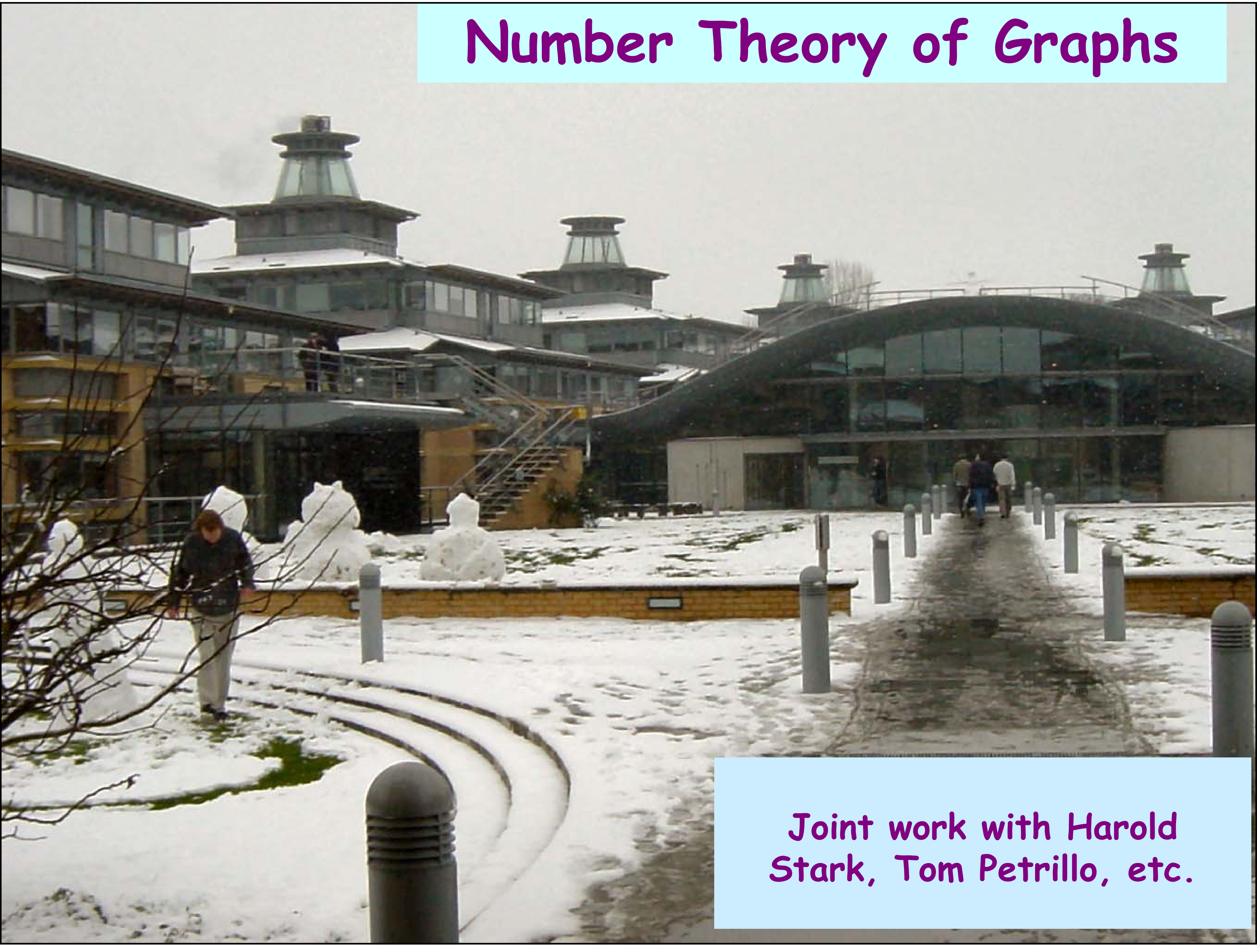
# Number Theory of Graphs



**Joint work with Harold Stark, Tom Petrillo, etc.**

$\mathbb{Q}(\sqrt{3})$
Number
&
Function
Fields
Riemann
Surfaces
Γ\H
Dynamical
Systems
Graphs
Tree of
Zetas
Dedekind
ζ
ζ
of
curve
over
$\mathbb{F}_q$
Selberg
ζ
Ruelle
ζ
Ihara
ζ

## The usual hypotheses:

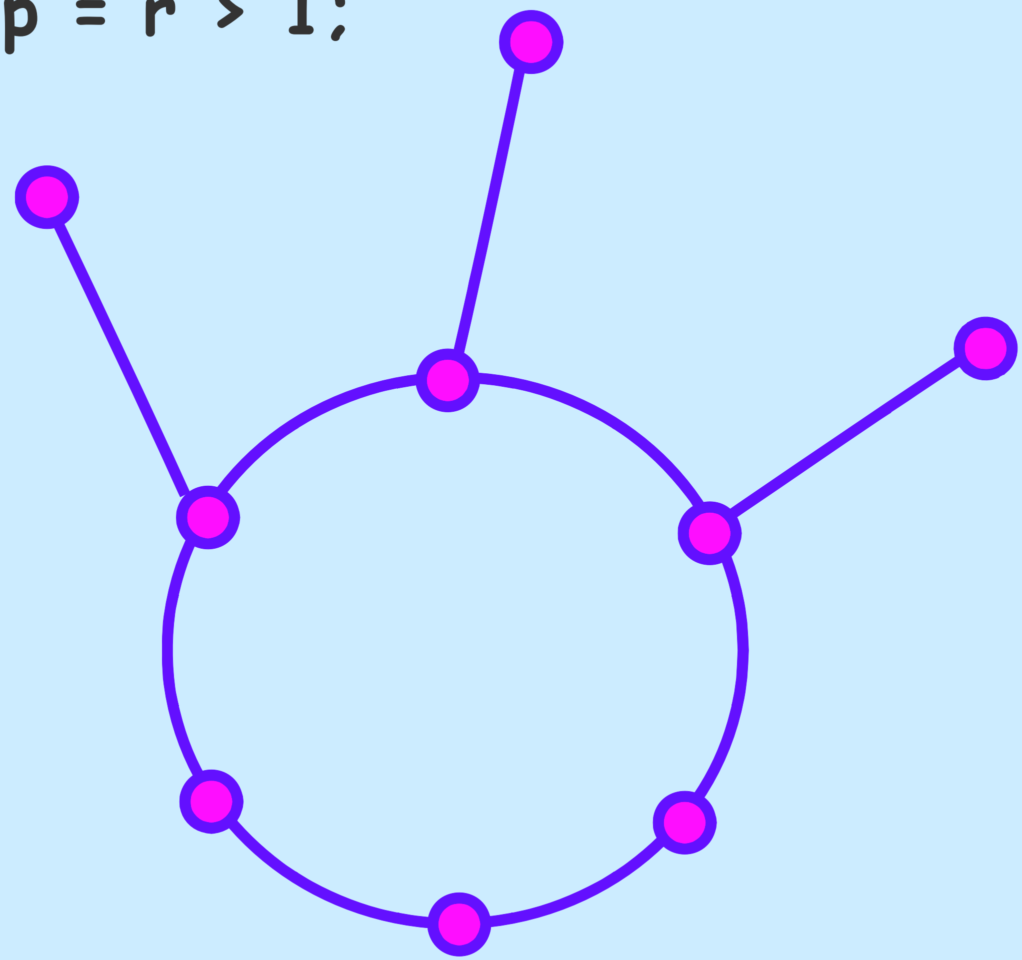
Graphs are finite,connected, not directed or weighted;

No degree 1 vertices;

