

Determinants associated to Zeta matrices of posets and their relation to graph theory

Cristina Ballantine*

Sharon Frechette

John Little

College of the Holy Cross

Special Session on Zeta Functions of Graphs, Ramanujan
Graphs, and Related Topics

San Diego

January 6, 2008

Introduction

- $P = (P, \leq)$ - poset (partially ordered set)

Examples: $n \in \mathbb{N}$, $[n] = \{1, 2, \dots, n\}$

- $([n], \leq)$ - *chain* (totally ordered)
- $P_n = \{A : A \subseteq [n]\}$, (P_n, \subseteq) - *boolean algebra* of rank n ;
 $P_n = 2^{[n]}$
- $D_n = \{d \in \mathbb{Z}^+ : d|n\}$, $i \leq j \iff i|j$ - *divisor poset*

Incidence algebra

- closed *interval* (defined if $x \leq y$):
 $[x, y] = \{z \in P : x \leq z \leq y\}$
- Incidence algebra: $I(P) = \{f : \text{Int}(P) \rightarrow \mathbb{C}\}$, convolution

$$fg(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$$

$I(P)$ associative \mathbb{C} -algebra. Identity: $\delta(x, y)$

Zeta function - Zeta Matrix

- $\zeta \in I(P)$, $\zeta(x, y) = 1$, $\forall x \leq y$ in P
- If P locally finite, $\zeta^{-1} = \mu$ Möbius function.
Möbius Inversion Formula
- $|P| = n$
Label elements of P : x_1, x_2, \dots, x_n such that $x_i < x_j \implies i < j$
- Zeta matrix Z_P of P : $n \times n$ unipotent, upper triangular matrix

$$(Z_P)_{ij} = \begin{cases} 1 & \text{if } x_i \leq x_j \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \zeta(x_i, x_j) & \text{if } x_i \leq x_j \\ 0 & \text{otherwise} \end{cases}$$

- $\mathfrak{Z}_P := Z_P + Z_P^t$

Goals

- (1) Combinatorial interpretation of $\det(\mathfrak{Z}_P)$ involving disjoint cycles in D_P (directed graph associated with P).
- (2) Combinatorial interpretation of $\det(\mathfrak{Z}_P)$ involving cycle decomp. of elements of S_n^P (subgp. of S_n associated with P).

Theorem (Boolean case)

If $n \geq 3$ is odd, then $\det(\mathfrak{Z}_n) = 0$.

If n is even, then $\det(\mathfrak{Z}_n) = 2^{\alpha_n}$,

where $\alpha_2 = 2$, and $\alpha_n = 4\alpha_{n-2} - 2$ for $n \geq 4$.

Corollary

If n is even, then $\det(\mathfrak{Z}_n) = 2^{\frac{2^n+2}{3}}$.

Combinatorial interpretation of $\det(\mathfrak{Z}_n)$

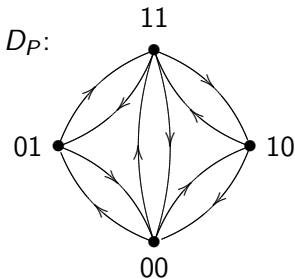
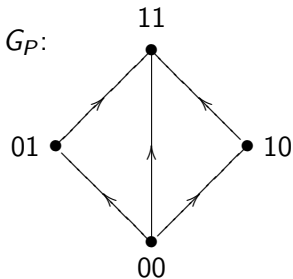
- $Y_P = Z_P - I_n$ adjacency matrix of the directed graph G_P associated with $<$ in P
vertices of G_P : elements of P
directed edge from x to $y \leftrightarrow x < y$
- $\mathfrak{Y}_P = Y_P + Y_P^t$ adjacency matrix of the directed graph D_P (edges in both directions if $x < y$)

Combinatorial interpretation of $\det(\mathfrak{Z}_n)$

- **Example:** $P = P_2$

$\emptyset \longleftrightarrow 00$, $\{1\} \longleftrightarrow 01$, $\{2\} \longleftrightarrow 10$, $\{1,2\} \longleftrightarrow 11$

labelling: $P = \{x_1 = 00, x_2 = 01, x_3 = 10, x_4 = 11\}$



Combinatorial interpretation of $\det(\mathfrak{Z}_n)$

- $\mathfrak{Z}_P = Z_P + Z_P^t = Y_P + Y_P^t + 2I_n = \mathfrak{Y}_P + 2I_n$
- $\det(\mathfrak{Z}_P) = \det(\mathfrak{Y}_P + 2I_n) = \chi(-2)$
 $\chi(t)$ characteristic polynomial of \mathfrak{Y}_P (also of D_P).

Combinatorial interpretation of $\chi(t)$

- $P = \{x_1, \dots, x_n\}$, $\mathfrak{A}_P - tI_n = (m_{ij})$

- $$\chi(t) = \sum_{\sigma \in S_n} (-1)^{\text{sign}(\sigma)} \prod_{j=1}^n m_{j\sigma(j)}$$

- Letting $k_\sigma = \#$ fixed points of the permutation σ , we have

$$\prod_{j=1}^n m_{j\sigma(j)} = \begin{cases} 0 & \text{if } x_j \text{ not comparable to } x_{\sigma(j)} \text{ for some } j, \\ (-t)^{k_\sigma} & \text{otherwise.} \end{cases}$$

- $S_n^P := \{\sigma \in S_n : x_j \text{ comparable to } x_{\sigma(j)}, \forall j = 1, \dots, n\}$

- $$\chi(t) = \sum_{\sigma \in S_n^P} (-1)^{\text{sign}(\sigma)} (-t)^{k_\sigma}$$

