Artin L-Functions of Graph Coverings

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Part 0. Outline.

The goal of this talk is to provide an introduction to my joint papers with Harold Stark on zeta and L-functions of graph coverings [*Advances in Math.*, 121 (1996) and 154 (2000)]. The motivation is to treat the graph zeta functions the same way as the number theory analogs. This requires a discussion of the graph theoretic version of Galois theory which is to be found in the 2nd Advances paper. Here we will not discuss the Galois theory but instead focus on examples and computation. The following picture shows the tree of zetas with the zetas appearing as roots of the tree. The branches indicate the parallel fields that benefit from these roots. Here we consider only the 2 fields of algebraic number theory and graph theory. In **part 1**, we discuss zeta and L-functions of algebraic number fields. More details for part 1 can be found in

> H. M. Stark, Galois theory, algebraic number theory & zeta functions, in *From Number Theory to Physics* (editors M. Waldschmidt et al), Springer-Verlag, 1992.

In **part 2**, the graph theory analogs are to be found. There are actually 3 kinds of graph zetas (vertex, edge and path). We will attempt to extol the computational advantages of the path zetas. The path and edge zetas have many variables and do not appear to have number theory analogs as yet.

Some History.

The theory of zeta functions of algebraic number fields was developed by Riemann (mid 1800s) for the rational number field, then Dedekind, Dirichlet, Hecke, Takagi, and Artin (early 1900s). Graph zeta functions appeared first from the point of view of p-adic groups in work of Ihara in the mid 1960s. Then Serre realized the graph theory interpretation. Papers of Sunada, Hashimoto and Bass further developed the theory. In particular, see Hashimoto, *Adv. Stud. Pure Math.*, 15, Academic, 1989, pages 211-280. More references can be found in the Advances papers mentioned above, as well as my book, *Fourier Analysis on Finite Groups and Applications*, Cambridge, 1999.


$\mathbb{Q}(\sqrt{3})$
Number & Function Fields
Riemann Surfaces $\Gamma\backslash H$
Dynamical Systems
Graphs
Tree of Zetas
Dedekind ζ
ζ of curve over $\mathbb{F}_q$
Selberg ζ
Ruelle ζ
Ihara ζ

Part I. The Algebraic Number Theory Zoo of Zetas.

Riemann zeta, **for Re(s)>1,**

$$z(s) = \sum_{n=1}^{\infty} \frac{1}{n^{s}} = \prod_{p = prime} \left(1 - p^{-s}\right)^{-1}.$$

- **Riemann extended to all complex s with pole at s=1.**
- **Functional equation relates value at s and 1-s**
- **Riemann hypothesis**
- **duality between primes and complex zeros of zeta**
- **prime number theorem**
- **See Davenport, *Multiplicative Number Theory*.**

Dedekind zeta **of algebraic number field K=Q(q)**

q **a root of a polynomial with coefficients in** Q

prime = prime ideal p in O_K, ring of integers of K

infinite product of terms $(1\text{-}Np^{-s})^{-1}$,

where Np = norm of p = #(O_K/p)

prime ideal theorem

SELBERG ZETA FUNCTION

associated to a compact **Riemannnian manifold** **M=**G**\H**

H = upper half plane with $ds^2=(dx^2+dy^2)y^{-2}$

G**=discrete subgroup** **of group of real fractional linear transformations**

primes **= primitive closed geodesics C in M of length** n**(C)**

(primitive means only go around once)

Duality **between spectrum of Laplacian** D **on M & lengths closed geodesics in M**

$$Z(s) = \prod_{[C]} \prod_{j \geq 0} \left(1 - e^{-(s+j)n(C)} \right)$$

Riemann hypothesis known to hold

Prime geodesic theorem

Z(s+1)/Z(s) is a closer analog of z**(s)**

We won't say more about this zeta here.