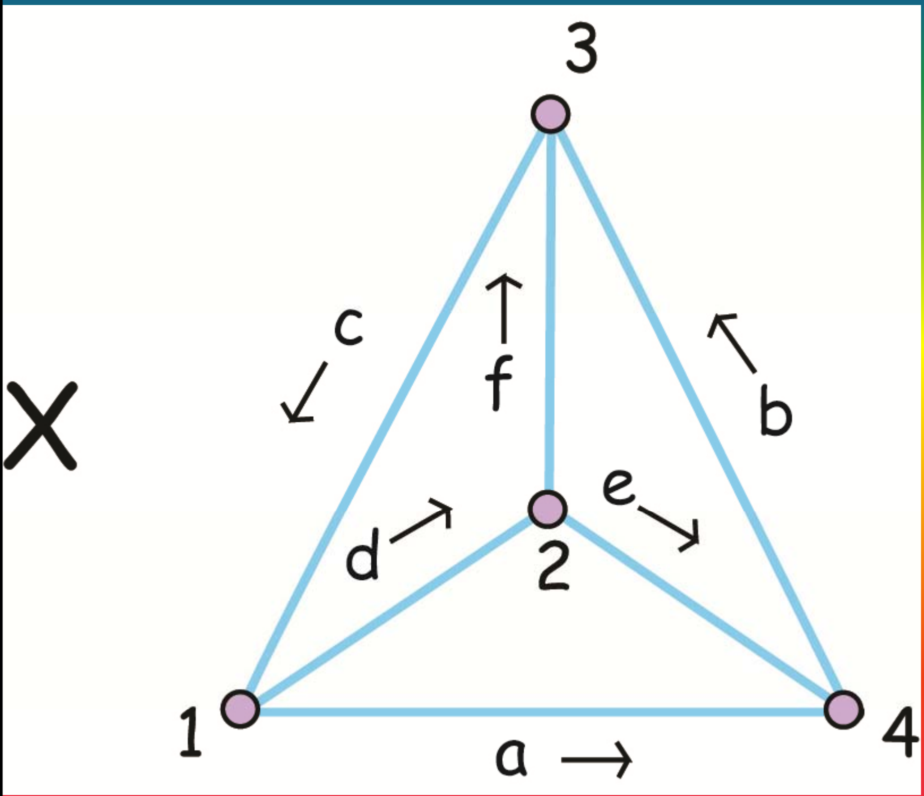
Rank fundamental group = r > 1;

$|E|-|V|=r-1>0$

Bad Graph



# Labeling Edges of Graphs




**Orient the m edges arbitarily. Label them as follows. Here the inverse edge has opposite orientation.**

**$e_1=a$, $e_2=b$, …, $e_6=f$;**

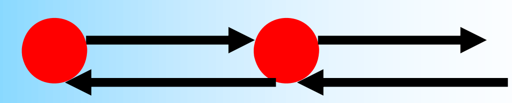
**$e_7=a^{-1}$, $e_8=b^{-1}$, …, $e_{12}=f^{-1}$**

# Primes in Graphs

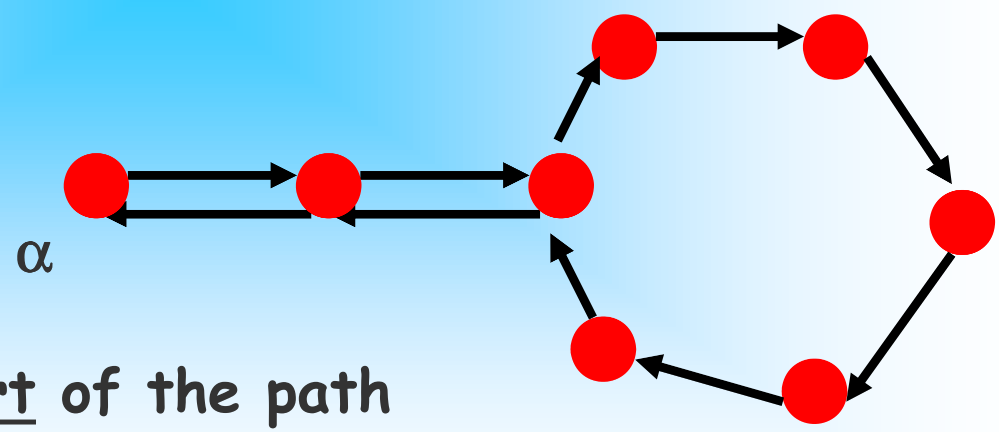
are equivalence classes [C] of closed backtrackless tailless primitive paths $C=e_1 e_2 \ldots e_s$ , $s=\nu(C)$

**DEFINITIONS** equivalence class: change starting point

backtrack $e_{i+1}=(e_i)^{-1}$



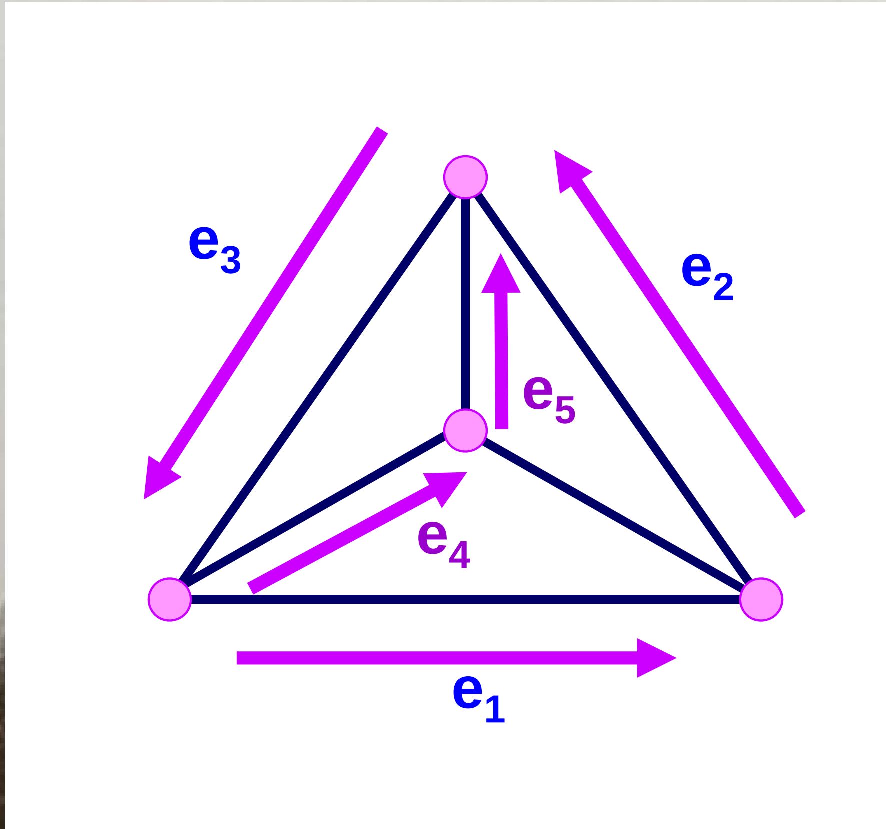
tail $e_s=(e_1)^{-1}$




Here α is the <u>start</u> of the path

non-primitive: go around path more than once

# EXAMPLES of Primes in a Graph




$[C] = [e_1 e_2 e_3]$

$[D] = [e_4 e_5 e_3]$

$[E] = [e_1 e_2 e_3 e_4 e_5 e_3]$

$\nu(C)$ = # edges in $C$

$\nu(C)=3$, $\nu(D)=3$, $\nu(E)=6$

E=CD
another prime $[C^n D]$, n=2,3,4, …
infinitely many primes

# Ihara Zeta Function

$$\zeta(u,X)=\prod_{\substack{[C]\\ prime}}\left(1-u^{\nu(C)}\right)^{-1}$$

|u| small enough

**Ihara's Theorem (Bass, Hashimoto, etc.)**

A = adjacency matrix of X

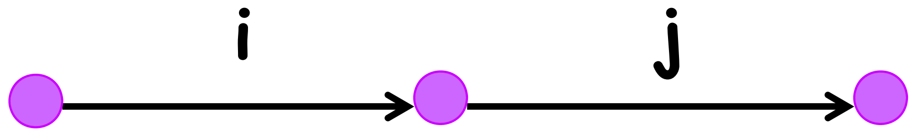
Q = diagonal matrix; jth diagonal entry
= degree jth vertex -1;

r = rank fundamental group = |E|-|V|+1

$$\zeta(u,X)^{-1}=(1-u^2)^{r-1}\det(I-Au+Qu^2)$$

# The Edge Matrix $W_1$

**Define** $W_1$ to be the $2|E|\times 2|E|$ matrix with i j entry 1 if edge i feeds into edge j, and $j\neq i^{-1}$ otherwise the i j entry is 0.




**Theorem.** $\zeta(u,X)^{-1}=\det(I-W_1u)$.

**Corollary.** The poles of Ihara zeta are the reciprocals of the eigenvalues of $W_1$.

The pole R of zeta is the closest to the origin. By Landau's theorem on power series with non-negative coefficients the pole R is simple. Moreover, R=1/Perron-Frobenius eigenvalue of $W_1$; i.e., the largest eigenvalue which has to be positive real and simple if the graph satisfies our usual hypotheses.

