http://www.math.washington.edu/%7Econroy/
for an animation zooming in on the Weierstrass function

$$f(t)=\sum_{k=1}^{\infty}\frac{1}{2^k}\sin(2^k t).$$

Problem 9. Show that $f(t)=\sum_{k=1}^{\infty}\lambda^{(s-2)k}\sin(\lambda^k t)$ is a continuous function on [0,1].



The following pictures of the Weierstrass function with $\lambda=2$ and s=1.9 were produced using the Mathematica commands below. I summed the first 150 terms of the Fourier series for the Weierstrass function and then plotted the result on the interval [0,1]. Why do 150 terms suffice? Look at the geometric series with x=2^(-.1). To get x^k<10^(-4), you need log(x^k)<log(10^-4) to get 4 significant digits. Then we want the next term which is the estimate for the remainder to be

k log(2^(-.1)) < -4 log 10 $\Leftrightarrow$ -.1 k log2 < -4 log 10
$\Leftrightarrow$ k > 40 log 10/log 2

40*Log[10]/Log[2]//N $\cong$132.877

The commands to sum the first 150 terms and make a function of it. Then plot on the interval [0,1].

```
fun[t_]:=fun[t]=Sum[2^(-k*(2-1.9))*Sin[t*2^k],{k,1,150}]
Plot[fun[t],{t,0,1}];
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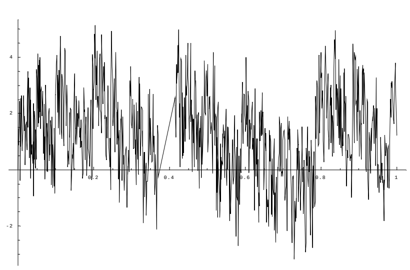
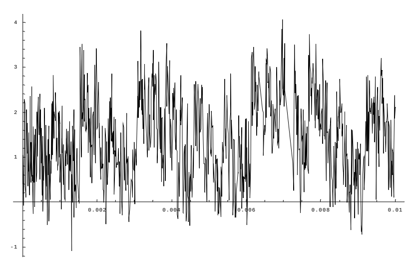
Weierstrass function $\lambda=2$ and s=1.9 plotted on the interval [0,1].



Same function plotted on the interval [0,0.01].



The essence of a fractal - the same behavior at all scales. This function also looks nowhere differentiable.

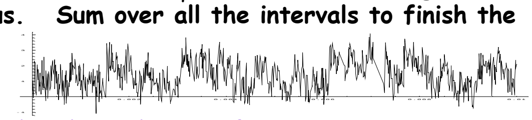
Preparations for computing the box dimension of a graph of a function.

Defn. $R_f[a,b]=\sup\{|f(t)-f(u)|,\ t,u \text{ in}[a,b]\}$.

Proposition 1. Let $f:[0,1]\to P$, be continuous. Suppose that $0<\delta<1$, and m is the least integer greater than or equal to $1/\delta$. If $N(\delta)$ is the number of squares of side δ that intersect the graph of f, we have

$$\delta^{-1}\sum_{k=1}^{m-1}R_f[k\delta,(k+1)\delta]\le N(\delta)\le 2m+\delta^{-1}\sum_{k=1}^{m-1}R_f[k\delta,(k+1)\delta].$$

Proof.
The number of squares of side δ in the column above the interval $[k\,\delta,(k+1)\,\delta]$ that intersect the graph of f is at least $R_f[k\delta,(k+1)\delta]/\delta$. It is at most $2+R_f[k\delta,(k+1)\delta]/\delta$, using the fact that f is continuous. Sum over all the intervals to finish the proof.

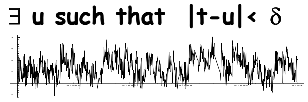


Problem 10. Fill in the details in this proof. Draw a picture.

Lemma 1. Let $f:[0,1]\to P$ be continuous. Assume the box dimension of the graph of f exists.

1) Suppose that $\exists$ $c>0$ and s, $1\le s\le 2$, such that
$|f(t)-f(u)|\le c|t-u|^{2-s}$, $\forall$ t,u in [0,1]. Then $\dim_B \text{graph} f\le s$.
This remains true if the inequality only holds for $|t-u|<\delta$, for some $\delta>0$.

2) Suppose that $\exists$ $c>0$, $\delta_0>0$ and s, $1\le s\le 2$ such that
$\forall$ t in [0,1], and $\forall$ δ with $0<\delta\le\delta_0$ $\exists$ u such that $|t-u|<\delta$ and $|f(t)-f(u)|\ge c\,\delta^{2-s}$.
Then $\dim_B \text{graph} f\ge s$.



Proof.
1) From the hypothesis of 1) we see that $R_f[a,b]\le c|a-b|^{2-s}$, for a,b in [0,1]. Using the notation of the proposition, we see that
$m<(1+\delta^{-1})$ and thus
$N(\delta)\le(1+\delta^{-1})(2+c\delta^{-1}\delta^{2-s})\le c_1\,\delta^{-s}$,
