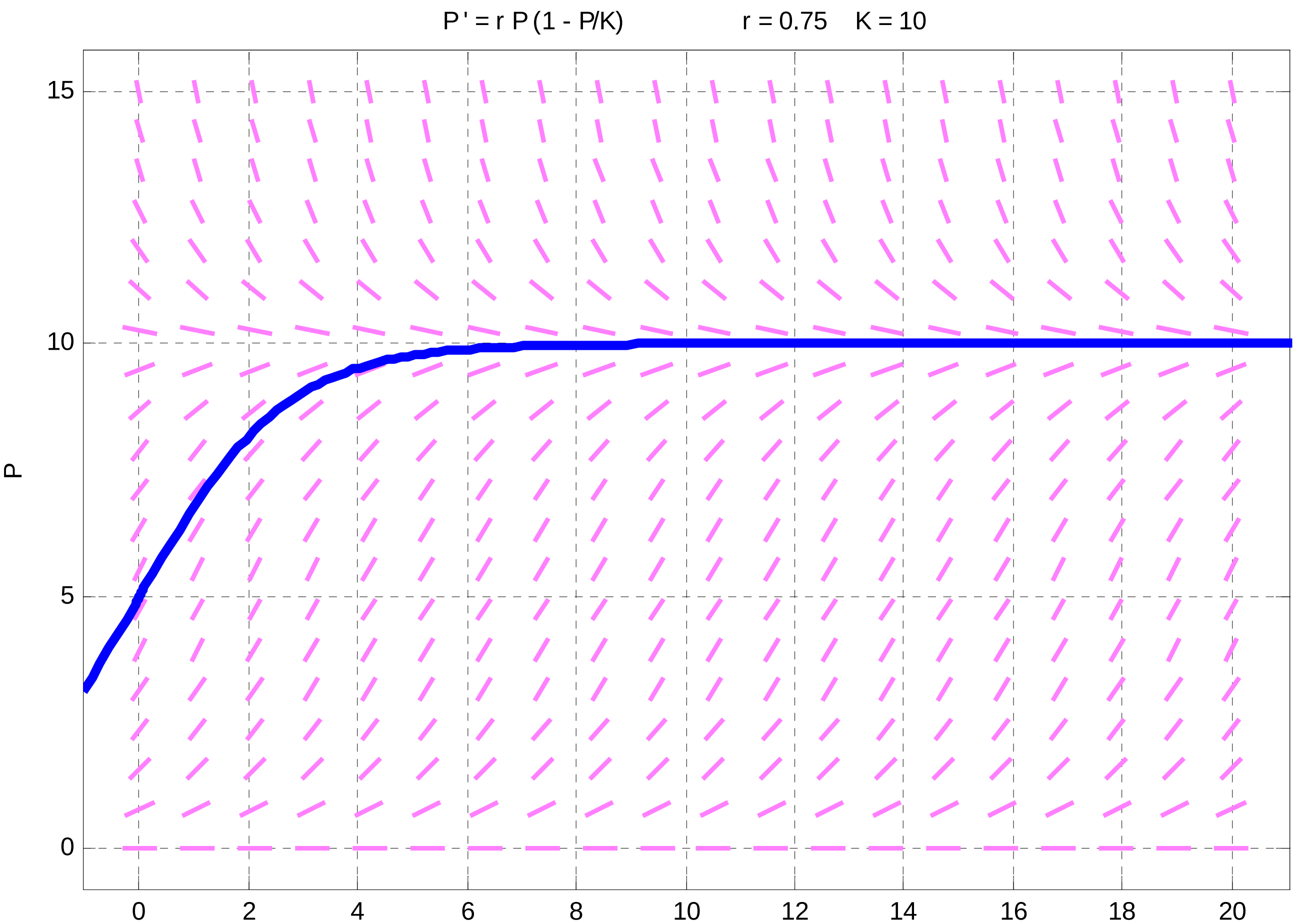
The Logistic Equation

(a model for population growth)

