Vibrating Things
Resonance is what happens when something is forced to vibrate at certain frequencies — unbounded oscillations.

In “real” life this would cause the spring to self destruct. Such things can happen in bridges or other structures. Most texts say the Tacoma Narrows bridge disaster was an example of such resonance. However, the latest research questions this conclusion.

The original Tacoma Narrows Bridge opened to traffic on July 1, 1940. It collapsed just four months later during a 42-mile-per-hour wind storm on Nov. 7.
M. Braun, Differential Equations and their Applications: “When the bridge began heaving violently, the authorities notified Professor F. B. Farquharson of the University of Washington. Professor Farquharson had conducted numerous tests on a simulated model of the bridge and had assured everyone of its stability. The professor was the last man on the bridge. Even when the span was tilting more than twenty-eight feet up and down, he was making scientific observations with little or no anticipation of the imminent collapse of the bridge. “

“A large sign near the bridge approach advertised a local bank with the slogan ‘as safe as the Tacoma Bridge.’”

“After the collapse of the Tacoma Bridge, the governor of the state of Washington made an emotional speech in which he declared ‘We are going to build the exact same bridge, exactly as before.’ Upon hearing this, the noted engineer Von Karman sent a telegram to the governor stating ‘If you build the exact same bridge exactly as before, it will fall into the exact same river exactly as before.’”
Non Forced Vibration of a String.
Let's look at the wave equation which describes the motion of a vibrating string. Assume the string has constant density $\rho$ and constant tension $\tau$. Then one can derive the following PDE known as the wave equation, using Newton's law of motion of the principle of least action:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\rho} \frac{\partial^2 u}{\partial x^2}, \quad 0<x<\pi, \quad 0<t.$$  \hspace{1cm} (1)

One assumes, for example, that the string is tied down at the boundary points giving the boundary conditions:

$$u(0,t)=u(\pi,t)=0, \quad \text{for all } t>0.$$ \hspace{1cm} (2)

And one may assume initial conditions

$$u(x,0)=f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0 \quad \text{for } 0<x<\pi.$$ \hspace{1cm} (3)
The method of separation of variables of Daniel Bernoulli says: look for a solution of the PDE in formula (1) of the form \( u(x,t) = X(x)T(t) \).

If you want this to satisfy (2), assume \( X(0) = X(\pi) \). If you want it to satisfy (3), you are in trouble for the 1st part, but the 2nd part becomes \( T'(0) = 0 \).

Now plug \( u(x,t) = X(x)T(t) \) into formula (1). You get (setting \( c = \tau / \rho \))

\[
X(x)T''(t) = c T(t)X''(x).
\]

Divide both sides by \( X(x)T(t) \) (hoping you are not dividing by 0). This gives:

\[
\frac{T''(t)}{T(t)} = c \frac{X''(x)}{X(x)} \quad (4)
\]

This implies each side is constant. Call the constant \( \lambda \). It is often called the separation constant. It is an eigenvalue in the 1st ODE below.

Exercise 1. Prove that each side in equation (4) must be constant.
Thus we now have 2 ODES to solve. Here assume $c=1$ in (4).

**ODE 1.**  $X''(x) = \lambda X(x), \quad 0 = X(0) = X(\pi)$.  

**ODE 2.**  $T''(t) = \lambda T(t), \quad T'(0) = 0$.  

Look at ODE1. The general solution from Math. 20D is

$$X(x) = a_1 \exp \left( x\sqrt{\lambda} \right) + a_2 \exp \left( -x\sqrt{\lambda} \right)$$

with constants $a_i$. To satisfy the boundary conditions, we need $\lambda < 0$. This means, since $e^{ix} = \cos x + is\sin x$ when $i = (-1)^{\frac{1}{2}}$, that we should write, for $\lambda = -\mu^2$,

$$X(x) = c_1 \cos (\mu x) + c_2 \sin (\mu x)$$

with constants $c_j$.

In order to satisfy the 1st boundary condition, we need $X(0) = c_1 = 0$. This makes $X(x) = c_2 \sin (\mu x)$. The 2nd boundary condition is $X(\pi) = c_2 \sin (\mu \pi) = 0$. This says

$$\mu \pi = n\pi, \quad \text{for some integer } n=1,2,3,\ldots$$

So we find the separation constant in equation (4) is

$$\lambda = -\mu^2 = n^2, \quad \text{for some integer } n=1,2,3,\ldots$$

Then

(5)  $X(x) = c_2 \sin (n x), \quad \text{for } n=1,2,3,\ldots$
Look at ODE2. Math 20D says that the general solution may be taken to be

\[ T(t) = b_1 \cos(nt) + b_2 \sin(nt) \]

Since \( T'(0) = nb_2 = 0 \), we see that

(6) \[ T(t) = b_1 \cos(nt), \quad \text{for } n=1,2,3,\ldots \]

Now turn to the problem of the 1\textsuperscript{st} part of the initial condition (3). For this you need to be able to write the function

\[ f(x)=X(x)T(0) = c_2 \sin(nx). \]

But, what should we do if the initial shape of the string is not a sine function; e.g., the plucked string pictured below.
To solve the plucked string problem you need to represent the function \( f(x) = u(x,0) \) as a Fourier sine series:

\[
f(x) = \sum_{n \geq 1} c_n \sin(nx)
\]  

(7)

Then the constants \( c_n \) are given by the formula

\[
c_n = \int_0^\pi f(y) \sin(ny) \, dy.
\]  

(8)

This is proved in Lang, p. 318, for sufficiently smooth functions \( f \). Of course, our plucked string function does not look smooth, just continuous. It has that sharp point, remember.