Note: the correspondence between theorems in analysis and linear algebra.

# Remarks for q+1-Regular Graphs Mostly

- Riemann Hypothesis, (non-trivial poles on circle of radius $q^{-1/2}$ center 0), means graph is Ramanujan i.e., non-trivial spectrum of adjacency matrix is contained in the interval $(-2\sqrt{q}, 2\sqrt{q})$ = spectrum for the universal covering tree [see Lubotzky, Phillips & Sarnak, *Combinatorica*, 8 (1988)].

- Ihara zeta has functional equations relating value at u and 1/(qu), q=degree - 1
Set $u=q^{-s}$ to get s goes to 1-s.

Alon conjecture says RH is true for "most" regular graphs but can be false. See Joel Friedman's website (www.math.ubc.ca/~jf) for a paper proving that a random regular graph is almost Ramanujan.

For irregular graphs there is an analog of the Riemann Hypothesis and of the Alon conjecture but no functional equation.
See my book also Friedman website.

The Prime Number Theorem (graphs satisfying usual hypotheses)

$\pi_X(m)$ = number of primes [C] in X of length m

$\Delta$ = g.c.d. of lengths of primes in X

R = radius of largest circle of convergence of $\zeta(u,X)$

R=1/q, if graph is q+1-regular

If $\Delta$ divides m, then

$$\pi_X(m) \sim \Delta R^{-m}/m, \quad \text{as } m \to \infty.$$

The proof comes from explicit formulas for $\pi_X(m)$ by analogous method to that of Rosen, *Number Theory in Function Fields*, page 56.

# Explicit Formulas

$N_m$=# closed paths C in X of length m with no backtracks, no tails;
$\pi_X(m)$ = number of primes [P] in X of length m

$$u\frac{d\log\zeta(u,X)}{du}=\sum_{m=1}^{\infty}N_m u^m$$

$$u\frac{d}{du}\log\zeta(u,X)=\sum_{m\geq 1}\left(\sum_{d|m}d\pi(d)\right)u^m$$

$$N_m=\sum_{d|m}d\pi(d)$$

$$\pi(m)=\frac{1}{m}\sum_{d|m}\mu\left(\frac{m}{d}\right)N_d$$

$$N_m=(r-1)\left((-1)^{m-1}-1\right)-\sum_{\lambda\in Spec(A)}\left(\alpha_\lambda^m+\beta_\lambda^m\right)$$

$$1-\lambda u+qu^2=(1-\alpha_\lambda u)(1-\beta_\lambda u)$$

For q+1-regular graphs

$$N_m=\sum_{\lambda\in SpecW_1}\lambda^m$$

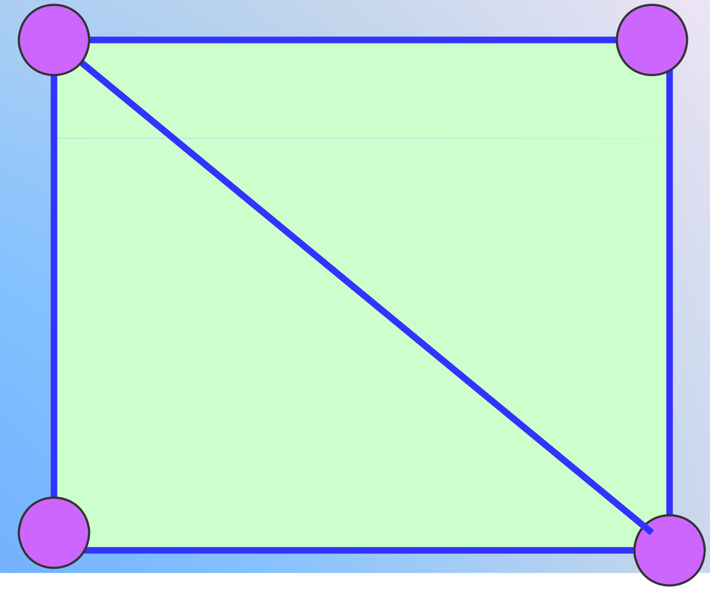
# Weil type explicit formulas

$$\sum_{\substack{\rho \\ pole\ \zeta}} \rho h(\rho) = \sum_{m \geq 1} N_m \hat{h}(m)$$

$$\hat{h}(m) = \oint_{|u|=a} u^m h(u) du$$

Here we need $0<a<R$ and $h$ very nice.
Example. $h(u)=u^{-n-1}$, n-1,2,3,…. This leads to one of our earlier formulas.

# $N_m$ for $K_4$-e



$x \, d/dx \ \log \zeta(x, K_4\text{-}e)$

$= 12x^3 + 8x^4 + 24x^6 + 28x^7 + 8x^8 + 48x^9 + 120x^{10}$

$+ \ . \ . \ .$

$\pi(3)=4$ $\pi(4)=2$

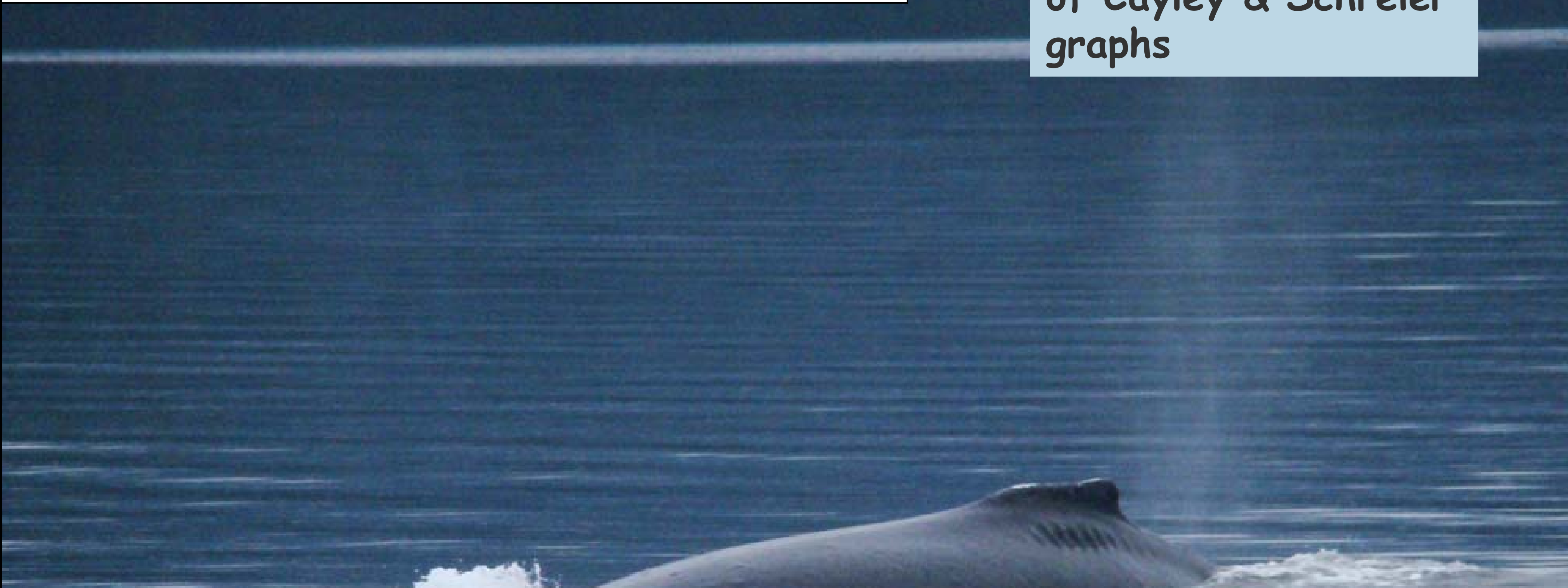
$\pi(5)=0$ $\pi(6)=2$

$$N_m = \sum_{d|m} d\pi(d)$$

# Artin L-Functions of Graphs

# Graph Galois Theory

Gives generalization of Cayley & Schreier graphs



Graph Y an unramified covering of Graph X means (assuming no loops or multiple edges) $\pi: Y \to X$ is an onto graph map such that

for every $x \in X$ & for every $y \in \pi^{-1}(x)$,

$\pi$ maps the points $z \in Y$ adjacent to y 1-1, onto the points $w \in X$ adjacent to x.