$P'(t) = r\,P(t)(1-P(t)/K)$

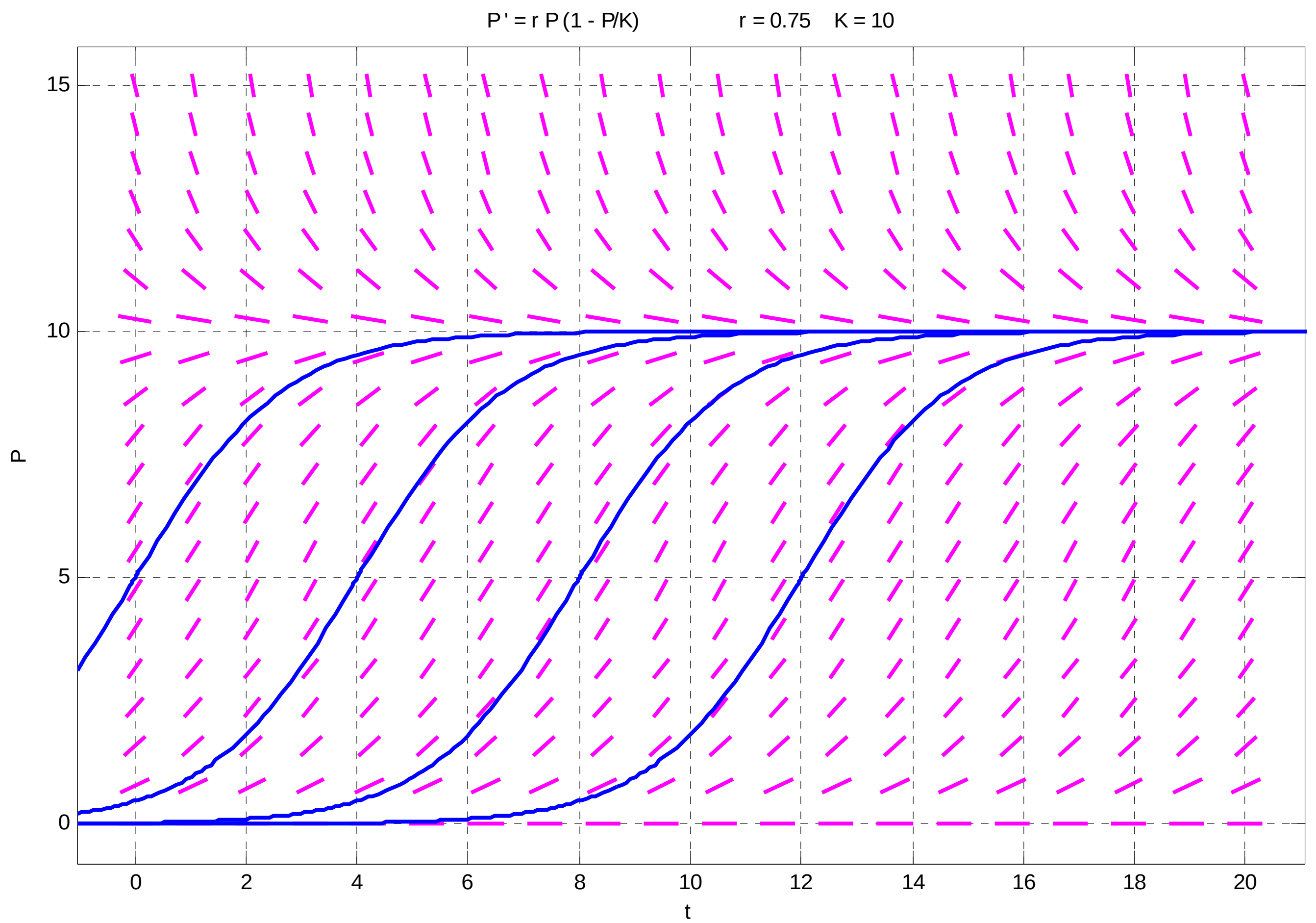
Here r=.75 and K=10.



We plot a solution curve corresponding to P(0)=5.

The Logistic Equation with arrows for direction field and more solution curves

Here y(0)=5 for the left-most curve.



The logistic equation is an example of an autonomous ODE since the right hand side is independent of t. This means if y(t) solves the ODE, so does y(t-c) for any constant c. The graph of y(t-c) looks the same as that of y(t) except shifted to the right by c. In the picture we shift a solution by 4, 8 and 12.