References:

D. Hejhal, *Duke Math. J.*, 43 (1976); A. Terras, *Harmonic Analysis on Symmetric Spaces & Applics., I*, Springer, 1985

Example 1. Quadratic Extension

field	ring	prime ideal	finite field
		g=# of such $\mathfrak{p}$	
$K=\mathbb{Q}(\sqrt{2})$	$O_K=\mathbb{Z}[\sqrt{2}]$	$\mathfrak{p} \supset pO_K$	$O_K/\mathfrak{p}$
2 $\Big\vert$	$\Big\vert$	$\Big\vert$	$\Big\vert$ **f=degree**
$F=\mathbb{Q}$	$\mathbb{Z}$	$p\mathbb{Z}$	$\mathbb{Z}/p\mathbb{Z}$

3 CASES

1) **p inert:** f=2. pO_K = prime ideal, $2 \neq x^2 \pmod{p}$

2) **p splits:** g=2. $pO_K = \mathfrak{p}\,\mathfrak{p}'$, $2 \equiv x^2 \pmod{p}$

3) **p ramifies:** e=2. $p=\mathfrak{p}^2$, p=2

Gal(K/F)={1,-1},
Frobenius automorphism = Legendre Symbol =

$$\left(\frac{2}{p}\right)=\begin{cases} -1, & \text{in case 1} \\ 1, & \text{in case 2} \\ 0, & \text{in case 3} \end{cases}$$

p odd implies p has 50% chance of being in Case 1

Zeta and L-Functions for Example 1

Dedekind Zeta

$$z_K(s) = \prod_{\mathfrak{p}} \left(1 - N\mathfrak{p}^{-s}\right)^{-s}$$ product over prime ideals in O_K

Riemann Zeta

$$z_Q(s) = \prod_{p} \left(1 - Np^{-s}\right)^{-1}$$ product over primes in **Z**

Dirichlet L-Function

$$L(s,c) = \prod_{p} \left(1 - c(p)p^{-s}\right)^{-1}, \text{ where } c(\mathrm{p}) = \left(\frac{2}{\mathrm{p}}\right)$$

product over primes in **Z**

Factorization

$$z_{Q(\sqrt{2})}(s) = z_Q(s)L(s,c)$$

Functional Equations: $\zeta_K(s)$ related to $\zeta_K(1\text{-}s)$
Hecke

Values at 0: $\zeta(0) = -1/2$, $\zeta_K(0) = -hR/w$

h = **class number** (measures how far O_K is from having unique factorization) =1 for **Z**[√2]

R = **regulator** (determinant of logs of units)
= log(1+√2) when K=**Q**(√2)

w = **number of roots of unity** in K is 2, when K=**Q**(√2)

Statistics of Prime Ideals and Zeros

- from information on zeros of $\zeta_K(s)$ obtain

  prime ideal theorem

$$\#\{p \text{ prime ideal in } O_K \mid Np \le x\} \sim \frac{x}{\log x}, \text{ as } x \to \infty$$

- there are an infinite number of primes such that $\left(\frac{2}{p}\right)=1.$
- Dirichlet theorem: there are an infinite number of primes p in the progression a, a+d, a+2d, a+3d, ..., when g.c.d.(a,d)=1.
- **Riemannn hypothesis still open:**

  GRH or ERH: $\zeta_K(s)=0$ implies Re(s)=1/2, assuming s is not real.

References: Lang or Neukirch, *Algebraic Number Theory*

See the **pulchritudinous primes** website for some interesting pictures made using programs involving primes, including prime island. The site belongs to Adrian J. F. Leatherland and the address is:

yoyo.cc.monash.edu.au/~bunyip/primes

Artin L-Functions

K ⊃ **F** ⊃ Q number fields with K/Q Galois

$O_K \supset O_F \supset Z$ rings of integers

P ⊃ **p** ⊃ pZ prime ideals (**p** unramified ,
i.e., P^2 does not contain **p**)

Frobenius Automorphism $\left(\frac{K/F}{P}\right) = s \in Gal(K/F)$

$$s_P(x) \equiv x^{Np} \,(\mathrm{mod}\, P),\ \text{for}\ x \in O_K,$$

when p is unramified.