Combinatorial interpretation of $\chi(t)$

- If $\chi(t) = \sum_{i=0}^n a_i (-t)^{n-i}$, then $a_0 = 1$ and for $i \geq 1$ (Harary and Mowshowitz)

$$a_i = \sum (-1)^{i+r} f_{D_P}(i_1, \dots, i_r)$$

sum over all partitions $i = i_1 + \dots + i_r$ with $i_j > 0$ for all j

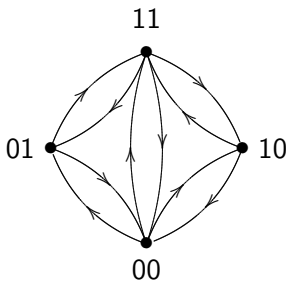
$f_{D_P}(i_1, \dots, i_r) = \#$ collections of disjoint directed cycles in D_P of lengths i_1, \dots, i_r

- No loops in $D_P \implies i_j \geq 2, \forall j$
- $a_1 = 0$ (no $\sigma \in S_n$ with $n - 1$ fixed points)

Example P_2

$$\mathfrak{Z}_P = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

D_P :

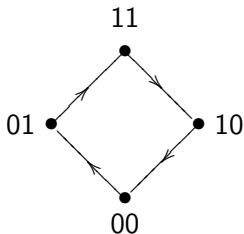
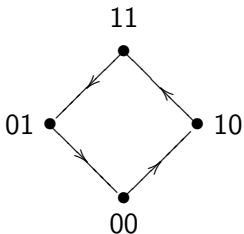


$$\chi(t) = \det(\mathfrak{Z}_P - tI_4) = \sum_{i=0}^4 a_i (-t)^{4-i} = (-t)^4 - 5(-t)^2 + 4(-t)$$

- $a_0 = 1, a_1 = 0, a_2 = -5, a_3 = 4, a_4 = 0$

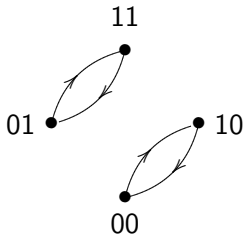
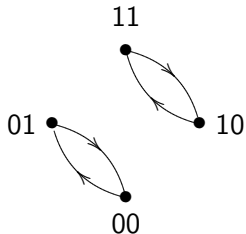
Example P_2 : $a_4 = 0$

- one term of $(-t)^0 = 1$ for each permutation in S_4^P w/o fixed points: 4-cycles or products of 2 transpositions
- partitions: $4 = 4$ and $4 = 2 + 2$
- $f_{D_P}(4)$ counts ($-$ sign)



Example P_2 : $a_4 = 0$

- $f_{D_P}(2, 2)$ counts (+ sign)



- Thus $a_4 = -2 + 2 = 0$

Combinatorial interpretation of $\det(\mathfrak{Z}_n)$

- $\chi(-2) = 2^n + \sum_{i=1}^n \left(\sum_{\sum i_j = i} (-1)^{i+r} f_{D_P}(i_1, \dots, i_r) \right) 2^{n-i}$
- $i = \sum_{j=1}^r i_j \leftrightarrow n - i$ vertices not contained in cycles
- $s = n - i + r$ total # of connected components (r cycles and $n - i$ disconnected vertices)

$$\bullet \chi(-2) = \sum_{s=1}^n (-1)^{n-s} \left(\sum_{\substack{r, (i_j) \\ s=r+n-\sum i_j}} f_{D_P}(i_1, \dots, i_r) 2^{s-r} \right) =$$

$$\sum_{s=1}^n (-1)^{n-s} c_s$$

Combinatorial interpretation of $\det(\mathfrak{Z}_n)$

- reinterpret $c_s = \sum_{\substack{r, (i_j) \\ s=r+n-\sum i_j}} f_{D_P}(i_1, \dots, i_r, 1, \dots, 1)$
 ($s - r$ ones)
- $c_s = \#$ collections of disjoint directed cycles of lengths $i_1, \dots, i_r, 1, \dots, 1$ containing *all* the vertices in directed graph $\overline{D_P}$ (add two loops at each vertex in D_P)
- $c_s = \sum_{\ell=0}^{s-1} \left| \left\{ \sigma \in S_n^P : \begin{array}{l} \sigma \text{ is a product of } s - \ell \text{ disjoint} \\ \text{cycles of length } > 1 \end{array} \right\} \right| 2^\ell$
 and $c_n = 2^n$

Recursive formula for $\det(\mathfrak{Z}_n)$ - boolean case P_n

Theorem

If $n \geq 3$ is odd, then $\det(\mathfrak{Z}_n) = 0$. If n is even, then $\det(\mathfrak{Z}_n) = 2^{\alpha_n}$ where $\alpha_2 = 2$, and $\alpha_n = 4\alpha_{n-2} - 2$ for $n \geq 4$.

- $Z_n = \begin{pmatrix} Z_{n-1} & Z_{n-1} \\ 0 & Z_{n-1} \end{pmatrix}$ and $\mathfrak{Z}_n = \begin{pmatrix} \mathfrak{Z}_{n-1} & Z_{n-1} \\ Z_{n-1}^t & \mathfrak{Z}_{n-1} \end{pmatrix}$ for $n \geq 2$
- Krattenthaler: introduce parameters

$$\mathfrak{Z}_n(x, y) = xZ_n + yZ_n^t = \begin{pmatrix} xZ_{n-1} + yZ_{n-1}^t & xZ_{n-1} \\ yZ_{n-1}^t & xZ_{n-1} + yZ_{n-1}^t \end{pmatrix}$$

- $\det(\mathfrak{Z}_{n+2}(x, y)) = (\det(\mathfrak{Z}_n(x, y)))^2 \det(\mathfrak{Z}_{n+1}(x, -y))$
- solve recurrence