Spectra of Heisenberg Graphs over Finite Rings: Histograms, Zeta Functions and Butterflies

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Abstract

We investigate spectra of Cayley graphs for the Heisenberg group over finite rings $\mathbb{Z}/p^n\mathbb{Z}$, where p is a prime. Emphasis is on graphs of degree four. We show that for odd p there is only one such connected graph up to isomorphism. When $p = 2$, there are at most two isomorphism classes. We study the spectra using two main tools - representations of the Heisenberg group and the theory of covering graphs. The former method allows us to produce histograms and butterfly diagrams of the spectra. The latter method allows us to compute the Ihara-Selberg zeta functions of the smallest of these graphs. The spectra and zeta functions of these nilpotent group graphs are compared with spectra and zeta functions of finite torus graphs which are Cayley graphs for the abelian groups $(\mathbb{Z}/p^n\mathbb{Z})^r$.

1 INTRODUCTION.

The aim of this paper is to study spectra of Cayley graphs attached to finite Heisenberg groups using histograms and Ihara zeta functions as well as Hofstadter butterfly figures. In order to do this, we will need to study the representations of the groups and the graph coverings provided by coverings of rings.

The **Heisenberg group** $H(R)$ over a ring R consists of upper triangular 3×3 matrices with entries in R and ones on the diagonal. When R is the field of real numbers $\mathbb{R}$, the group makes its presence known in quantum physics, in particular, when considering the uncertainty principle. It also is important in the theory of radar cross-ambiguity functions. See W. Schempp [21]. When the ring R is $\mathbb{Z}$, the ring of integers, there are degree 4 and 6 Cayley graphs (see the next paragraph) associated to $H(\mathbb{Z})$ whose spectra (i.e., eigenvalues of the adjacency matrix) have been much studied starting with D. R. Hofstadter's work on energy levels of Bloch electrons [8] which includes a picture of the Hofstadter butterfly. This subject also goes under the name of the spectrum of the almost Mathieu operator or the Harper operator. Or one can just look at the finite difference equation corresponding to Mathieu's equation $y'' - 2\theta\cos(2x)y = -ay$. M. P. Lamoureux's web site (http://www.math.ucalgary.ca/~mikel/mathieu.html) has a picture and references. For results concerning the Cantor-set structure of the spectra, see M. D. Choi, G. A. Elliott and Noriko Yui [2]. Other references are C. Béguin, A. Valette and A. Zuk [1] as well as M. Kotani and T. Sunada [10]. Approximations to these spectra can be pictured as Hofstadter butterflies. Compare these with the last figures in this paper which use the same method to depict the spectrum of a Cayley graph for a finite Heisenberg group.

If S is a subset of a finite group G, the **Cayley graph** $X(G,S)$ has as its vertex set the set G. Edges connect vertices $g \in G$ and gs, for all $s \in S$. Usually we will assume that $s \in S$ implies $s^{-1} \in S$ so that the graph is undirected. And we will normally assume that S is a set of generators of G so that the graph will be connected. It is not hard to see that the spectrum of the adjacency matrix of $X(G,S)$ is contained in the interval $[-k,k]$, if $k = |S|$.

Heisenberg groups over finite fields have provided a tool in the search for random number generators (see Maria Zack [30]) as well as the search for Ramanujan graphs (see Perla Myers [19]). **Ramanujan graphs** were defined by A. Lubotzky, R. Phillips and P. Sarnak [15] to be finite connected k-regular graphs such that the eigenvalues λ of the adjacency matrix satisfy $|\lambda| \leq 2\sqrt{k-1}$. Other references are P. Diaconis and L. Saloff-Coste [5] and Terras [26]. As shown in the last reference, the size of the second largest (in absolute value) eigenvalue of the adjacency matrix governs the speed of convergence to uniform for the standard random walk on a connected regular graph. Ramanujan graphs have the best possible eigenvalue bound for connected regular graphs of fixed degree in an infinite sequence of graphs with number of vertices going to infinity. For such graphs, the random walker gets lost as quickly as possible. Equivalently, this says that such graphs can be used to build efficient communication networks.

There are more reasons to study the Heisenberg group. First, as a nilpotent group (see A. Terras [26] for the definition), it may be viewed as the closest to abelian. Second, it is an important subgroup of $GL(3,R)$ (the general linear group of matrices x such that x and x^{-1} have entries in the ring R) for those interested in creating a finite model of the symmetric space of the real $GL(3,\mathsf{R})$ analogous to the finite upper half plane model of the Poincaré upper half plane.