σ_P determined by **p** up to conjugation if **P** / **p** unramified

f (**P**/**p**)= order of σ_P = $[O_K / P : O_F/p]$

Artin L-Function for s∈C, π is a representation of Gal(K/F)

$$L(s,p)\text{"}=\text{"}\prod_p \left(1 - p\left(\frac{K/F}{P}\right) Np^{-s}\right)^{-1}$$

where "=" means we only give the formula for unramified primes **p** of **F**. Here we pick **P** a prime in O_K dividing **p**,

Applications

- **Factorization**

$$z_K(s) = \prod_{\substack{p \\ \text{irreducible} \\ \text{degree } d_p}} L(s,p)^{d_p}$$

- **Chebotarev Density Theorem**

  " s in Gal(K/F), $ ¥ -ly many prime ideals p of O_F

  such that $ P in O_K dividing p with Frobenius

$$\left(\frac{K/F}{P}\right) = s$$

- **Artin Conjecture: L(s,p) entire for non-trivial irreducible rep p**
- **Stark Conjectures: p not containing trivial rep**

$$\lim_{s \to 0} s^a L(s,p) = \Theta(p) * R(p)$$

  = algebraic number * determinant of a×a matrix in linear forms with alg. coeffs. of logs of units of K and its conjugate fields /Q.

References:

Stark's paper in *From Number Theory to Physics*, edited by Waldschmidt et al

Stark, Adv. in Math., *Advances in Math.*, 17 (1975), 60-92

Lang or Neukirch, *Algebraic Number Theory*

Chebotarev Density Theorem for K/ℚ normal.

For a set S of rational primes, define the analytic density of S $\lim_{s\to 1+}\left(\dfrac{\sum_{p\in S} p^{-s}}{\log {}^{1}\!/\!_{(s-1)}}\right)$**. In the following proof, one needs to know that $L(s,\rho)$ continues to s=1 with no pole or zero if $\rho \neq 1$, while $L(s,1)=\zeta(s)$=Riemann zeta.**

Theorem. For every conjugacy class C in G=Gal(K/ℚ), the analytic density of the set of rational primes p such that C(p)=the conjugacy class of the Frobenius auto of a prime ideal P of K dividing p is |C|/|G|.

Proof. Sum the logs of the Artin L-functions × conjugate of characters χ_ρ over all irreducible reps ρ of G. As $s\to 1+$,

$$\log\frac{1}{s-1} \sim \sum_{\rho} \log L(s,\rho)\,\overline{\chi_\rho(C)}$$

$$\sim \sum_{\rho}\sum_{p} \chi_\rho(C(p))\,p^{-s}\,\overline{\chi_\rho(C)}$$

$$\sim \frac{|G|}{|C|}\sum_{\substack{p\\ C(p)=C}} p^{-s}$$

by the orthogonality relations of the characters of the irreducible representations ρ of G. Here C(p) denotes the conjugacy class of the Frobenius auto of the prime of K above p.

Example 2. Galois Extension of Non-normal Cubic

	field	ring	prime ideal	finite field
			$g(\mathfrak{P}/\mathfrak{p})$ = # of such $\mathfrak{P}$	
	$K=F(e^{2\pi i/3})$	O_K	$\mathfrak{P}$	$O_K/\mathfrak{P}$
3	|	|	|	|
	$F=\mathbb{Q}\left(\sqrt[3]{2}\right)$	O_F	$\mathfrak{p}$	$O_F/\mathfrak{p}$
2	|	|	|	|
	$\mathbb{Q}$	$\mathbb{Z}$	$p\mathbb{Z}$	$\mathbb{Z}/p\mathbb{Z}$

$f(\mathfrak{P}/\mathfrak{p})$=degree$(O_K/\mathfrak{P} : O_F/\mathfrak{p})$

More details are in Stark's article in *From Number Theory to Physics*, edited by Waldschmidt et al