Exact Solution. **The Logistic equation is separable.**

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right)$$

$$\frac{dy}{y\left(1 - \frac{y}{K}\right)} = rdt$$

Partial fractions gives

$$\left(\frac{1}{y} + \frac{1/K}{1 - y/K}\right) dy = rdt$$

$$\left(\frac{1}{y} + \frac{1}{K - y}\right) dy = rdt$$

Solving

$\ln|y| - \ln|K-y| = rt + C'$

$\ln|y/(K-y)| = rt + C'$

exponentiating

$y/(K-y) = \pm e^{rx+C'} = \pm e^{rt} e^{C'}$

$y/(K-y) = C e^{rt}$ **where** $C = \pm e^{C'}$

Now to find y

$y/(K-y) = C e^{rt}$

$y = C(K-y) e^{rt} = CKe^{rt} - Cye^{rt}$

$y + Cye^{rt} = CK e^{rt}$

$y(1 + Ce^{rt}) = CK e^{rt}$

$y = CKe^{rt} /(1+C e^{rt})$

Determine the constant C from the initial condition. If the initial condition is $y(0)=K/2$, for example, then

$K/2 = CK e^{r0} /(1+C e^{r0})$

which implies

$1=2C/(1+C)$

$1+C=2C$

$1=C$

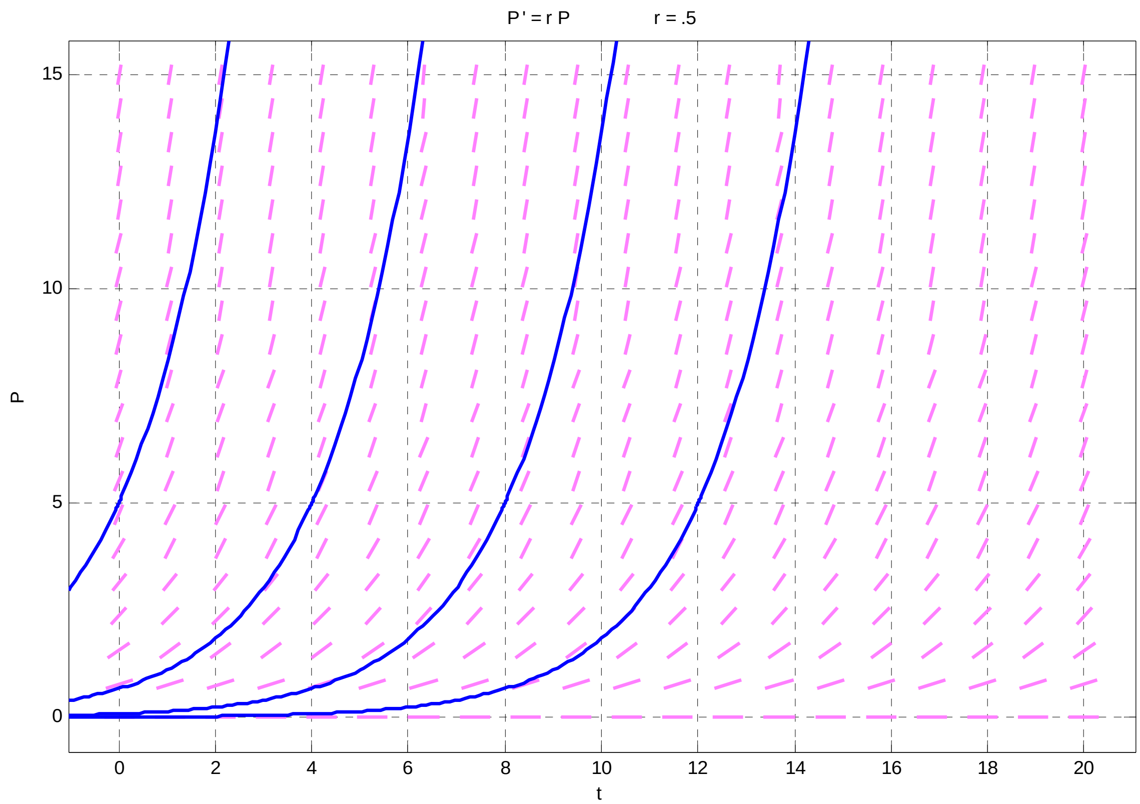
$C=1$

So for this initial condition, our solution is

$$y = K\frac{e^{rt}}{1+e^{rt}} = K\frac{1}{e^{-rt}+1}$$

Long time behavior: As $t \to \infty$, $y \to K$

There are lots of population models.
The simplest is $y'(t)=ry(t)$, $r>0$ constant.
The solution is $y(t)=Ce^{rt}$. Here $C=y(0)$.
This means the population grows exponentially.