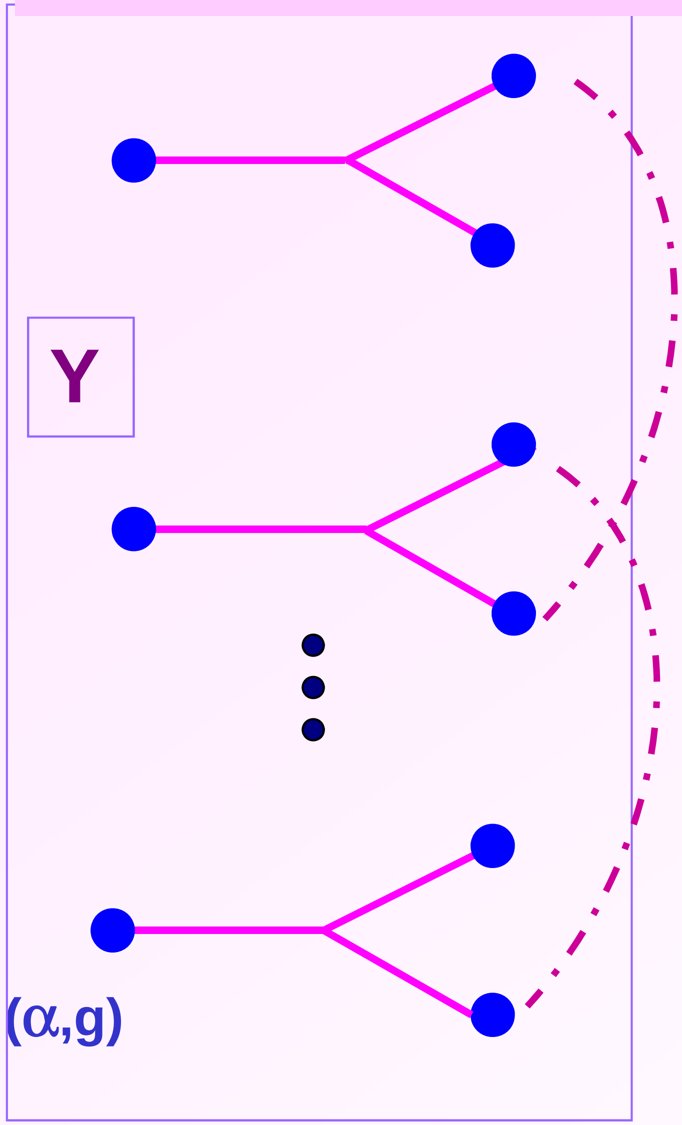
Normal d-sheeted Covering means:

$\exists$ d graph isomorphisms $g_1, \ldots, g_d$ mapping $Y \to Y$

such that $\pi\, g_j(y) = \pi(y)$, $\forall\, y \in Y$

Galois group $G(Y/X) = \{ g_1, \ldots, g_d \}$.

## How to Label the Sheets of a Galois Covering

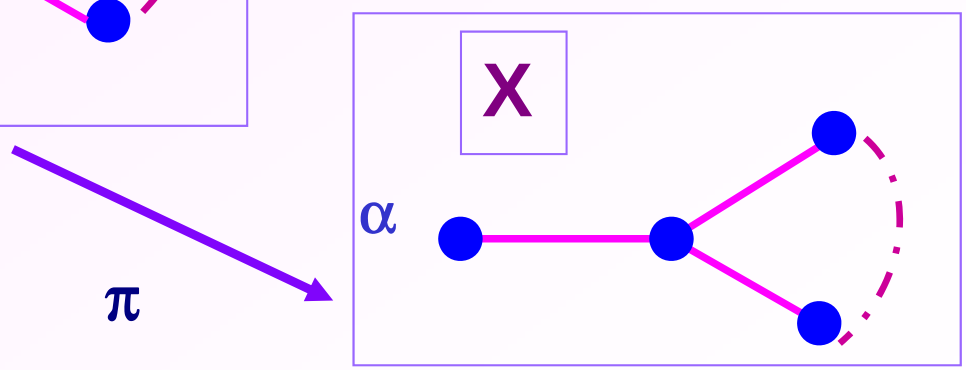



First pick a spanning tree in X (no cycles, connected, includes all vertices of X).

Second make $n=|G|$ copies of the tree T in X. These are the sheets of Y. Label the sheets with $g \in G$. Then

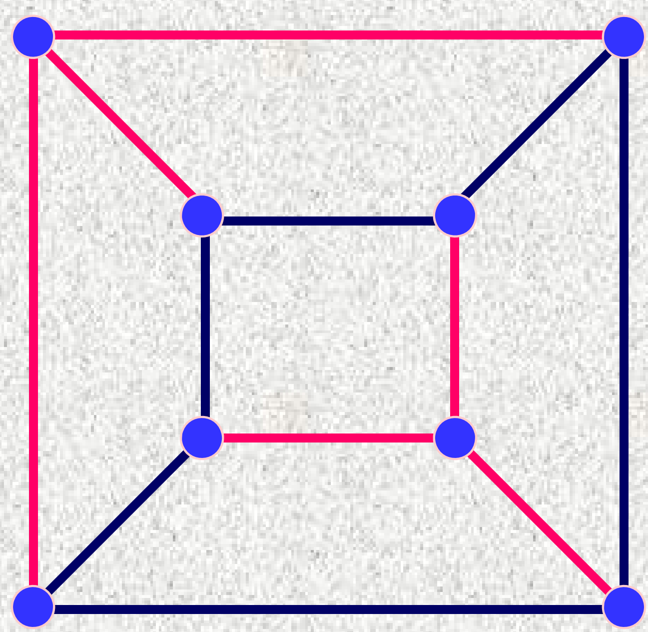
$$g(\text{sheet } h)=\text{sheet}(gh)$$
$$g(\alpha,h)=(\alpha,gh)$$

$$g(\text{path from } (\alpha,h) \text{ to } (\beta,j)) = \text{path from } (\alpha,gh) \text{ to } (\beta,gj)$$

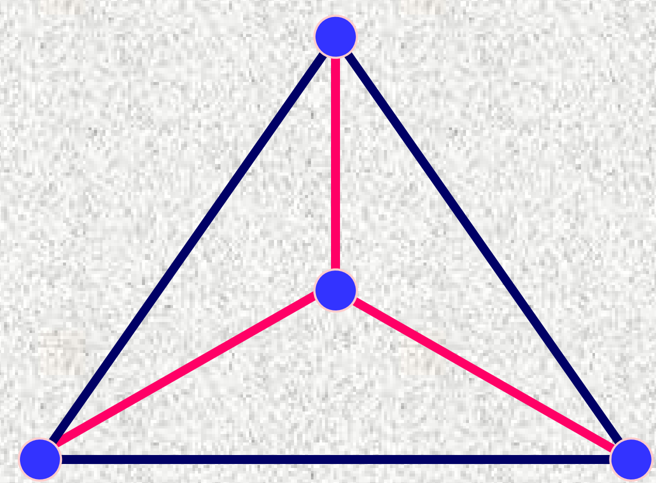



Given G, get examples Y by giving permutation representation of generators of G to lift edges of X left out of T.

# Example 1. Quadratic Cover



Cube covers Tetrahedron

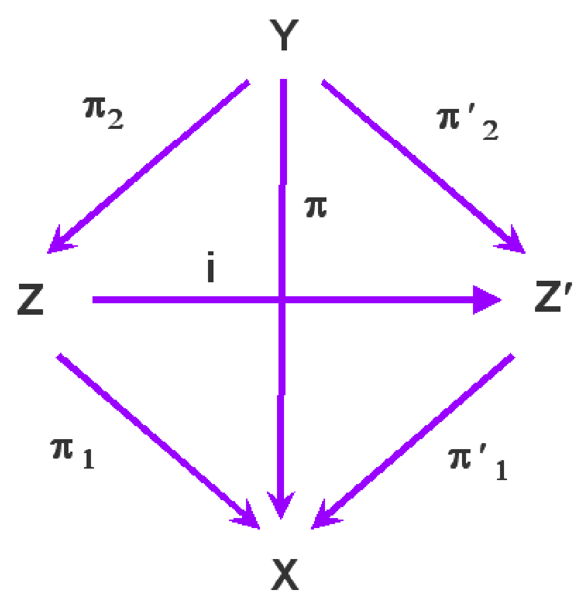


Spanning Tree in X is red.
Corresponding sheets of Y are also red

**Evil Exercise.** State and prove the fundamental theorems of graph Galois theory. See my book for some answers.
www.math.ucsd.edu/~aterras/newbook.pdf
about to appear from Cambridge U. Press

**Definition.** A graph Z is an **intermediate covering** to the covering Y/X with projection map $\pi: Y \to X$ means there are projection maps $\pi_1: Z \to X$ and $\pi_2: Y \to Z$ so that $\pi = \pi_1 \circ \pi_2$.