Splitting of Rational Primes in O_F

Type 1. $pO_F= \mathfrak{p}_1\, \mathfrak{p}_2\, \mathfrak{p}_3$, with distinct $\mathfrak{p}_i$ of degree 1 (p=31 is 1st example), Frobenius of prime $\mathfrak{P}$ above $\mathfrak{p}_i$ has order 1

density 1/6 by Chebotarev

Type 2. $pO_F= \mathfrak{p}_1\, \mathfrak{p}_2$, with $\mathfrak{p}_1$ of degree 1, $\mathfrak{p}_2$ of degree 2 (p=5 is 1st example), Frobenius of prime $\mathfrak{P}$ above $\mathfrak{p}_i$ has order 2

density 1/2 by Chebotarev

Type 3. $pO_F= \mathfrak{p}$, with $\mathfrak{p}$ of degree 3, (p=7 is 1st example), Frobenius of $\mathfrak{P}$ above $\mathfrak{p}_i$ has order 3

density 1/3 by Chebotarev

Part II. The Graph Theory Zoo of Zetas

References:

- **Harold M. Stark and Audrey Terras, *Adv. in Math.*, Vol. 121 (1996); Vol. 154 (2000)**

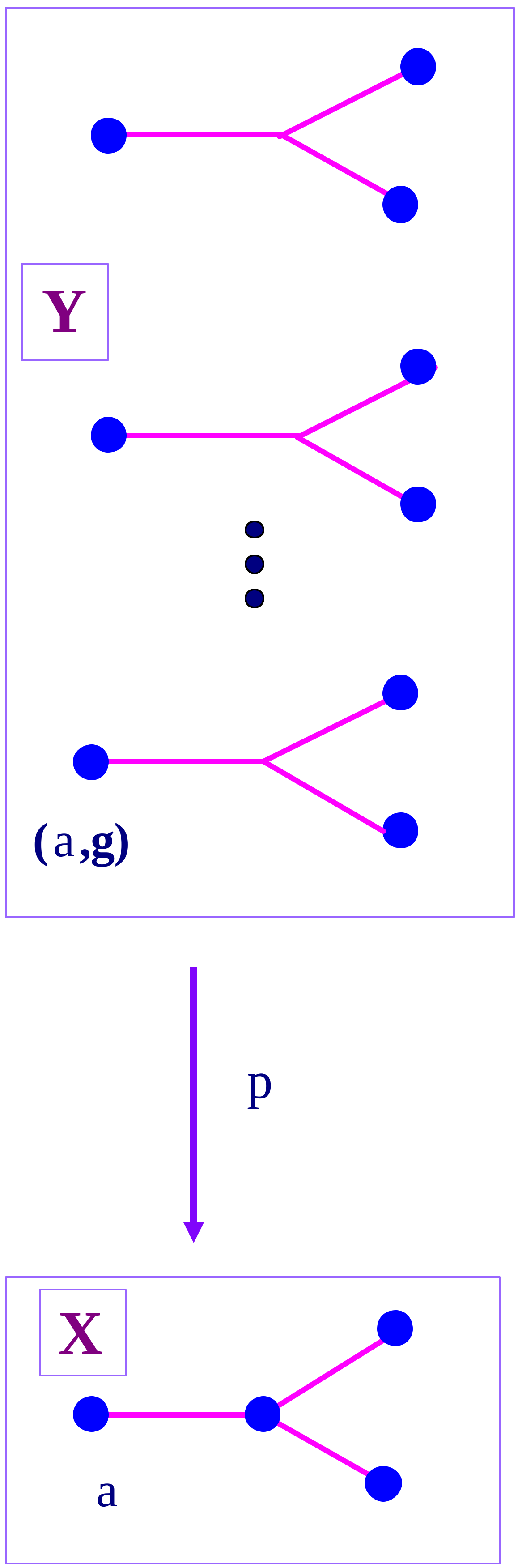
- **K. Hashimoto, *Internatl. J. Math.*, 1992, Vol.3.**

Definitions.