Anyway our final solution to the vibrating string problem is:

\[
u(x,t) = \sum_{n \geq 1} c_n \sin(nx) \cos(nt).
\]  

(9)

Here the constants \( c_n \) are from formula (8) and we assume \( \tau = \rho \) in the PDE (1). Exercise 2. Check (9) solves the PDE.
Forced Motion of a Vibrating String

Assume the vibrating string is as before but now apply an external force of the form $f(x)\cos(\omega t)$. This leads to the PDE:

$$u_{tt} - \frac{\tau}{\rho} u_{xx} = f(x)\cos(\omega t)$$

$$u(0, t) = u(\pi, t) = 0; \quad u(x, 0) = u_t(x, 0) = 0.$$ 

Assume $1 = \tau/\rho$, for simplicity. So our problem is now

(10) $$u_{tt} - u_{xx} = f(x)\cos(\omega t)$$

$$u(0, t) = u(\pi, t) = 0; \quad u(x, 0) = u_t(x, 0) = 0.$$ 

To solve (10), plug in

$$u(x, t) = \sum_{n \geq 1} c_n(t) \sin(nx).$$
Define the inner product for piecewise continuous functions $f, g$ on $[0, \pi]$ by

\begin{equation}
\langle f, g \rangle = \int_{0}^{\pi} f(x)g(x)dx.
\end{equation}

Then

\begin{equation}
c_n(t) = \left\langle u(*, t), \frac{\sin(n\pi)}{\sqrt{\pi/2}} \right\rangle = \int_{0}^{\pi} u(x, t) \frac{\sin(nx)}{\sqrt{\pi/2}} dx
\end{equation}

Here we use the fact that

\begin{equation}
\int_{0}^{\pi} \sin^2(nx)dx = \frac{\pi}{2}.
\end{equation}

Exercise 3. Prove the last formula and then use it to show that

\begin{equation}
v_n(x) = \frac{\sin(nx)}{\sqrt{\pi/2}}, n=1, 2, 3, \ldots
\end{equation}

forms an orthonormal family for the inner product space of piecewise continuous functions on $[0, \pi]$ using the inner product defined by formula (12). Here we use the terminology in Lang, p. 301.
If we plug formula (11) into (10) without worrying about our issues of interchange of derivative and summation, we see that we need

\[ c''_n (t) = \int_0^\pi u''(c, t)v_n(x)dx = \int_0^\pi (u_{xx}(x, t) + f(x)\cos(\omega t))v_n(x)dx \]

\[ = \int_0^\pi u(x, t)v''_n(x)dx + \cos(\omega t)\int_0^\pi f(x)v_n(x)dx. \]

Exercise 4. Show this last formula holds using integration by parts and the boundary conditions.

So from our last formula, we have

\[ c''_n (t) = -n \int_0^\pi u(x, t)v_n(x)dx + \cos(\omega t)\int_0^\pi f(x)v_n(x)dx \]

\[ = -n^2c_n(t) + \langle f, v_n \rangle \cos(\omega t). \]

So now we need to solve an ODE:

\[ c''_n (t) = -n^2c_n(t) + \langle f, v_n \rangle \cos(\omega t), \quad c_n(0) = c'_n(0) = 0 \]
You can solve this using methods from Math. 20D – for example, variation of parameters. The result is:

\[ c_n(t) = \frac{\langle f, v_n \rangle}{n^2 - \omega^2} (\cos(\omega t) - \cos(nt)). \]

Exercise 5. Check this answer.

Thus a solution for the problem posed by formula (10) is

\[ u(x,t) = \sum_{n=1}^{\infty} a_n \frac{\cos(\omega t) - \cos(nt)}{n^2 - \omega^2} v_n(x), \quad a_n = \langle f, v_n \rangle, \quad v_n(x) = \frac{\sin(nx)}{\sqrt{\pi/2}}. \]

Exercise 6. What happens to formula (13) as \( \omega \to \pm n \)??
You need to compute

\[ \lim_{\omega \to n} \frac{\cos(\omega t) - \cos(nt)}{\omega^2 - n^2}. \]

There is a similar result as \( \omega \to -n \). You should be able to deduce from this that unless \( a_n = 0 \), the function \( u(x,t) \) blows up as \( t \) goes to infinity. This is resonance.
Mathematica plots a vibrating square drum:

\[ \text{Animate}[\text{Plot3D}[(\sin(5\pi x)\sin(3\pi y) - \sqrt{2/3}\sin(3\pi x)\sin(5\pi y))\cos(\sqrt{2}\pi t/10000),
\{x,0,3/2\},\{y,0,3/2\},\text{PlotPoints}->80,\text{ColorFunction}->\text{Hue},\text{Mesh}->\text{False},\text{FaceGrids}->\text{None},\text{Axes}->\text{False},\text{Boxed}->\text{False}],\{t,0,100000\}] \]
The End