Nonhomogeneous Systems (7.9)

As in Chapter 3, Sections 6 and 7, we have a matrix version of undetermined coefficients and variation of parameters. I will only discuss the 2nd. It has the advantage that it always works – not to mention the lack of guessing.

Suppose our ODE is:
$$\frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{g}(t)$$

Assume A(t) is an nxn matrix of functions and g(t) (the inhomogeneous term is a vector of functions in n-space).

Put the n fundamental solutions $v_1,\ldots,v_n$ to the homogenous equation
$$\frac{d\vec{x}}{dt} = A(t)\vec{x}$$
in as the columns of the fundamental matrix $\Psi(t)$. Then it is easy to check that a solution to the nonhomogeneous ODE above is

$$\vec{x} = \Psi(t)\vec{c} + \Psi(t)\int \Psi^{-1}(t)\vec{g}(t)dt$$

Note that here we are integrating a vector function meaning that you integrate each coordinate separately. It is an algebraically neat formula, but it conceals inverting an nxn matrix of functions and then integrating the product of the inverse matrix and the inhomogeneous term g. We can probably only manage 2x2 matrices by hand.

Example. (7.9.6)

Solve X'=Ax+g, where

$$A = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \quad and \quad \vec{g} = \begin{pmatrix} t^{-1} \\ 4+2t^{-1} \end{pmatrix}, \quad t > 0.$$

First find the eigenvalues of A.

$$\det(A - \lambda I) = \det\begin{pmatrix} -4-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix}$$

$$= (4+\lambda)(1+\lambda) - 4 = \lambda^2 + 5\lambda.$$

So the eigenvalues of A are 0 and -5.

Find the corresponding eigenvectors:

eigenvector for the eigenvalue 0:

$$A-0I=\begin{pmatrix}-4 & 2\\ 2 & -1\end{pmatrix}\to\begin{pmatrix}-2 & 1\\ 2 & -1\end{pmatrix}\to\begin{pmatrix}-2 & 1\\ 0 & 0\end{pmatrix}$$

Here we multiplied the 1st row by ½. Then we replaced row 2 by row2+row1.

So the coefficients of our eigenvector satisfy

$-2x_1+x_2=0$. Take $x_2=2$ and then $x_1=1$.

The eigenvector is any scalar multiple of $\begin{pmatrix}1\\ 2\end{pmatrix}$.

eigenvector for the eigenvalue -5

$$A+5I=\begin{pmatrix}-4 & 2\\ 2 & -1\end{pmatrix}+\begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}\to\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\to\begin{pmatrix}1 & 2\\ 0 & 0\end{pmatrix}.$$

Here we replaced row 2 by row 2 -2row1.

So the eigenvector coefficients satisfy

$x_1+2x_2=0$. Take $x_1=2$. Then $x_2=-1$.

So our 2 linearly independent solutions to the homogeneous equation are:

$$\vec{u}_1 = e^{0t}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}, \quad \vec{u}_2 = e^{-5t}\begin{pmatrix}2\\-1\end{pmatrix}$$

This means that the fundamental matrix is

$$\Psi(t) = \begin{pmatrix}\vec{u}_1 & \vec{u}_2\end{pmatrix} = \begin{pmatrix}1 & 2e^{-5t}\\ 2 & -e^{-5t}\end{pmatrix}.$$

We must invert this matrix:

$$\Psi(t)^{-1} = \frac{1}{\det\Psi}\begin{pmatrix}-e^{-5t} & -2e^{-5t}\\ -2 & 1\end{pmatrix}$$

$$= \frac{1}{-5e^{-5t}}\begin{pmatrix}-e^{-5t} & -2e^{-5t}\\ -2 & 1\end{pmatrix} = \begin{pmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2e^{5t}}{5} & \frac{-e^{5t}}{5}\end{pmatrix}.$$

Then we must multiply this matrix times the vector g representing the nonhomogeneous term.

$$\Psi(t)^{-1}\vec{g} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2e^{5t}}{5} & \frac{-e^{5t}}{5} \end{pmatrix} \begin{pmatrix} t^{-1} \\ 4+2t^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5}t^{-1} + \frac{2}{5}(4+2t^{-1}) \\ \frac{2}{5}e^{5t}t^{-1} + \frac{-e^{5t}}{5}(4+2t^{-1}) \end{pmatrix} = \begin{pmatrix} t^{-1} + \frac{8}{5} \\ \frac{-4}{5}e^{5t} \end{pmatrix}.$$

Finally our formula for the solution

$$\vec{x} = \Psi(t)\vec{c} + \Psi(t)\int \Psi^{-1}(t)\vec{g}(t)dt$$

requires us

to integrate this last vector

$$\begin{pmatrix} \int\left(t^{-1} + \frac{8}{5}\right) \\ \int \frac{-4}{5}e^{5t} \end{pmatrix} = \begin{pmatrix} \ln(t) + \frac{8}{5}t + c_1 \\ \frac{-4}{25}e^{5t} + c_2 \end{pmatrix}$$

So now (loud music) the solution is

$$\vec{x} = \begin{pmatrix} 1 & 2e^{-5t} \\ 2 & -e^{-5t} \end{pmatrix} \begin{pmatrix} \ln(t) + \frac{8}{5}t + c_1 \\ \frac{-4}{25}e^{5t} + c_2 \end{pmatrix}$$

$$= \begin{pmatrix} \ln(t) + \frac{8}{5}t + 2e^{-5t}c_2 + c_1 - \frac{8}{25} \\ 2\ln t + \frac{16}{5}t - e^{-5t}c_2 + 2c_1 + \frac{4}{25} \end{pmatrix}.$$

It was a long calculation but now it is done and we check with the answer in the back of the book where c_2 is replaced by $-c_2$. So the answers agree.

The Sun God smiles !