Of course there is more to the spectrum than the second largest eigenvalue in absolute value. Quantum physicists have long been interested in the distribution of eigenvalues or energy levels of Schrödinger operators as well as finite matrix approximations to Schrödinger operators. For example, in the 1950s E. Wigner considered spectra of real symmetric $n \times n$ matrices whose entries are independent Gaussian random variables [29]. He found that the histogram of the eigenvalues looks like a semi-circle. The histograms from our finite Heisenberg graphs can be found in Figures 1 - 4. They are definitely not semi-circles or even semi-ellipses.

Quantum chaoticists study histograms of various spectra. This MSRI website (http://www.msri.org/) has movies and transparencies of many talks from 1999 on the subject. See, for example, the talks of Sarnak from Spring, 1999. Other references are Sarnak [20] and Terras [27], [28]. In this context, one also studies the distribution function of the histogram of level spacings (differences of adjacent eigenvalues). See Fan Chung [3], J. D. Lafferty and D. Rockmore [11], Winnie Li [12], Lubotzky, [14], and Terras [25], [26] for more examples.

There are alternatives to histograms for the depiction of spectra of k-regular graphs, as was demonstrated by Hofstadter [8], who separates spectra into bands which for us correspond to higher dimensional irreducible representations of the Heisenberg group. These Hofstadter butterflies are quite beautiful. We will consider such depictions of the spectrum in the last section. See Figures 10 and 11.

Yet another view of the spectrum of a graph comes from the Ihara-Selberg zeta function. This is an analogue of the **Riemann zeta function** $\zeta(s)$. The latter is defined for $\mathrm{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Thanks to this Euler product, the zeros of zeta are important for any work on the statistics of the primes. The zeros of $\zeta(s)$ have been of interest to number theorists since the work of Riemann. For example now there is a million dollar prize problem to prove the Riemann hypothesis which says that the non-real zeros of the analytic continuation of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = 1/2$. For a report of experimental verification for the first 30 billion zeros, see the web site http://www.hipilib.de/zeta/index.html. Quantum chaoticists have experimental evidence that the zeros of zeta behave analogously to the eigenvalues of a random Hermitian matrix. See N. Katz and P. Sarnak [9] for a discussion of various zeta functions whose zeros and poles have been studied in the same manner that the physicists study energy levels of physical systems.

To define a graph-theoretical analogue of $\zeta(s)$, we must define "prime" in a graph X. Modelling the idea of the Selberg zeta function of a Riemannian manifold, we use the prime cycles $[C]$ in X. Orient the edges of X, which we assume is a finite connected graph. A **prime** $[C]$ in X is an equivalence class of tailless backtrackless primitive cycles C in X. Here $C = a_1 a_2 \cdots a_s$, where the a_j are oriented edges of X. The **length** of C is $\nu(C) = s$. “Backtrackless” means that $a_{i+1} \neq a_i^{-1}$, for all i. “Tailless” means that $a_s^{-1} \neq a_1$. The “equivalence class” of C is $[C]$ which consists of all cycles

$a_i a_{i+1} \cdots a_s a_1 a_2 \cdots a_{i-1}$; i.e., the same path with all possible starting points. We call the class $[C]$ "primitive" if you only go around once; i.e., $C \neq D^m$, for all integers $m \geq 2$ and all paths D in X.

The **Ihara zeta function** of a connected graph X is defined for $u \in \mathbb{C}$, with $|u|$ sufficiently small, by

$$\zeta_X(u) = \prod_{\substack{[C] \text{ prime} \\ \text{cycle in } X}} \left(1 - u^{\nu(C)}\right)^{-1}. \tag{1}$$

The connection with the adjacency matrix A of X is given by **Ihara's theorem** which says

$$\zeta_X(u)^{-1} = (1-u^2)^{r-1} \det(I - Au + Qu^2), \tag{2}$$

where $r = |E| - |V| - 1$=rank of fundamental group of X and Q is the diagonal matrix whose jth diagonal entry is Q_{jj} =(-1+degree of jth vertex). Proofs of the Ihara theorem can be found in [23], [24], [26]. In the first two papers, edge and path zeta functions of more than one variable are also discussed. The most elementary proof of formula (2) was found by Bass and involves the edge zeta function associated to more than one variable for which the analogous determinant formula is easy to prove. See [24] pages 168 and 172.