Graph Y an <u>unramified covering</u> of Graph X
means (assuming no loops or multiple edges)
$\pi: Y \to X$ **is an onto graph map such that**
for every $x \in X$ **& for every** $y \in \pi^{-1}(x)$,
π **maps the points** $z \in Y$ **adjacent to y**
1-1, onto the points $w \in X$ **adjacent to x.**

Normal d-sheeted Covering means:
$\exists$ **d graph isomorphisms**
$g_1, \ldots, g_d$ **mapping** $Y \to Y$
such that $\pi\, g_j(y) = \pi(y)$ **for all** $y \in Y$
The Galois group $G(Y/X) = \{ g_1, \ldots, g_d \}$.

Note: We do not assume graphs are regular!



How to label points on Y covering X with Galois group G=Gal(Y/X)

Second make n=|G| copies of the tree in X. These are the sheets of Y. Label the sheets with $g \in G$. Then

g(sheet h)=sheet(gh)

g(a,h)=(a,gh)

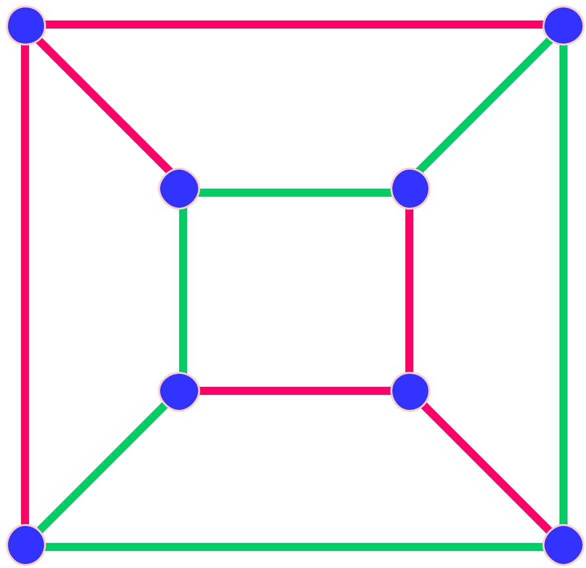
g(path from (a,h) to (b,j)) = path from (a,gh) to (b,gj)

First pick a spanning tree in X (no cycles, connected, includes all vertices of X).

Example 1. Quadratic Cover

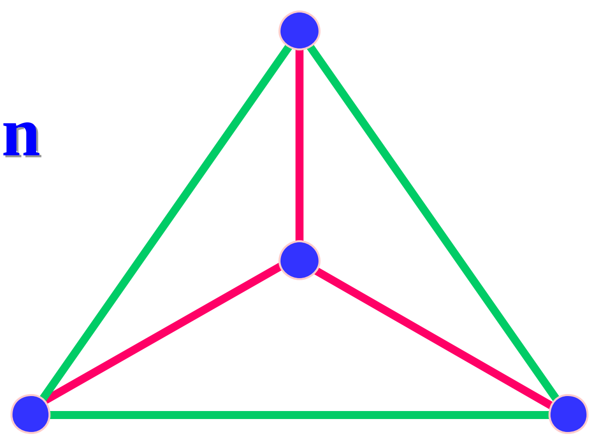
This is analogous to Example 1 in Part 1.

Cube



covers

Tetrahedron



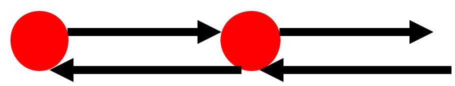
Spanning Tree in X is red.
Corresponding sheets of Y are also red