The number of people on the earth in 1961 was around 3 billion and increasing at a rate of $r=.02$ per year. One can check whether this predicts the population now.

$$y(45) = 3 * (10^9) * (e^{.02*45}) \cong 7 * 10^9$$

This isn't too far off. As of May 4, 2006, the U.S. Census Bureau says the population of the world is about 6,513,823,130. See the website:
http://www.census.gov/ipc/www/world.html
The equation predicts the population $3.6*10^{15}$ by the year 2670. But the total surface area of the planet is only $1.8*10^{15}$square feet !!!!
The logistic equation reflects the fact that the population cannot increase indefinitely.

Exact 1st Order ODEs

Suppose we have an equation

$u(x,y)=c$, c=constant and $y=y(x)$

Using the chain rule for functions of several variables, differentiate the equation with respect to x and get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}y'(x)=0$$

Such an ODE is called exact.

Definition. $M+N(dy/dx) = 0$ is exact $\Leftrightarrow$ there is a function $u(x,y)$ so that

$$M = u_x \quad \text{and} \quad N = u_y.$$

Assuming the partials continuous on some open rectangle, then $u_{xy}=u_{yx}$ and this means $M_y=N_x$. It turns out the converse is also true; meaning that (assuming M, N, M_y, N_x continuous in an open rectangle)

$$M+N(dy/dx) = 0 \text{ is exact } \Leftrightarrow M_y=N_x.$$

There is a proof of this theorem in Section 2.6 of the text.

Example. $(3x^2+2xy)dx+(x^2+3y^2)dy=0$

Here $M = 3x^2+2xy$ and $N= x^2+3y^2$

$M_y=2x=N_x$ so the ODE is exact.

How to find u such that $M=u_x$, $N=u_y$?

$$u_x = M = 3x^2+2xy$$

Integrate with respect to x, holding y fixed

$$u(x,y) = \int (3x^2 + 2xy)dx + K(y)$$

(1) $u = x^3+x^2y+K(y)$.

To find K(y), recall

(2) $u_y = N = x^2+3y^2$.

Formula (1) implies $u_y = x^2+ K'(y)$.

So using (2) $x^2 + 3y^2 = x^2 + K'(y)$.

This means $3y^2= K'(y)$.

