What are primes in graphs and how many of them have a given length?

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A **graph** is a bunch of vertices connected by edges. The simplest example for the talk is the tetrahedron K_4.

A **prime** in a graph is a closed path in the graph minimizing the number of edges traversed. This means no backtrack, no tails. Go around only once. Orientation counts. Starting point doesn't count.

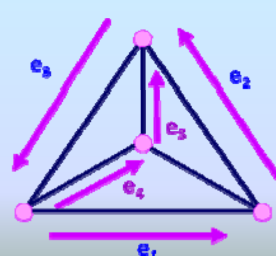
The **length** of a path is the number of edges in the path.

The talk concerns the prime number theorem in this context.

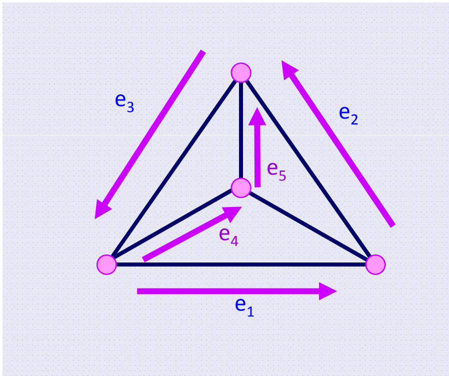
Degree of a vertex is number of edges incident to it.

Graph is **regular** if each vertex has same degree. K_4. is 3-regular.

We **assume** graphs connected, no degree 1 vertices, and graph is not a cycle.



Examples of primes in K_4



Here are 2 of them in K_4:

$[C] = [e_1 e_2 e_3]$

$= \{e_1 e_2 e_3,\ e_2 e_3 e_1,\ e_3 e_1 e_2\}$

The [] means that

it does not matter where the path starts.

$[D] = [e_4 e_5 e_3]$

$[E] = [e_1 e_2 e_3 e_4 e_5 e_3]$

$\nu(C)$ = length C = # edges in C

$\nu(C)=3$, $\nu(D)=3$, $\nu(E)=6$

E=CD
another prime $[C^n D]$, n=2,3,4, …
infinitely many primes
assuming the graph is not a cycle or a cycle with hair.

Ihara Zeta Function

Definition

$$\zeta(u,X) = \prod_{\substack{[C] \\ prime}} \left(1 - u^{\nu(C)}\right)^{-1}$$

u complex number

|u| small enough

Ihara's Theorem (Bass, Hashimoto, etc.)

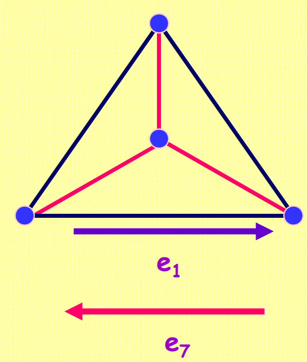
A = adjacency matrix of X= |V|x|V| matrix of 0s and 1s with i,j entry 1 iff vertex i adjacent to vertex j

Q = diagonal matrix; jth diagonal entry = degree jth vertex -1;

r = |E|-|V|+1

$$\zeta(u,X)^{-1} = (1-u^2)^{r-1} \det(I - Au + Qu^2)$$

Labeling Edges of Graphs



Orient the m edges; i.e., put arrows on them.
Label them as follows.
Here the inverse edge has opposite orientation.

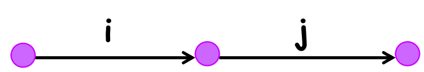
$e_1, e_2, \ldots, e_m,$

$e_{m+1}=(e_1)^{-1}, \ldots, e_{2m}=(e_m)^{-1}$

Note that these directed edges are our alphabet needed to express paths in the graph.

The Edge Matrix W

Define W to be the $2|E|\times 2|E|$ matrix with i j entry 1 if edge i feeds into edge j, (end vertex of i is start vertex of j) provided that $j \neq$ the inverse of i, otherwise the i j entry is 0.



Theorem. $\zeta(u,X)^{-1}=\det(I-Wu)$.

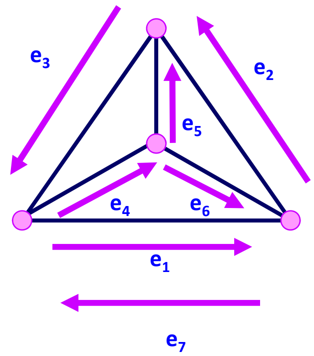
Corollary. The poles of Ihara zeta are the reciprocals of the eigenvalues of W.

The pole R of zeta is the closest to 0 in absolute value.

R=1/Perron-Frobenius eigenvalue of W; i.e., the largest eigenvalue which has to be positive real. See Horn & Johnson, Matrix Analysis, Chapter 8.