In Section 3 we consider the Ihara-Selberg zeta functions of the finite Heisenberg graphs. From (2), we know that these zeta functions are reciprocals of polynomials. When the graph is connected and regular, one sees that it is a Ramanujan graph if (and only if) the Ihara-Selberg zeta function satisfies **the Riemann hypothesis** in the sense that the zeros of the polynomial satisfy $|u| = q^{-1/2}$, where the degree of the graph is $q+1$.

Special values or residues of the Ihara-Selberg zeta function give graph theoretic constants such as the number of spanning trees. There are connections with famous polynomials such as the Alexander polynomials of knots. See X.-S. Lin and Z. Wang [13].

Here we consider Cayley graphs $\mathcal{H}_S(q) = X(G,S)$ with vertex set the **Heisenberg group** $G = Heis(\mathbb{Z}/q\mathbb{Z})$ consisting of matrices $(x,y,z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, where $x,y,z \in \mathbb{Z}/q\mathbb{Z}$, $q = p^n$ and p is prime. In Section 2 of this paper the edge set S is at first chosen to have 4 elements $S = \left\{ X^{\pm 1}, A^{\pm 1} \right\}$, where $X = (x,y,z)$ and $A = (a,b,c)$. We assume that $ay \not\equiv bx (\bmod\, p)$ to insure that the graph is connected. See Theorem 1. For p odd, all these graphs are isomorphic. When $p = 2$, there are only two isomorphism classes. See Theorem 2. Define the **degree 4 Heisenberg graph**

$$\mathcal{H}(q) = X(\mathrm{Heis}(\mathbb{Z}/q\mathbb{Z}), \{(\pm 1, 0, 0), (0, \pm 1, 0)\}). \tag{3}$$

As $q = p^n$ goes to infinity, the spectra of the degree 4 Heisenberg graphs approach a continuous line segment from -4 to $+4$. See Theorem 3. Figures 1 - 4 show some examples of histograms of Heisenberg graph spectra for degree 4 and degree 6 graphs. These figures were made using the representations of the Heisenberg group to block diagonalize the adjacency matrix of $\mathcal{H}_S(q)$. This changes the size of the eigenvalue problem from a $p^{3n} \times p^{3n}$ matrix problem to a collection of $p^n \times p^n$ matrix problems. See Propositions 1 and 2. The histograms clearly do not approach a semi-circle nor are the graphs Ramanujan. The histograms in Figures 1 and 2 appear more like the eigenvalue distributions encountered when replacing 2-dimensional finite Euclidean space by 3-dimensional finite Euclidean space (see Terras [26], p. 92) or the result of replacing fields by rings in our finite upper half plane graphs (see Terras [26], p. 359). The histograms in Figures 1 - 4 should also be compared with those for the **finite torus graphs**

$$\mathcal{T}^{(n)}(q) = X\left((\mathbb{Z}/q\mathbb{Z})^n, \{\pm e_1, \pm e_2, \ldots, \pm e_n\}\right), \tag{4}$$

where e_i denotes a unit vector with ith component 1 and the rest 0. See Figures 7-9.

In Section 3 we introduce the theory of covering graphs. Taking $S = \{\pm(1,0,0), \pm(0,1,0)\}$, the graph $\mathcal{H}_S(p^{n+1})$ covers the graph $\mathcal{H}_S(p^n)$ in the usual sense of covering spaces in topology. See Theorem 4. The covering is unramified and normal or Galois with Abelian Galois group isomorphic to the subgroup of (x,y,z) in $\mathrm{Heis}(\mathbb{Z}/p^{n+1}\mathbb{Z})$ such that $x, y,$

and z are all congruent to 0 modulo p^n. This implies that the spectrum of the adjacency operator on $\mathcal{H}_S(p^n)$ is contained in that of $\mathcal{H}_S(p^{n+1})$. Moreover it says that the adjacency matrix of $\mathcal{H}_S(p^{n+1})$ can be block diagonalized with blocks the size of the adjacency matrix of $\mathcal{H}_S(p^n)$ associated to the characters of the Galois group. See Proposition 3. This reduces the $p^{3(n+1)} \times p^{3(n+1)}$ matrix eigenvalue problem to a collection of p^3 sparse $p^{3n} \times p^{3n}$ matrix problems. It is really the beginning of an FFT algorithm. Thus we find an alternative numerical method for computing the spectra besides that of using the representations of Heis($\mathbb{Z}/p^n\mathbb{Z}$) to block diagonalize the adjacency matrix.