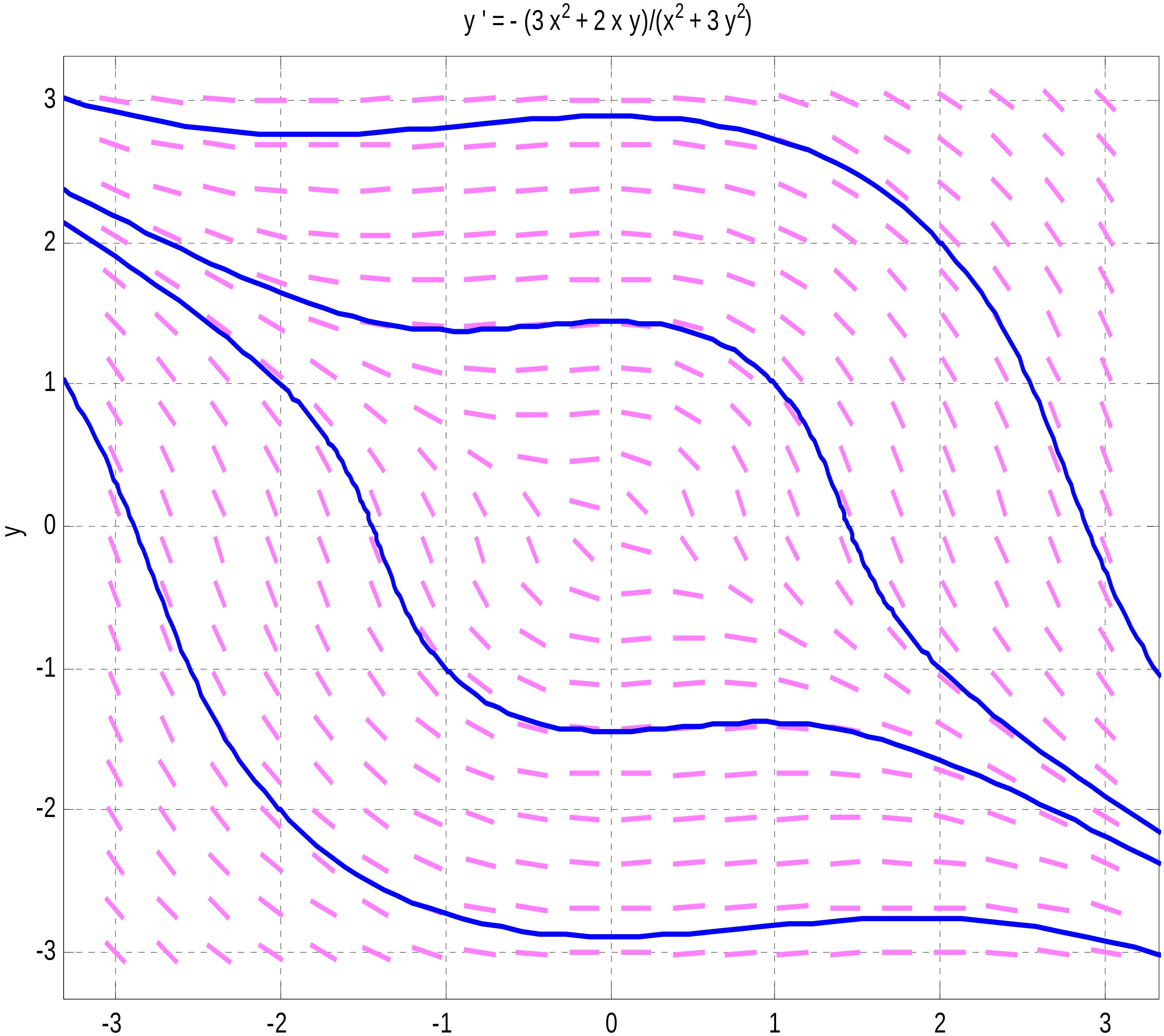
It follows that $K(y)=y^3+c'$.

Therefore by (1) we can take

$$u = x^3+x^2y+ y^3.$$

Solutions of our ODE are given implicitly by $x^3+x^2y+y^3=c$ where c is a constant.

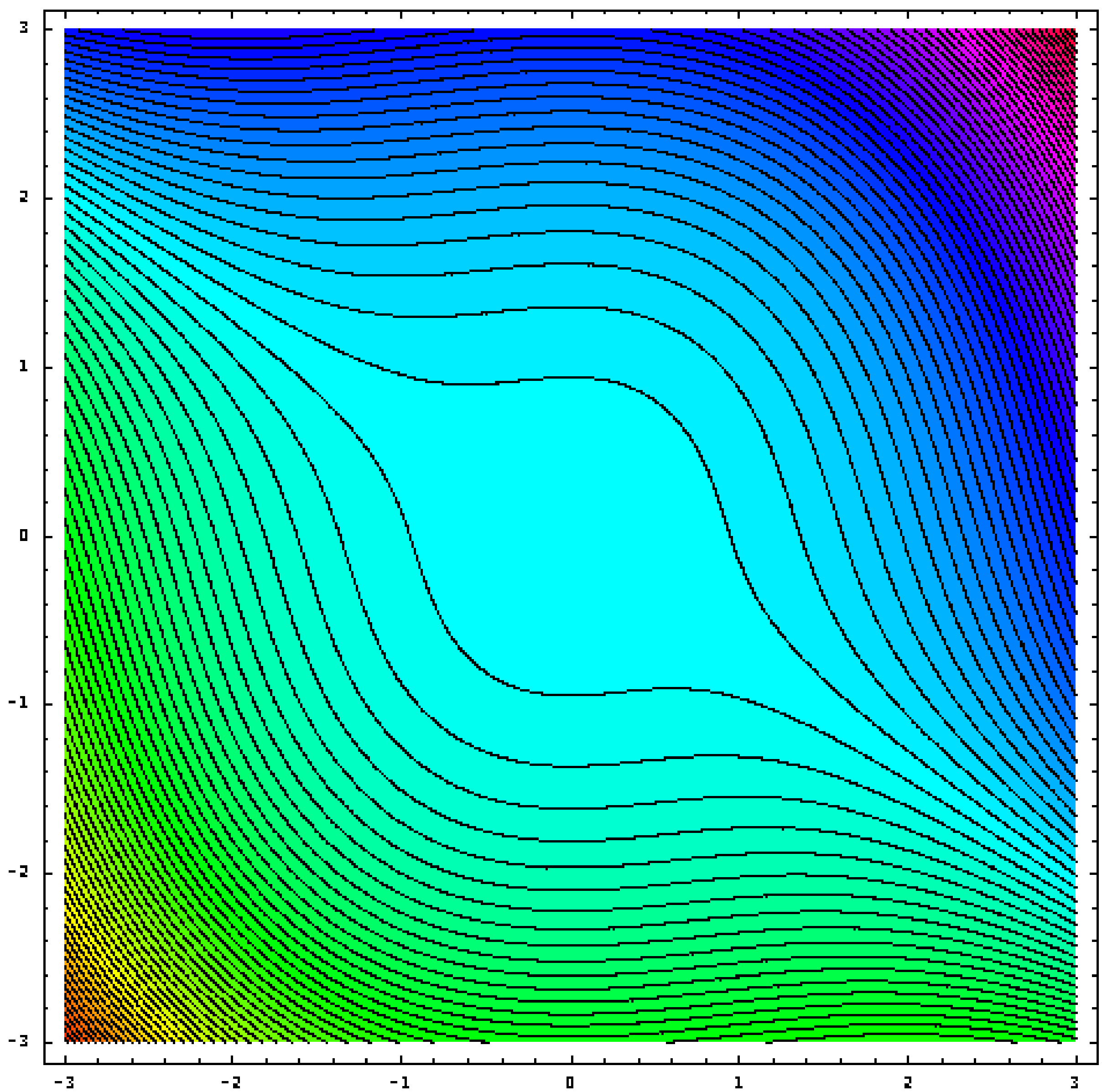
Matlab draws the direction field for the exact equation $(3x^2+2xy)dx+(x^2+3y^2)dy=0$