Example. W for the Tetrahedron

Label the edges
The inverse of
edge j is edge j+6.



$$W = \begin{pmatrix} 0&1&0&0&0&0&0&0&0&0&0&1\\ 0&0&1&0&0&0&0&0&0&0&1&0\\ 1&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&0&1&0&0&1&0\\ 1&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0&0&1&0&0\\ 0&0&0&0&1&0&0&0&0&1&0&0 \end{pmatrix}$$

There are elementary proofs of the 2 determinant formulas for zeta, even for X irregular.
See my book on my website www.math.ucsd.edu/~aterras/newbook.pdf

If you are willing to believe these formulas, we can give a rather easy proof of the graph theory prime number theorem.

But first state the prime number thm and give some examples.

The Prime Number Theorem

$\pi_X(m)$ = # {primes [C] in X of length m}

Δ = greatest common divisor of lengths of primes in X

R = radius of largest circle of convergence of $\zeta(u,X)$

If Δ divides m, then

$$\pi_X(m) \sim \Delta R^{-m}/m, \quad \text{as } m \to \infty.$$

R=1/q, if graph is q+1-regular

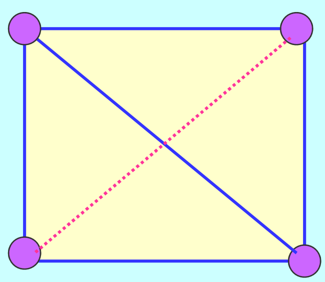
The proof involves formulas like the following, defining

N_m = # {closed paths of length m where we count starting point and orientation}

$$u\frac{d\log\zeta(u,X)}{du} = \sum_{m=1}^{\infty} N_m u^m$$

The proof is similar to one in Rosen, *Number Theory in Function Fields*, p. 56.

2 Examples
K_4 and
K_4-edge



$$\zeta(u,K_4)^{-1} =$$
$$(1-u^2)^2(1-u)(1-2u)(1+u+2u^2)^3$$

$$\zeta(u,K_4-e)^{-1} =$$
$$(1-u^2)(1-u)(1+u^2)(1+u+2u^2)(1-u^2-2u^3)$$

N_m for the examples

Example 1. The Tetrahedron K_4.
N_m=# closed paths of length m

$$x\frac{d\log\zeta(x,G)}{dx}=\sum_{m=1}^{\infty}N_m x^m$$

x d/dx log ζ (x,K_4) = $24x^3 + 24x^4 + 96x^6 + 168x^7 + 168x^8 + 528x^9 + \cdots$

$\pi(3)=8$ (orientation counts) $\pi(4)=6$ $\pi(5)=0$

we will show that:

$$N_6=\sum_{d|6}d\pi(d)=\pi(1)+2\pi(2)+3\pi(3)+6\pi(6)$$

$$\pi(6)=24$$

Δ = g.c.d. lengths of primes = 1

Example 2. The Tetrahedron minus an edge.

x d/dx log ζ (x,K_4-e) = $12x^3 + 8x^4 + 24x^6 + 28x^7 + 8x^8 + 48x^9 + \cdots$

$\pi(3)=4$ $\pi(4)=2$ $\pi(5)=0$ $\pi(6)=2$

Δ = g.c.d. lengths of primes = 1

Poles of Zeta for K_4 are
$\{1,1,1,-1,-1,\frac{1}{2},r_+,r_+,r_+,r_-,r_-,r_-\}$

where $r_\pm=(-1\pm\sqrt{-7})/4$ and $|r|=1/\sqrt{2}$
R=½=Pole closest to 0
The prime number thm $\pi(m) \sim \Delta R^{-m}/m$, as $m \to\infty$.
becomes

| K_4 $\pi(m) \sim 2^m/m$, as $m \to\infty$. |

Poles of zeta for K_4-e are
$\{1,1,-1,i,-i,r_+,r_-,\alpha,\beta,\bar\beta\}$
R = α real root of cubic ≅ .6573
β complex root of cubic

The prime number thm becomes for $1/\alpha \cong 1.5$

| K_4-e $\pi(m) \sim 1.5^m/m$, as $m \to\infty$. |

Proof of Prime Number Theorem

Start with the definition of zeta as a product over primes.
Take log and a derivative. Use Taylor series.

$$\zeta(u,X) = \prod_{\substack{[C]\\ prime}} \left(1-u^{\nu(C)}\right)^{-1}$$

$$u\frac{d}{du}\log\zeta(u,X) = -u\frac{d}{du}\sum_{\substack{[C]\\ prime}} \log(1-u^{\nu(C)})$$

$$= u\frac{d}{du}\sum_{[C]}\sum_{j\geq 1}\frac{1}{j}u^{\nu(C)j} = \sum_{\substack{[C]\\ prime}} \nu(C)\sum_{j\geq 1} u^{\nu(C)j}$$

$$u\frac{d}{du}\log\zeta(u,X) = \sum_{j\geq 1}\sum_{\substack{C\\ prime\\ path}} u^{\nu(C^j)}$$

$$= \sum_{m\geq 1} N_m u^m$$

Taylor Series for log(1-x)