where c_1 is independent of δ. The result follows from the definition of box dimension.
2) Similarly the hypothesis of 2) implies that $R_f[a,b]\ge c|a-b|^{2-s}$.
Since $\delta^{-1}\le m$, we have from Proposition 1 that $N(\delta)\ge\delta^{-1}\,\delta^{-1}\,c\,\delta^{2-s}=c\delta^{-s}$. Again, the result follows from the definition of box dimension.

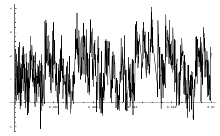
Problem 11. Suppose $f:[a,b] \rightarrow$P has a continuous derivative. Show $\dim_B \text{graph} f = 1$. Hint. Use the mean value theorem and Lemma 1.

Now we prove Theorem 3 which says that λ large enough implies that the box dimension of the graph of the Weierstrass function is s.

Given h in (0,1), let N be the integer such that

$$\lambda^{-(N+1)} \le h < \lambda^{-N}.$$

Use our definition $$f(t)=\sum_{k=1}^{\infty}\lambda^{(s-2)k}\sin(\lambda^k t).$$



Splitting sums up into 2 parts we get:

$$\left|f(t+h)-f(t)\right| = \left|\sum_{k=1}^{\infty}\lambda^{(s-2)k}\left[\sin\left(\lambda^k(t+h)\right)-\sin\left(\lambda^k t\right)\right]\right|$$

$$\le \sum_{k=1}^{N}\lambda^{(s-2)k}\left|\sin\left(\lambda^k(t+h)\right)-\sin\left(\lambda^k t\right)\right| + \sum_{k=N+1}^{\infty}\lambda^{(s-2)k}\left|\sin\left(\lambda^k(t+h)\right)-\sin\left(\lambda^k t\right)\right|$$

$$\le \sum_{k=1}^{N}\lambda^{(s-2)k}\lambda^k h + \sum_{k=N+1}^{\infty}2\lambda^{(s-2)k}.$$

Here we used the mean value theorem on the first N terms and that $|\sin x| \le 1$ for the rest of the terms.
Then we sum the geometric series to see that

$$\left|f(t+h)-f(t)\right| \le \frac{h\lambda^{(s-1)N}}{1-\lambda^{1-s}} + 2\frac{\lambda^{(s-2)(N+1)}}{1-\lambda^{s-2}} \le ch^{2-s},$$

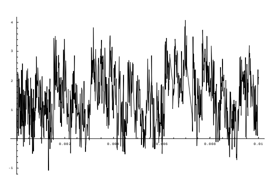
where c is independent of h.
This implies that the $\dim_B$(graph f) $\le$ s by the Lemma above.

To go the other way, take our sum defining f(t+h)-f(t) and split it into 3 parts, the first N-1 terms, the Nth term, and the rest. This implies that

$$\text{if} \quad \lambda^{-(N+1)} \le h < \lambda^{N}, \quad \text{then}$$

$$\left|f(t+h)-f(t)-\lambda^{(s-2)N}\sin(\lambda^N(t+h))-\sin\left(\lambda^N t\right)\right|$$

(*) $$\le \frac{\lambda^{(s-2)N-s+1}}{1-\lambda^{1-s}} + \frac{2\lambda^{(s-2)(N+1)}}{1-\lambda^{s-2}}.$$



Suppose that $\lambda>2$ is large enough that

$$\textit{right hand side of } (*) \textit{ is } < \frac{1}{20}\lambda^{(s-2)N}, \quad \forall N.$$

For $\delta<1/\lambda$, take N such that $\lambda^{-N} \le \delta < \lambda^{-(N-1)}$.

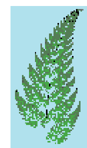
$\forall$ t, $\exists$ h, such that $\lambda^{-(N+1)} \le h < \lambda^{N}$

and such that $\left|\sin\left[\lambda^{N}(t+h)\right]-\sin\left[\lambda^{N}t\right]\right| > \frac{1}{10}.$

All this implies that

$$|f(t+h)-f(t)| \ge \frac{1}{20}\lambda^{(s-2)N} \ge \frac{1}{20}\lambda^{s-2}\delta^{2-s}.$$

It follows from the preceding Lemma that $\dim_B(\text{graph} f) \ge s$



Problem 12. Show that any function satisfying condition 2 of Lemma 1 with s>1 must be nowhere differentiable. It follows that the Weierstrass function is continuous but nowhere differentiable.

There is lots more to say about fractals, but we will stop here. I leave you with a picture of the Mandelbrot set from Wikipedia.