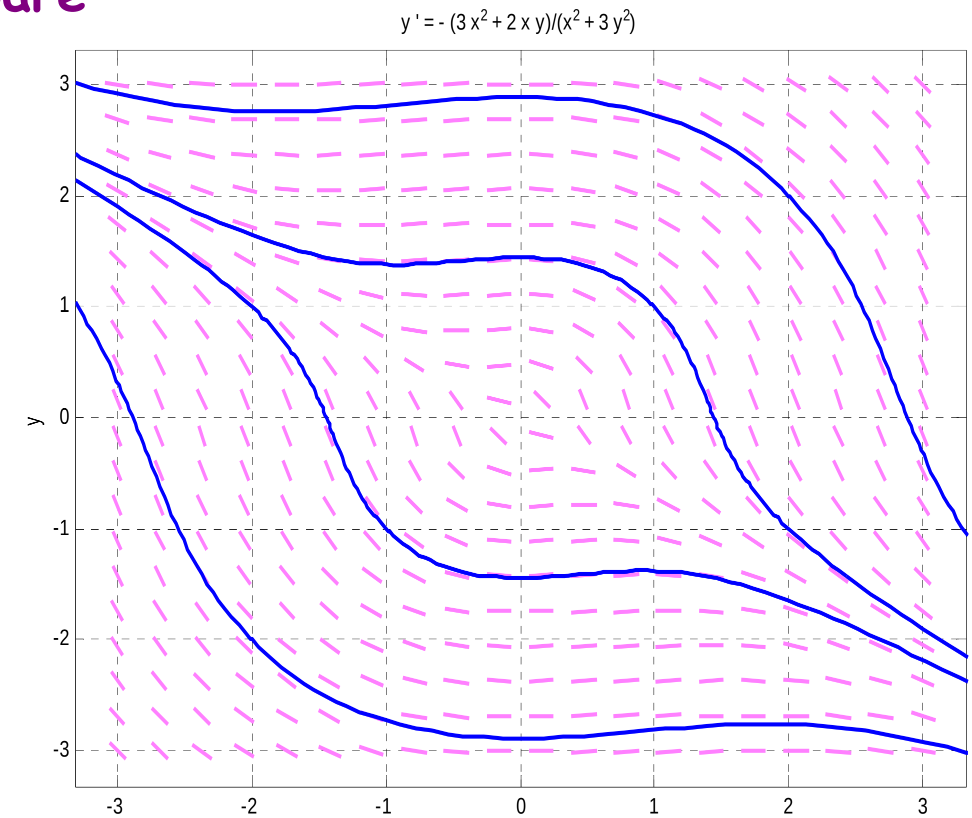
Mathematica draws the contours for the implicit solution