#[C]=ν(C)

non-primes are powers of primes

This completes the proof of the first formula

(1) $$u\frac{d}{du}\log\zeta(u,X) = \sum_{m\geq 1} N_m u^m$$

Next we note another formula for the zeta function coming from the original definition.

$$\zeta(u,X) = \prod_{\substack{[C]\\ prime}} \left(1-u^{\nu(C)}\right)^{-1}$$

Recall
π(n) = # primes [C] with ν(C)=n

$$\zeta(u,X) = \prod_{n\geq 1}\left(1-u^n\right)^{-\pi(n)}.$$

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$$u\frac{d}{du}\log\zeta(u,X) = \sum_{n\geq 1}\frac{n\pi(n)u^n}{1-u^n} = \sum_{m\geq 1}\left(\sum_{d|m} d\pi(d)\right)u^m$$

Taylor Series for $(1-x)^{-1}$

If you combine this with formula 1, which was

$$u\frac{d}{du}\log\zeta(u,X) = \sum_{m\geq 1} N_m u^m$$

you get our 2nd formula saying N_m is a sum over the positive divisors of m:

(2) $$N_m = \sum_{d|m} d\pi(d)$$

$$N_m = \sum_{d|m} d\pi(d)$$

This is a math 104 (number theory) -type formula and there is a way to invert it using the Mobius function defined by:

$$\mu(n) = \begin{cases} 1, & \text{if} \quad n=1 \\ 0, & \text{if} \quad n \quad \text{not square} - \text{free} \\ (-1)^r, & n = p_1 \cdots p_r, \quad \text{with } p_i \text{ distinct primes} \end{cases}$$

(3) $$\pi(m) = \frac{1}{m}\sum_{d|m}\mu\left(\frac{m}{d}\right)N_d$$

To complete the proof, we need to use one of our 2 determinant formulas for zeta.

$\zeta(u,X)^{-1}=\det(I-Wu)$.

Fact from linear algebra - Schur Decomposition of a Matrix (Math 102)

There is an orthogonal matrix Q (i.e., $QQ^{t}=I$) and
T=upper triangular with eigenvalues of W along the diagonal such that $W=QTQ^{-1}$.

$$T=\begin{pmatrix} \lambda_1 & * & \cdots & * & * \\ 0 & \lambda_2 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{m-1} & * \\ 0 & 0 & \cdots & 0 & \lambda_m \end{pmatrix}$$

So we see that

$$Det(I-uW)=Det(I-uQTQ^{-1})$$
$$=Det(I-uT)=\prod_{i=1}^{m}\left(1-u\lambda_i\right)$$

$$\zeta(u,X)^{-1}=\det(I-Wu)=\prod_{\substack{\lambda \\ \text{eigenvalue of } W}}\left(1-\lambda u\right)$$

$$u\frac{d}{du}\log\zeta(u,X)=u\frac{d}{du}\sum_{\substack{\lambda \\ \text{eigenvalue of } W}} u\frac{d}{du}\log(1-\lambda u)$$

$$=\sum_{\substack{\lambda \\ \text{eigenvalue of } W}}\sum_{m=1}^{\infty}(\lambda u)^m$$

It follows that

(4) $$N_m=\sum_{\substack{\lambda \\ \text{eigenvalue of } W}}\lambda^m$$

$$N_m = \sum_{\substack{\lambda \\ \text{eigenvalue of } W}} \lambda^m$$

The main terms in this sum come from the largest eigenvalues of W in absolute value.

There is a theorem in linear algebra that you don't learn in a 1st course called the Perron-Frobenius theorem. (Horn & Johnson, Matrix Analysis, Chapter 8).
It applies to our W matrices assuming we are looking at connected graphs (no degree 1 vertices) & not cycles.

The easiest case is that Δ=g.c.d. lengths of primes = 1.

Then there is only 1 eigenvalue of W of largest absolute value. It is positive and is called the Perron-Frobenius eigenvalue. Moreover it is 1/R, R=closest pole of zeta to 0.

So we find that if $\Delta=1$, $N_m \approx R^{-m}$, as $m \to \infty$.

If $\Delta=1$, $N_m \approx R^{-m}$, as $m \to \infty$.

To figure out what happens to $\pi(m)$, use

$$\pi(m) = \frac{1}{m} \sum_{d|m} \mu\left(\frac{m}{d}\right) N_d$$

This allows you to prove the prime number theorem when $\Delta=1$

$$p_X(m) \sim R^{-m}/m, \text{ as } m \to \infty.$$

For the general case, see my book

www.math.ucsd/~aterras/newbook.pdf


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