"PRIMES in GRAPHS" are

equivalence classes of closed backtrackless tailless primitive paths

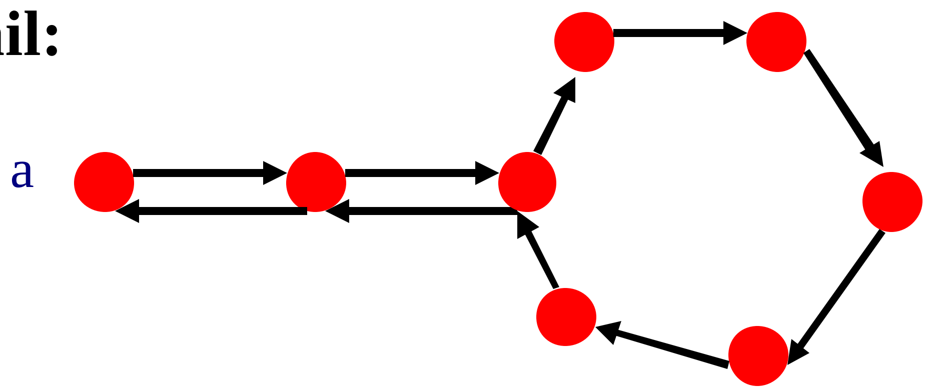
DEFINITIONS

backtrack



equivalence class: change starting point

tail:

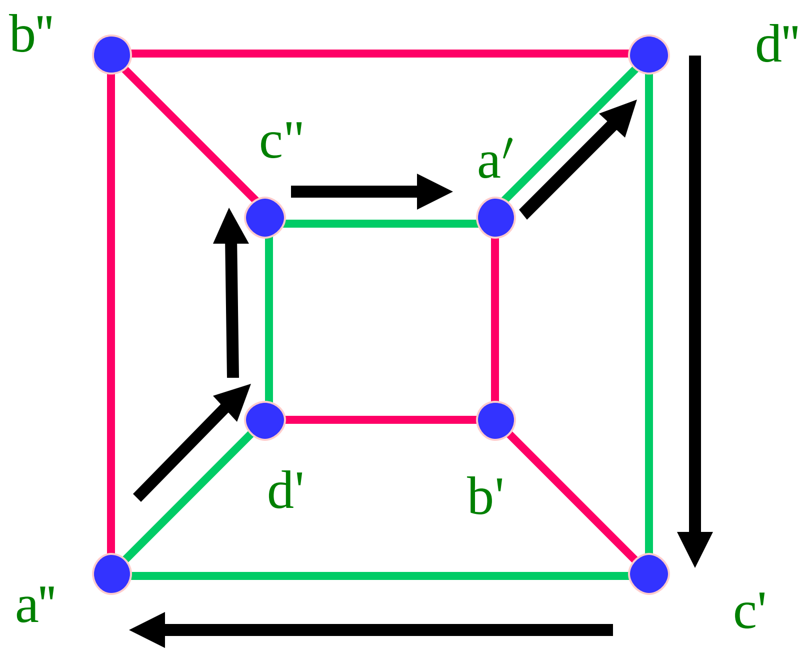


Here a is the start of the path

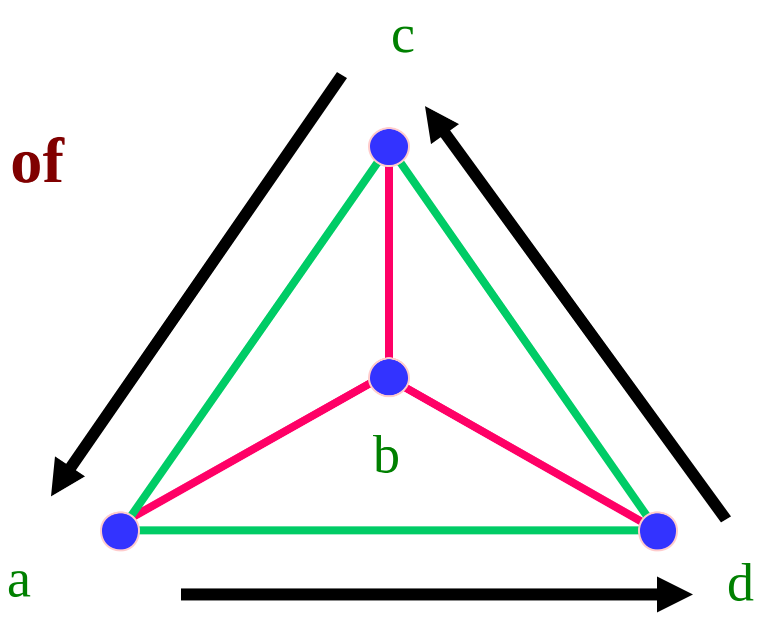
non-primitive: go around path more than once