**Definitions.** 2 intermediate covers Z and Z' to Y/X are **covering isomorphic** (conjugate) if there is an graph isomorphism $i: Z \to Z'$ such that $\pi'_1 \circ i = \pi_1$ .




For us to say **Z=Z'** we need also $i \circ \pi_2 = \pi'_2$ .

# Fundamental Theorems of Graph Galois Theory

**Theorem.** Suppose Y/X is an unramified normal covering with Galois group G=G(Y/X).

1) We have a 1-1 correspondence between subgroups H of G and graphs Z(H) intermediate to Y/X. Each intermediate graph Z to Y/X corresponds to some subgroup H(Z) of G.

   Write $Z \leftrightarrow H$ when intermediate graph Z corresponds to subgroup H.

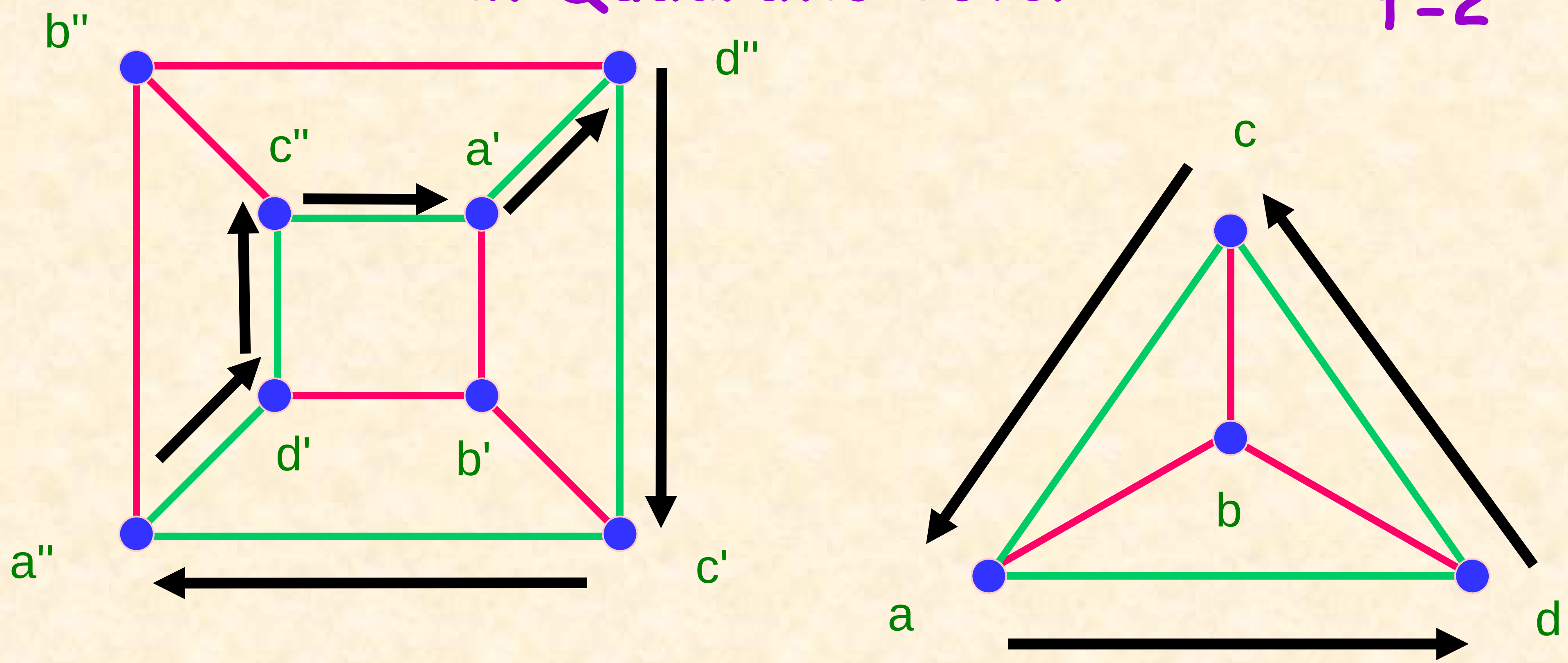
2) If $Z_i \leftrightarrow H_i$, for i=1,2, then $Z_1$ is intermediate to $Y/Z_2$ iff $H_1 \subset H_2$.

3) 2 intermediate graphs to Y/X correspond to conjugate subgroups of G iff they are covering isomorphic.

4) If Z is an intermediate covering to Y/X, then Z is itself a normal covering of X iff the corresponding subgroup H of G is a normal subgroup. Then $Gal(Z/X) \cong G/H$.

# Example of Splitting of Primes in Quadratic Cover

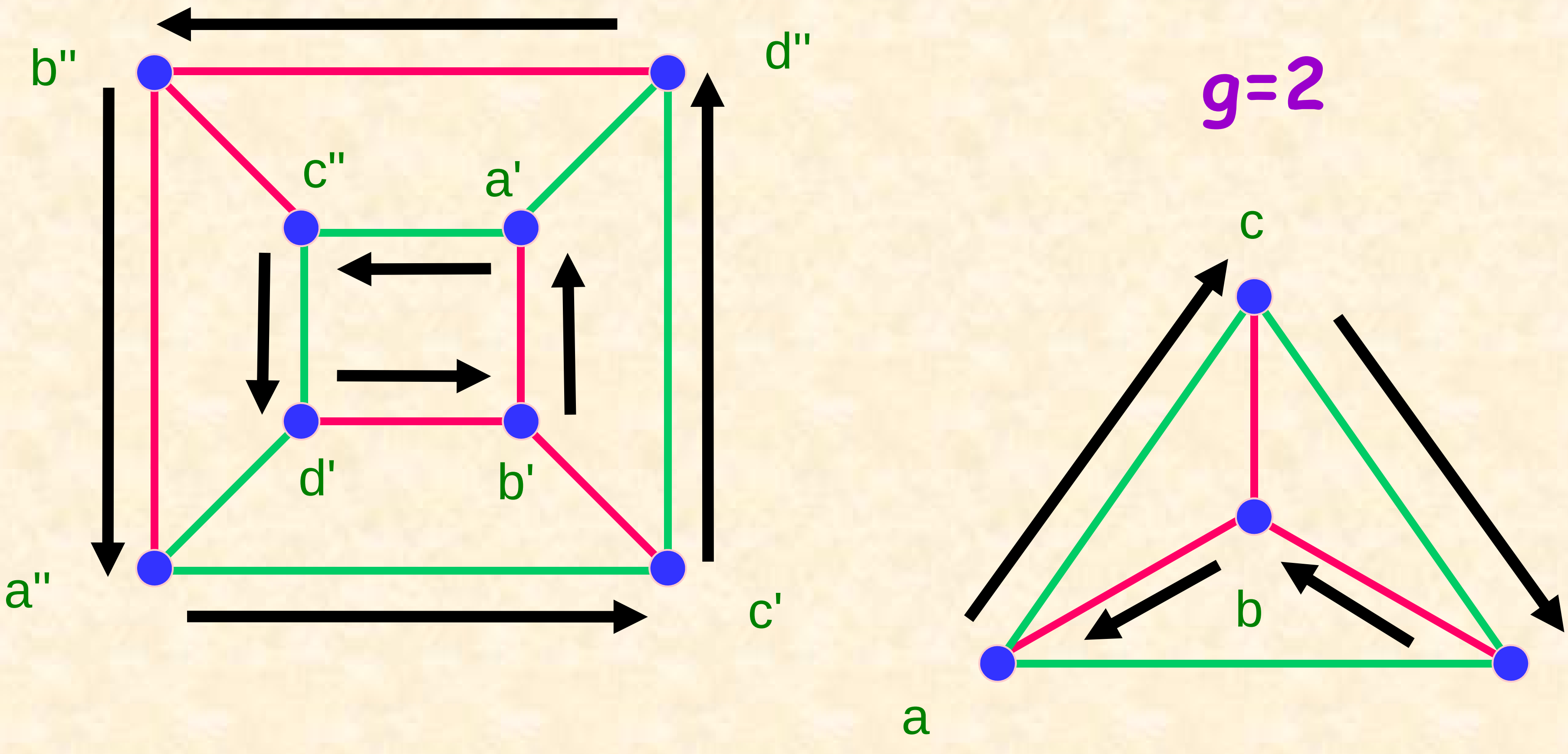
f=2




Picture of Splitting of Prime which is inert; i.e., f=2, g=1, e=1

1 prime cycle D above, & D is lift of $C^2$.