The same result that implies Proposition 3 implies that the Ihara zeta function of $\mathcal{H}_S(p^{n+1})$ factors as a product of Artin-Ihara L-functions $L(u,\chi)$ corresponding to the characters χ of irreducible representations ρ of the Galois group of the covering. See K. Hashimoto [7] or H. Stark and A. Terras [24]. We use this factorization to compute the Ihara-Selberg zeta function for the smallest Heisenberg graphs. See formulas (19) and (20).

Section 4 of this paper concerns comparisons of spectra and zeta functions for Cayley graphs of the Heisenberg group with analogous Cayley graphs for finite abelian groups as well as Hofstadter butterfly graphs of the Heisenberg spectra. See Figures 10 and 11. We also find that the zeta functions of the smallest degree four Heisenberg and torus graphs can be compared using the following formula

$$\zeta_{\mathcal{H}(4)}(u)^{-1}/\zeta_{\mathcal{T}^{(2)}(4)}(u)^{-1} = (1-u^2)^{48}(3u^2+1)^{20}(3u^2-2u+1)^4(3u^2+2u+1)^4(9u^4-2u^2+1)^{10}. \quad (5)$$

In an earlier version of this paper, we considered graphs generalizing some of those in A. Medrano [17]. We looked at $X(Heis(\mathbb{Z}/p^n\mathbb{Z}, S))$, with edge set

$$S = \left\{(x,y,z) \,\middle|\, x,y,x \in \mathbb{Z}/p\mathbb{Z},\ 2 \text{ of } x,y,z \text{ are } 0,\ \text{the 3rd a unit } (\operatorname{mod} p) \right\}$$

with of degree $3(p^n - p^{n-1})$. The spectrum has very few entries. As these graphs require a somewhat different analysis, we will publish these results elsewhere. Myers [19] considers Cayley graphs for $Heis(\mathbb{Z}/p\mathbb{Z})$ with large edge sets as well. Other references related to this work are the Ph.D. theses of some of the co-authors [4],[16], [17], and [18]. The authors would like to thank J. Schulte for a correction to Proposition 1.

2 Spectra via Group Representations and Histograms

As we have said, one method we use to study graph spectra comes from the basic fact that the Fourier transform on the finite group $G = $ Heis($\mathbb{Z}/p^n\mathbb{Z}$) will block diagonalize the adjacency operator on a Cayley graph with vertex set G. See Proposition 2.

So we need to find a complete set of inequivalent irreducible unitary representations of G. One can find these representations for the case $n = 1$ in the paper of Schempp [21] or the book of Terras [26]. However the case of arbitrary n is a bit more complicated. Let us sketch the results we need - in particular, we need the matrix entries of the representations to write our programs. It is not sufficient to know the characters.

Notation.

We write $\mathbb{Z}/m\mathbb{Z}$ for the ring of integers mod m. And $(\mathbb{Z}/m\mathbb{Z})^*$ is the set of units or invertible elements for multiplication in this ring; i.e., the integers $x(\operatorname{mod} m)$ which are relatively prime to m. In the rest of the paper we use the notation $p^f \mid x$, for an integer x, if p^f divides x. We write $p^f \parallel x$ if f is the highest power such that p^f divides x. When $f = 0$, this means $p \nmid x$. When $f = n$ this means $x = 0$.

Here $\widehat{G}$ denotes a complete set of **inequivalent irreducible unitary representations** of G.

Lemma 1 Conjugacy Classes of Heis($\mathbb{Z}/p^n\mathbb{Z}$). The conjugacy classes partition themselves into $n+1$ types defined as follows for $0 \le e \le n$: $C^{(e)}_{x,y,v} = C^{(e)} = \left\{(x,y,z) \,\middle|\, z \equiv up^{n-e} + v(\operatorname{mod} p^n), u(\operatorname{mod} p^e) \right\}$, for $v(\operatorname{mod} p^{n-e})$, $p^{n-e} \parallel g.c.d.(x,y)$. So $|C^{(e)}| = p^e$ and the number of distinct classes $C^{(e)}$ is $p^{n-e}\left(p^{2e} - p^{2(e-1)}\right)$, for $e \neq 0$. The number of distinct classes is p^n for $e = 0$.