Example of Splitting of Primes in Quadratic Cover, f=2

D prime above C of length 6



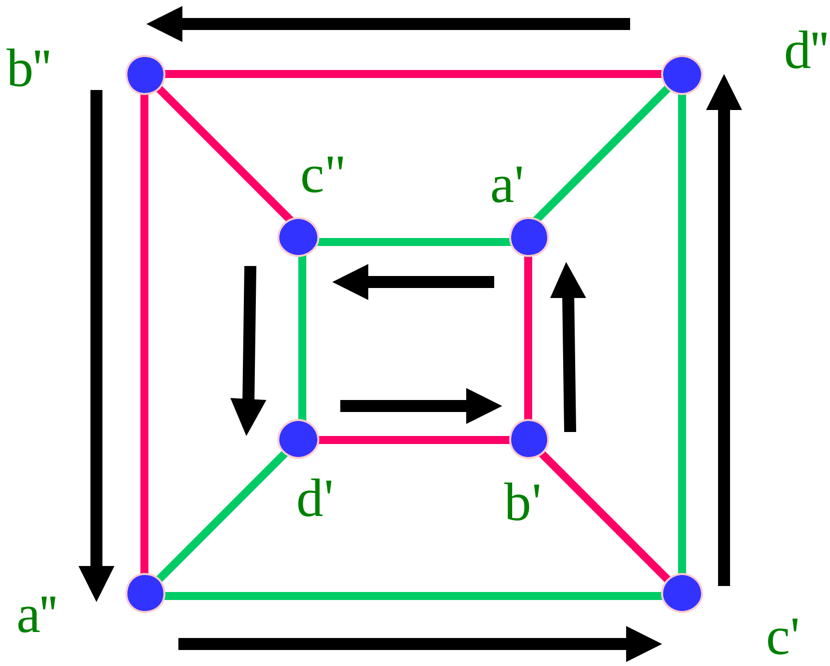
C prime of length 3



Picture of Splitting of Prime
which is inert; i.e., **f=2**, g=1, e=1
1 prime cycle D above, & D is lift of C^2.

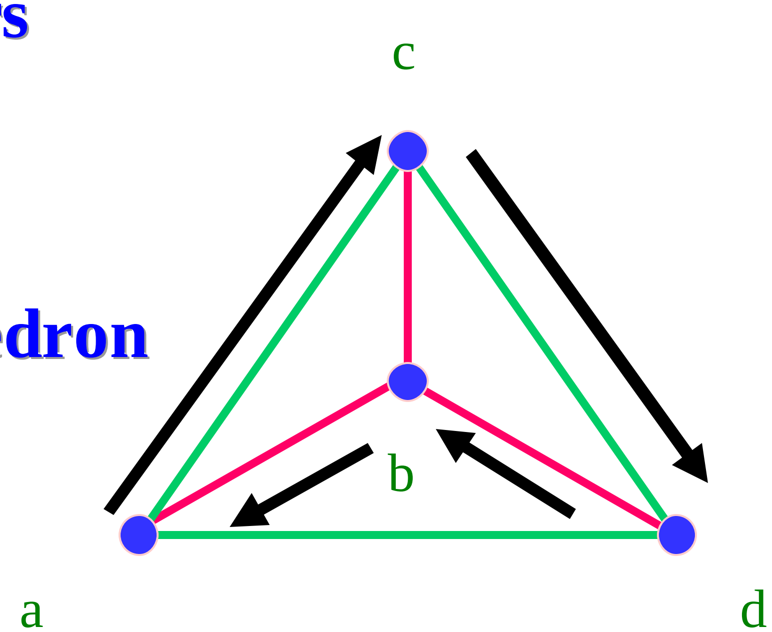
Example of Splitting of Primes in Quadratic Cover, g=2