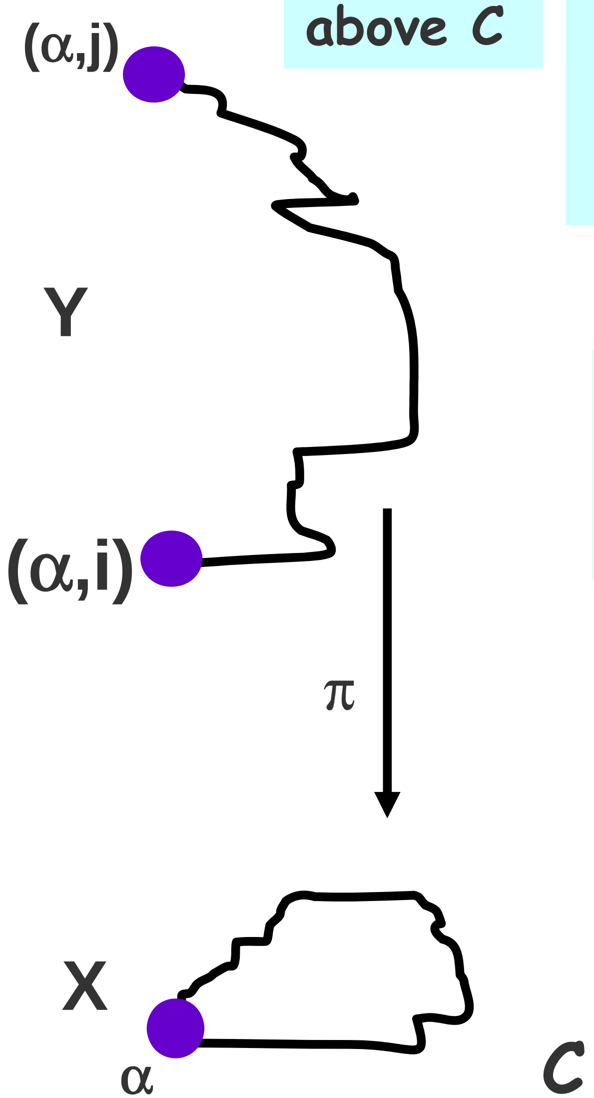
# Example of Splitting of Primes in Quadratic Cover




Picture of Splitting of Prime which splits completely;
i.e., f=1, g=2, e=1
2 primes cycles above

# Frobenius Automorphism

D a prime above C




$$\mathrm{Frob}(D) = \left(\frac{Y/X}{D}\right) = ji^{-1} \in G=\mathrm{Gal}(Y/X)$$

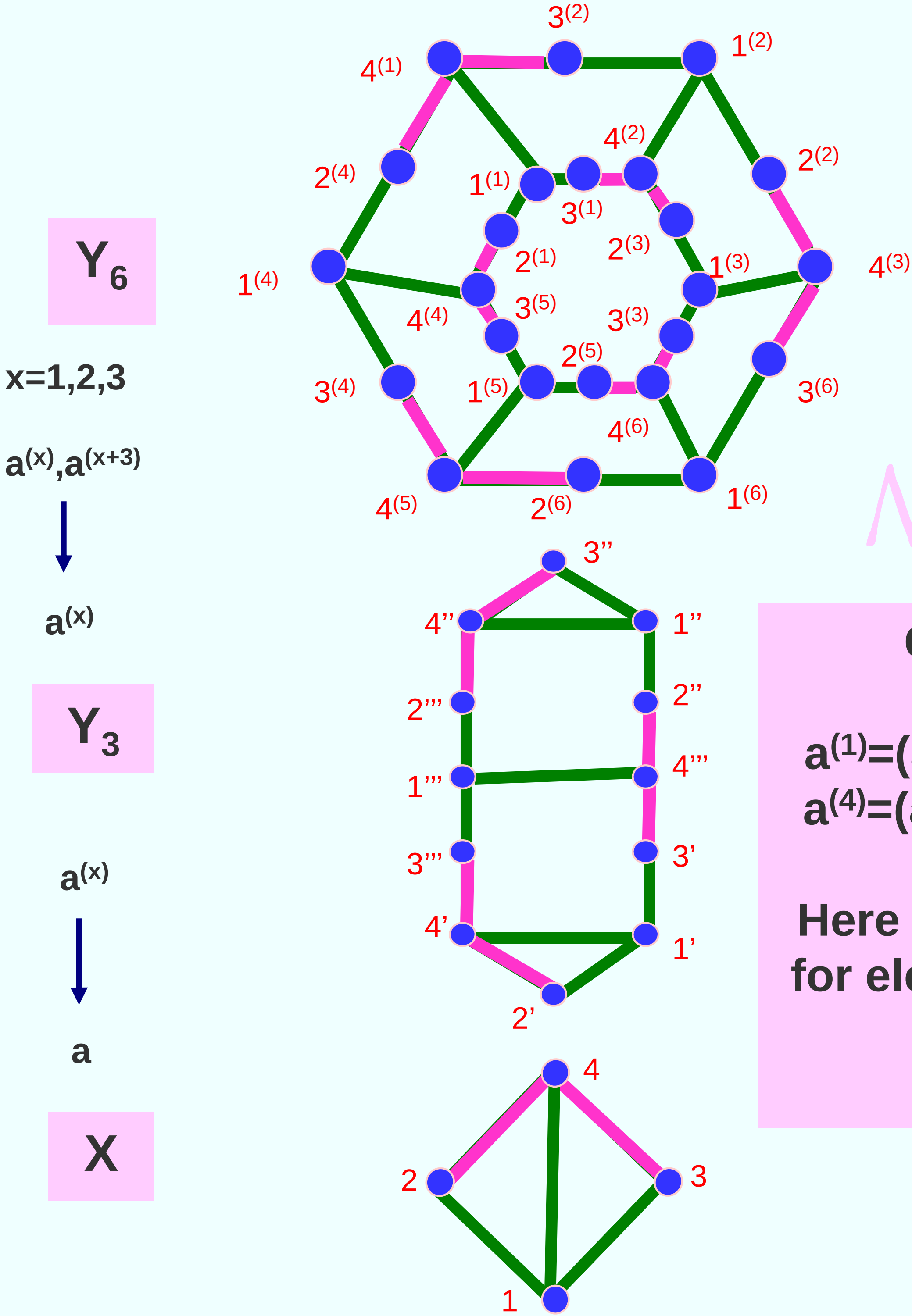
where $ji^{-1}$ maps sheet i to sheet j

The unique lift of C in Y starts at $(\alpha,i)$ ends at $(\alpha,j)$

Normalized Frobenius $\sigma(C)=g$ means start lift on sheet 1 and end on sheet g

Exercise: Compute $\sigma(C)$ on preceding pages, $G=\{1,g\}$.

# Galois Cover of Non-Normal Cubic




**G=$S_3$, H={(1),(23)} fixes $Y_3$.**

**$a^{(1)}$=(a,(1)), $a^{(2)}$=(a,(13)), $a^{(3)}$=(a,(132), $a^{(4)}$=(a,(23)),$a^{(5)}$=(a,(123)),$a^{(6)}$=(a,(12)).**

**Here we use standard cycle notation for elements of the symmetric group.**

# Prime Splitting Completely in $Y_3/X$

path in X (list vertices) 14312412431

Length is 10

f=1, g=3 3 primes above in $Y_3$

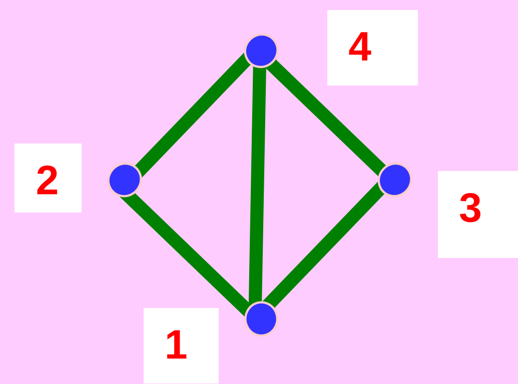
1'4'3'''1'''2'''4''1''2''4'''3'1'

1''4''3''1''2''4'''1'''2'''4''3''1''

1'''4'''3'1'2'4'1'2'4'3'''1'''

Frobenius trivial $\Rightarrow$ density 1/6

Question: Is 10 the minimal length of a prime of X splitting completely in $Y_6/X$?