Proof. Note that

$$(a,b,c)(x,y,z)(a,b,c)^{-1} = (x,y,z+(ay-bx)).$$

This means that if $p^{n-e} \parallel g.c.d.(x,y)$, we can change z by adding multiples of p^{n-e} without changing the conjugacy class. So we obtain the formula above for the conjugacy class of (x,y,z).

Next we need to count the number of such classes. When $e=n$, we must count the $(x,y) \bmod p^n$ such that $p \nmid$ g.c.d.(x,y). If $p \nmid x$, then y is arbitrary and there are $(p^n - p^{n-1})p^n$ such pairs. If $p \mid x$ then $p \nmid y$ and there are $p^{n-1}\left(p^n - p^{n-1}\right)$ such pairs. The total number is $p^{2n} - p^{2(n-1)}$.

Suppose $e \neq 0$ or n. To see the formula for the number of classes, set $f = n-e$ and note that $p^f \parallel g.c.d.(x,y)$ iff $x = p^f u$ and $y = p^f v$ with $(u,v) \bmod p^e$ and $p \nmid g.c.d.(u,v)$. Thus the number of (x,y) with $p^f \parallel g.c.d.(x,y)$ is $p^{2e} - p^{2(e-1)}$. Multiply this by p^{n-e} to get the number of classes of type $C^{(e)}$. ■

Now we use the little group method of Mackey and Wigner to find the **irreducible unitary representations** of $Heis(\mathsf{Z}/p^n\mathsf{Z})$. See Serre [22]. We have a semi-direct product

$$G = (\mathsf{Z}/p^n\mathsf{Z}) = A \cdot H, \text{ where } A = \left\{ \ (0,y,z) \in G \ \right\} \text{ and } H = \left\{ \ (x,0,0) \in G \ \right\}.$$

We find that

$$(x,0,0)^{-1}(0,y,z)(x,0,0) = (0,y,z-xy).$$

The characters of A have the form

$$\lambda_{r,s}(0,y,z) = \exp\left(\ 2\pi i(ry+sz)/p^n \ \right) \text{ in } \widehat{A}.$$

The action of $h_x = (x,0,0) \in H$ on $\lambda_{r,s}$ is given by

$$h_x \lambda_{r,s}(0,y,z) = \lambda_{r,s}(0,y,z-xy) = \lambda_{r-sx,s}(0,y,z).$$

It follows that

$$\widehat{A}/H = \left\{ \ \lambda_{0,s} \ \middle| \ p \nmid s \ \right\} \cup \bigcup_{f=1}^{n-1} \left\{ \ \lambda_{r,s} \ \middle| \ p^f \parallel s, \ \ r \bmod p^f \ \right\} \cup \left\{ \ \lambda_{r,0} \ \middle| \ r \in \mathsf{Z}/p^n\mathsf{Z} \ \right\}.$$

Define $H_{r,s} = \left\{ \ (x,0,0) \ \middle| \ \lambda_{r,s} = \lambda_{r-sx,s} \ \right\}$. Thus the representations split into types we will call $\Theta^{(f)}_{r,s,t}$, where $0 \le f \le n$.

Look at case f. Here $p^f \parallel s, \ 0 \le f \le n$. Then set

$$H^{(f)}_{r,s} = \left\{ \ (x,0,0) \ \middle| \ p^{n-f} \text{ divides } x \ \right\}.$$

This group has order p^f. Define $G^{(f)}_{r,s} = A \cdot H^{(f)}_{r,s}$. Extend $\lambda_{r,s}$ to $G^{(f)}_{r,s}$ by making it constant on $H^{(f)}_{r,s}$. Let ρ_t be a character on $H^{(f)}_{r,s}$ and extend it to $G^{(f)}_{r,s}$ by making it constant on A. Here $\rho_t(x,0,0) = \exp(2\pi i t x/p^n)$, where p^{n-f} divides x. Then $\Theta^{(f)}_{r,s,t}$ is an induced representation defined by inducing the representation $\lambda_{r,s} \otimes \rho_t$ on $G^{(f)}_{r,s}$ up to the full Heisenberg group G; i.e.

$$\Theta^{(f)}_{r,s,t} = \mathrm{Ind}^{G}_{G^{(f)}_{r,s}}(\lambda_{r,s} \otimes \rho_t).$$

Here we take r and $t \bmod p^f$, $s = p^f s_1$, where p does not divide s_1 and $s_1 (\bmod p^{n-f})$. The degree of $\Theta^{(f)}_{r,s,t}$ is p^{n-f}.