Cube



covers

Tetrahedron



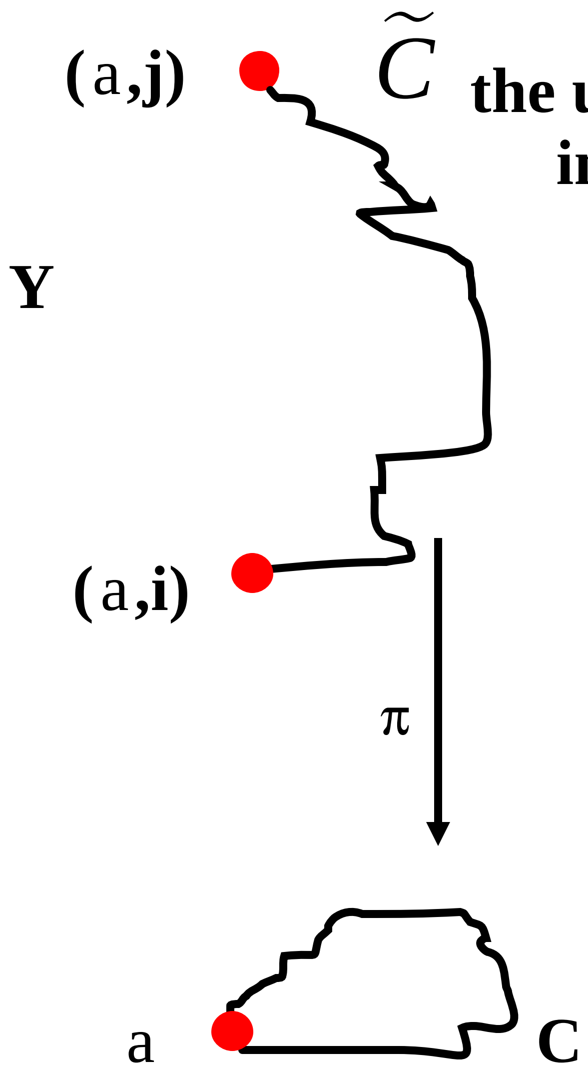
Picture of Splitting of Prime which splits completely; i.e., f=1, **g=2**, e=1
2 primes cycles above

FROBENIUS AUTOMORPHISM

C a "prime" in X, D a prime over C in Y

$$\textbf{Frob(D)} = \left(\frac{Y/X}{D}\right) = ji^{-1} \in G = Gal(Y/X)$$

where ji^{-1} maps sheet i to sheet j



the unique lift of C in Y starting at (a,i)

$\widetilde{C}$ **not necessarily closed**

$$n\left(\widetilde{C}\right) = n\left(C\right)$$

(D the prime above C is closed)

Exercise: Compute Frob(D) on preceding pages, G={1,g}.

Answers.
preceding page: 1,
page before that: g

Properties of Frobenius

1) Replace (a,i) with (a,hi). Then
Frob(D) = ji^{-1} is replaced with $hji^{-1}h^{-1}$.
Conjugacy class of Frob(D) Î Gal(Y/X)
does not change.

2) Varying a does not change Frob(D).

3) $\text{Frob}(D)^j = \text{Frob}(D)^j$.

Artin L-Function

r = representation of G=Gal(Y/X), u Î C, |u| small

$$L(u, r, Y/X) = \prod_{[C]} \det\left(1 - r\left(\frac{Y/X}{D}\right) u^{n(C)}\right)^{-1}$$

[C]=equivalence class of primes of X
n(C)=length C,   D a prime in Y over C