## Properties of Frobenius

1) Replace $(\alpha, i)$ with $(\alpha, hi)$. Then $\mathrm{Frob}(D) = \left(\frac{Y/X}{D}\right) = ji^{-1}$

is replaced with $hji^{-1}h^{-1}$. Or replace D with different prime above C and see that

Conjugacy class of $\mathrm{Frob}(D) \in \mathrm{Gal}(Y/X)$ unchanged.

2) Varying $\alpha$=start of C does not change Frob(D).

3) $\mathrm{Frob}(D)^j = \mathrm{Frob}(D^j)$ .

## Artin L-Function

$\rho$ = representation of $G=\mathrm{Gal}(Y/X)$, u complex, |u| small

$$L(u,\rho,Y/X) = \prod_{[C]} \det\left(1-\rho\left(\frac{Y/X}{D}\right)u^{\nu(C)}\right)^{-1}$$

[C]=primes of X

$\nu(C)$=length C, D a prime in Y over C

# Properties of Artin L-Functions

1) $L(u,1,Y/X) = \zeta(u,X)$ = **Ihara zeta function** of X

2)
$$\zeta(u,Y)=\prod_{\rho} L(u,\rho,Y/X)^{d_\rho}$$

product over all irreducible representations of G,

$d_\rho$=degree $\rho$

# Det(I-uW$_1$) formula for Artin L-Functions

Set $W_1=(w_{ef})$ and call the normalized Frobenius automorphism of an edge $\sigma(e)$.

$$\left(W_{1,\rho}\right)_{ef}=\left(w_{ef}\,\rho(\sigma(e))\right)$$

$$L(u,\rho,Y/X)=\det(I-uW_{1,\rho})$$

# Ihara Theorem for L-Functions

$$L(u,\chi_\rho,Y/X)^{-1}$$

$$=(1-u^2)^{(r-1)d_\rho}\det(I'-A'_\rho u+Q'u^2)$$

r=rank fundamental group of X = |E|-|V|+1

ρ= representation of G = Gal(Y/X), d = $d_\rho$ = degree ρ

**Definitions.** nd×nd matrices A', Q', I', n=|X|

nxn matrix A(g), g ∈ Gal(Y/X), has entry for α,β∈X given by

$(A(g))_{\alpha,\beta}$ = # { edges in Y from (α,e) to (β,g) },

e=identity ∈ G.

$$A'_\rho=\sum_{g\in G}A(g)\otimes\rho(g)$$

Q = diagonal matrix, jth diagonal entry = $q_j$

= (degree of jth vertex in X)-1,

Q' = $Q\otimes I_d$ , I' = $I_{nd}$ = identity matrix.

EXAMPLE

**Y=cube, X=tetrahedron:** $G = \{e,g\}$

representations of $G$ are 1 and ρ: $\rho(e) = 1$, $\rho(g) = -1$

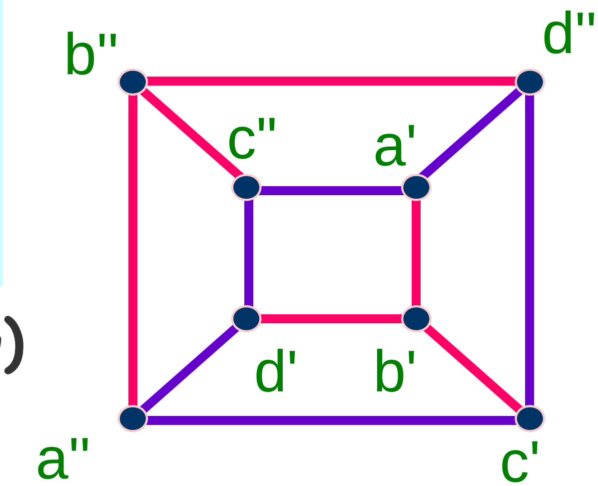
$A(e)_{u,v}$ = #{ length 1 paths u' to v' in Y}

$A(g)_{u,v}$ = #{ length 1 paths u' to v'' in Y}

(u,e)=u'

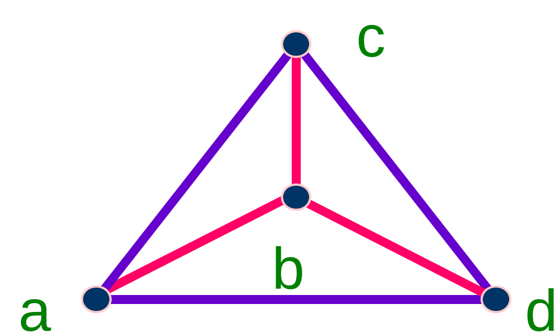
(u,g)=u''

$$A(e) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad A(g) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

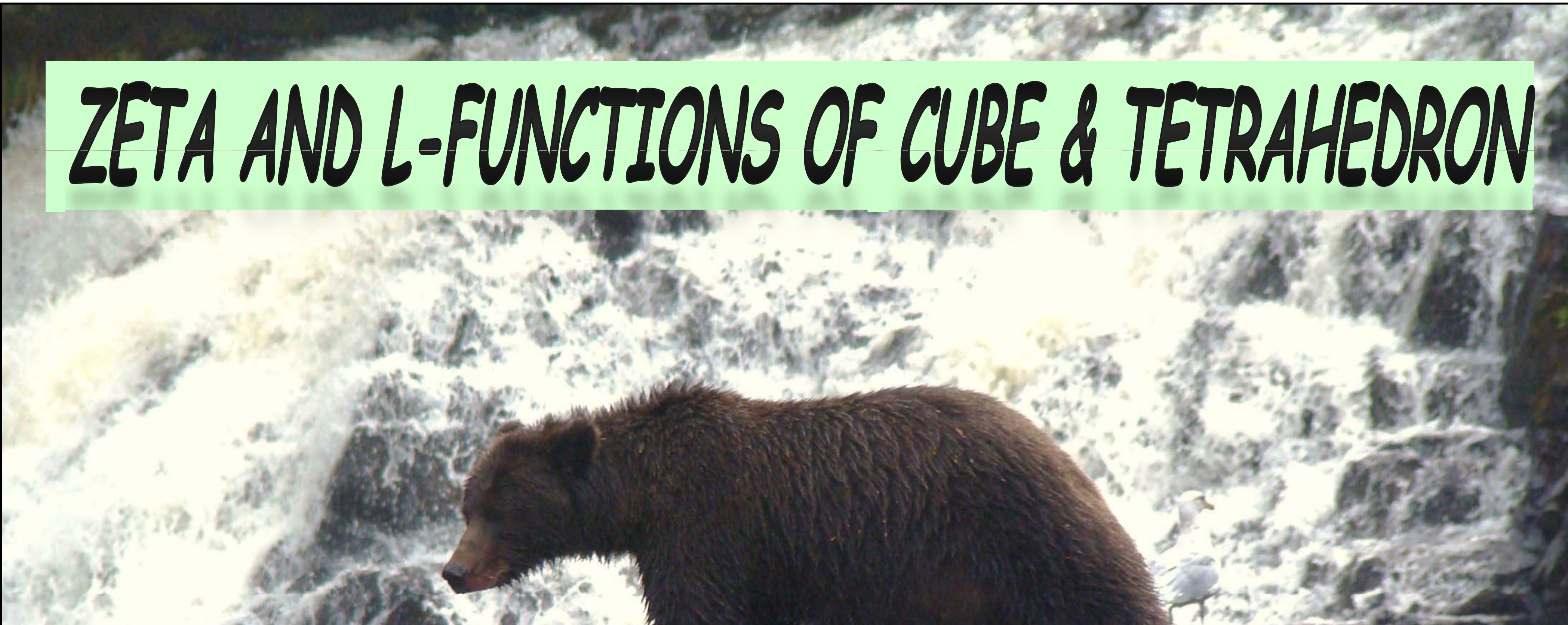



$A'_1 = A$ = adjacency matrix of X = $A(e)+A(g)$

$$A'_\rho = A(e) - A(g) = \begin{pmatrix} 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$