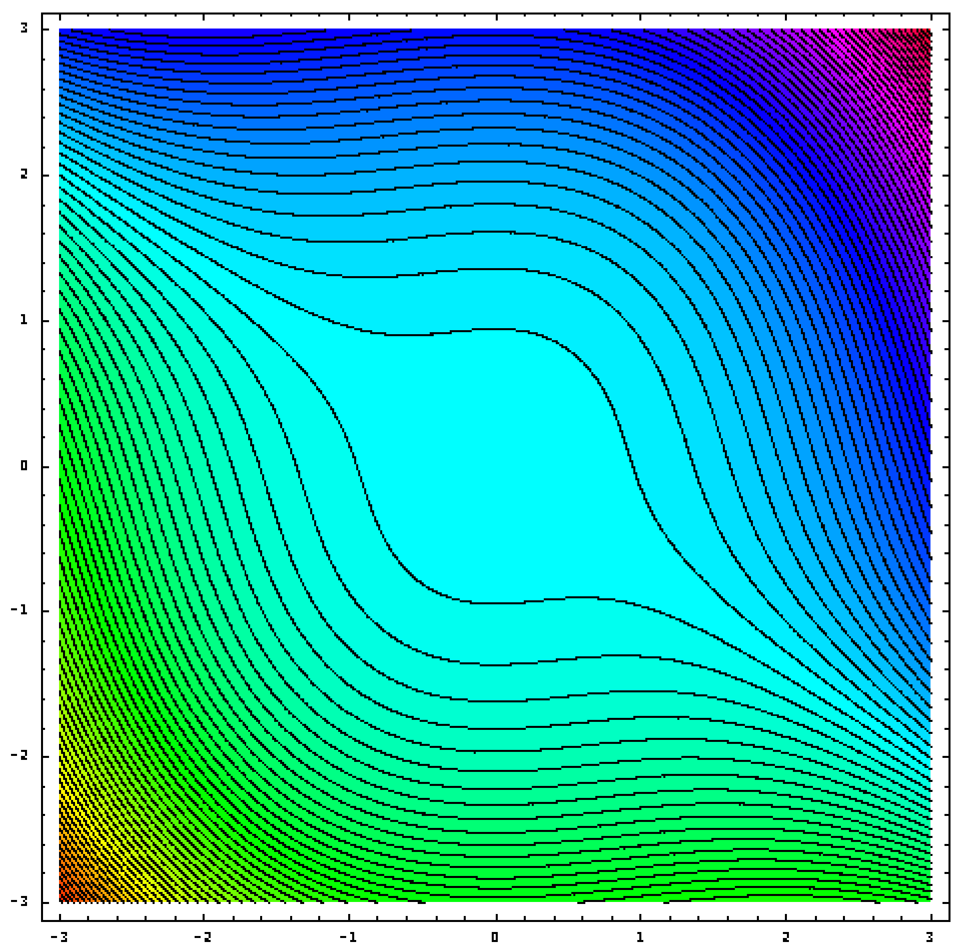
$$u = x^3 + x^2 y + y^3.$$

Each line represents u = constant and the colors tell how big the constant is.



Compare





We can extend the method of exact ODEs using an integrating factor.

If $\quad M\,dx + N\,dy = 0$

is not exact, sometimes one can multiply by some function $v(x,y)$ so that

$$v\,M\,dx + v\,N\,dy = 0$$

is exact.

Example. $y\,dx - (x+y^3)dy = 0$

is not exact since

$$M_y = 1 \quad \text{and} \quad N_x = -1.$$

Rewrite the ODE as

$$y\,dx - x\,dy = y^3\,dy.$$

Recall that $$d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}.$$

This makes us want to multiply the ODE by $v = 1/(y^2)$, assuming $y \neq 0$.

That gives $\quad d(x/y) = y\,dy.$

Integration yields $\quad x/y = (1/2)\,y^2 + c.$

So we take $\quad u(x,y) = (x/y) - (1/2)\,y^2.$

The implicit solution of our ODE is (for $y \neq 0$)

$$(x/y) - (1/2)\,y^2 = c.$$