# ZETA AND L-FUNCTIONS OF CUBE & TETRAHEDRON



- $\zeta(u,Y)^{-1} = L(u,\rho,Y/X)^{-1}\,\zeta(u,X)^{-1}$

- $L(u,\rho,Y/X)^{-1} = (1-u^2)(1+u)(1+2u)(1-u+2u^2)^3$

- $\zeta(u,X)^{-1} = (1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3$

# Explicit Formulas for L-functions (T. Petrillo)

Assume Y/X Galois with G(Y/X)=G. Let $\sigma(C)$ denote the normalized Frobenius automorphism.

$$N_{m,g} = \#\{C \mid C \text{ is closed, no backtrack, no tail, length m, } \sigma(C) = g\}$$

$$u\frac{d}{du}LogL(u,\rho,Y/X) = \sum_{m\geq 1}\sum_{g\in G(Y/X)} N_{m,g}\chi_\rho(g)u^m$$

$$\pi(m,g) = \#\{[P] \mid [P] \text{ prime, length}(P) = m,\ \sigma(P) = g\}$$

We want to find the minimum length of a prime that splits completely in Y. This means we want $\sigma(P)=e$ the identity of G.

$$u\frac{d}{du}LogL(u,\rho,Y/X) = \sum_{m\geq 1}\sum_{g\in G(Y/X)}\sum_{d\mid m} d\pi(d,g)\chi_\rho(g^{m/d})u^m$$

$$u\frac{d}{du}LogL(u,\rho,Y/X) = \sum_{m\geq 1}\sum_{\lambda\in Spec(W_{1,\rho})} \lambda^m u^m$$

To get rid of the sum over G, use a trick that goes back to Dirichlet. Sum the formulas times conjugate of $\chi_\rho(g)$. Use the **dual orthogonality relations** for characters of irreducible unitary representations of G=G(Y/X).

$$\sum_{\rho \in \hat{G}} \overline{\chi_\rho(g)}\chi_\rho(s) = \begin{cases} \frac{|G|}{|\{s\}|}, & if \;\; g \in \{s\} = \{xsx^{-1} \mid x \in G\} \\ 0, & otherwise \end{cases}$$

This leads to the formulas:

$$\frac{1}{|G|}\sum_{\rho \in \widehat{G}} \overline{\chi_\rho(s)}\; u\frac{d}{du}\log L(u,\rho,Y/X) = \sum_{m \geq 1} N_{m,s}u^m$$

$$N_{m,s} = \frac{1}{|G|}\sum_{\rho \in \widehat{G}} \overline{\chi_\rho(s)} \sum_{\lambda \in Spec(W_{1,\rho})} \lambda^m$$

With this sort of information one obtains the **Chebotarev density theorem**.

Define the **analytic density** of a set S of primes in the graph X to be:

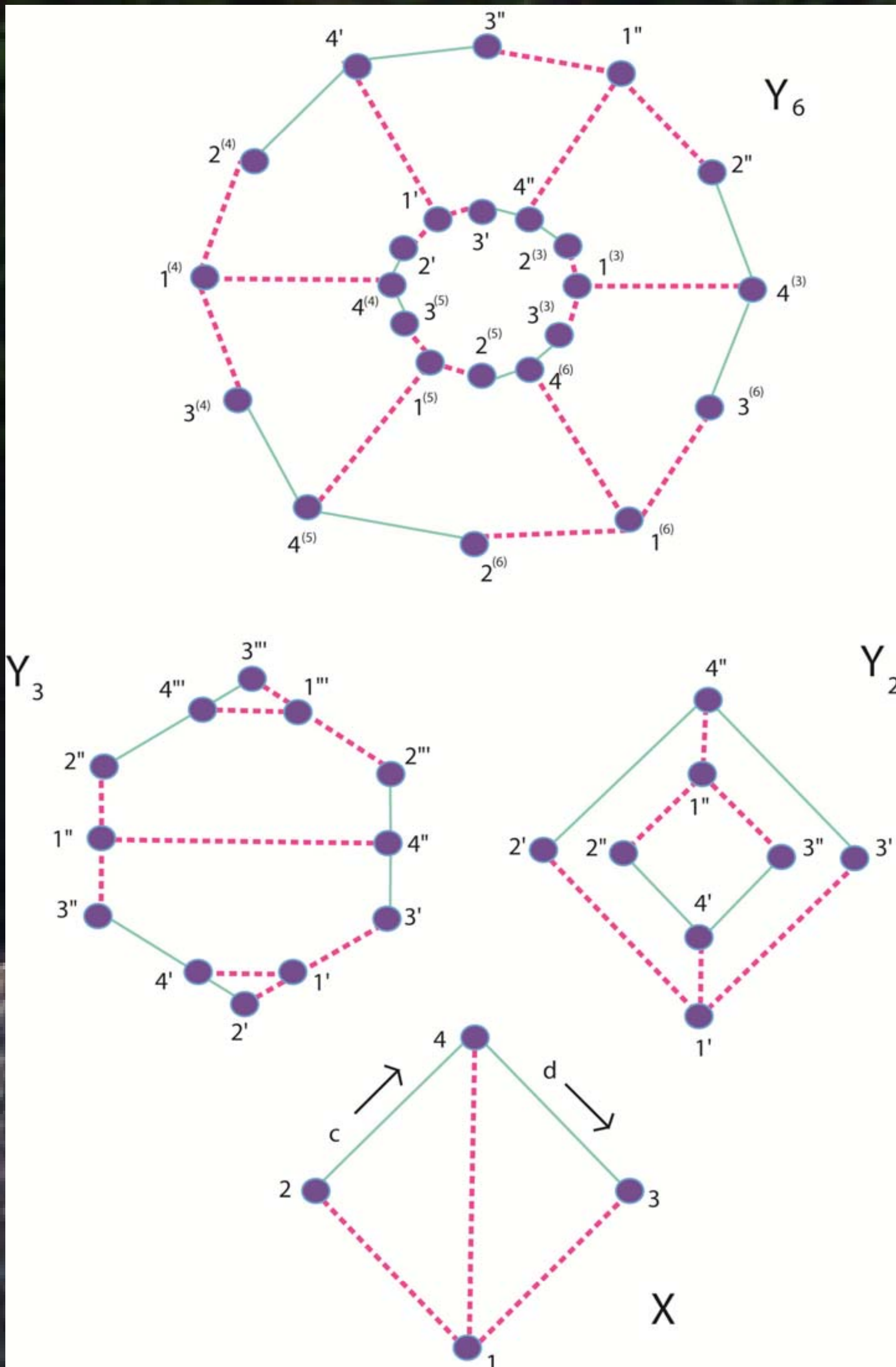
$$\delta(S) = \lim_{u \to R_X -} \frac{\sum_{[C] \in S} u^{\nu(C)}}{\sum_{[C]} u^{\nu(C)}}$$

Assume the graphs satisfy our usual hypotheses.

If Y/X is normal and {g} is a fixed conjugacy class in the Galois group G=G(Y/X), then

$$\delta\{[P] \mid \{\sigma(P)\} = \{g\}\} = \frac{|\{g\}|}{|G|}$$

We can also work out examples such as $Y_6/X$




Question: Is 10 the minimal length prime of X splitting completely in $Y_6/X$?

$$u\frac{d}{du}\log L(u,1,Y/X) = 12u^3+8u^4+24u^6+28u^7+8u^8+48u^9+120u^{10}+\ldots$$

$$u\frac{d}{du}\log L(u,\chi,Y/X) = -12u^3+8u^4+24u^6-28u^7+8u^8-48u^9+120u^{10}+\ldots$$

$$u\frac{d}{du}\log L(u,\rho,Y/X) = -8u^4+12u^6-8u^8-60u^{10}+\ldots$$

$$\frac{1}{|G|}\sum_{\rho\in\widehat{G}}\chi_\rho(e)u\frac{d}{du}\log L(u,\rho,Y/X) = \sum_{m\geq 1} N_{m,e}u^m = 12u^6+20u^{10}+\ldots$$

So we have to look for paths of length 6 and 10. But the paths of length 6 that lift to closed paths in $Y_6$ are from $C^2$ not primes. The the minimal length of a prime that splits completely in $Y_6/X$ is indeed 10.

References: 3 papers with Harold Stark in Advances in Math.

See my website for draft of a book:

www.math.ucsd.edu/~aterras/newbook.pdf

Soon to be a real book from Cambridge U. Press

Tom Petrillo, Ph.D. Thesis, UCSD, 2010