Next let us compute the character of $\Theta^{(f)}_{r,s,t}$ using the Frobenius formula for the character of an induced representation. See Terras [26], p. 271. The formula says, if $\psi = \lambda_{r,s} \otimes \rho_t$:

$$\chi_{\Theta^{(f)}_{r,s,t}}(x,y,z) = \sum_{a \in \mathsf{Z}/p^{n-f}\mathsf{Z}} \widetilde{\psi}\left(\ (a,0,0)(x,y,z)(-a,0,0) \ \right).$$

The elements $(a,0,0)$ which are summed over are representatives of the quotient $G/G_{r,s}^{(f)}$. The tilda on the character means that the function is 0 when the argument does not lie in the subgroup $G_{r,s}^{(f)}$. The argument is

$$(a,0,0)(x,y,z)(-a,0,0) = (x,y,z+ay).$$

This is in $G_{r,s}^{(f)}$ when p^{n-f} divides x. So

$$\widetilde{\psi}(x,y,z+ay) = \begin{cases} 0, & \text{if } \ p^{n-f} \nmid x, \\ \exp\left(\ 2\pi i(ry + s(z+ay) + tx)/p^n \right) , & \text{if } \ p^{n-f} \mid x. \end{cases}$$

It follows upon summing over $a (\operatorname{mod} p^{n-f})$ that for $r, t (\operatorname{mod} p^f)$, $s = p^f s_1$, where $p \nmid s_1$, we have

$$\chi_{_{\Theta_{r,s,t}^{(f)}}}(x,y,z) = \begin{cases} 0, & \text{if } \ p^{n-f} \nmid x \ \text{ or } \ p^{n-f} \nmid y, \\ p^{n-f} \exp\left(\ 2\pi i(sz + ry + tx)/p^n \right) , & \text{if } \ p^{n-f} \mid x \text{ and } y. \end{cases}$$

We can also compute the matrix entries of the representations as in Terras [26], p. 270. The matrices are $p^{n-f} \times p^{n-f}$, indexed by elements $a, b \in \mathsf{Z}/p^{n-f}\mathsf{Z}$. We find the matrix entries

$$\left(\ \Theta_{r,s,t}^{(f)} \ \right)_{a,b}(x,y,z) = \begin{cases} 0, & \text{if } \quad p^{n-f} \nmid (a-b+x), \\ \exp\left(\ 2\pi i \frac{t(a-b+x)+ry+s(z+ay)}{p^n} \ \right), & \text{if } \ p^{n-f} \mid (a-b+x). \end{cases}$$

In particular, we obtain the following result.

Proposition 1 When $G = \mathrm{Heis}(\mathsf{Z}/p^n\mathsf{Z})$ the set $\widehat{G}$ of inequivalent irreducible unitary representations of G consists of degree p^{n-f} representations $\Theta_{r,s,t}^{(f)}$ with $0 \le f \le n, \quad r,t \in \mathsf{Z}/p^f\mathsf{Z}, \quad s = p^f s_1, \quad s_1 \in (\mathsf{Z}/p^{n-f}\mathsf{Z})^*$ defined by

$$\Theta_{r,s,t}^{(f)}(x,y,z) = \exp\left(\frac{2\pi i r y}{p^n}\right) \exp\left(\frac{2\pi i s_1 z}{p^{n-f}}\right) D_f^{sy} W_f(x).$$

Here we need two $p^{n-f} \times p^{n-f}$ matrices: the diagonal matrix D_f involving powers of $w = \exp(2\pi i/p^{n-f})$ given by

$$D_f = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & w & 0 & \cdots & 0 & 0 \\ 0 & 0 & w^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & w^{p^{n-f}-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & w^{p^{n-f}-1} \end{pmatrix},$$

and the matrix $W_f(x)$ whose a,b entry is

$$(W_f(x))_{a,b} = \begin{cases} 0, & \text{if } \quad p^{n-f} \nmid (a-b+x), \\ \exp\left(\ 2\pi i \frac{t(a-b+x)}{p^n} \ \right), & \text{if } \ p^{n-f} \mid (a-b+x). \end{cases}$$

Note that when $f = 0$ (in the case of the highest dimensional representation), $W_f(x) = W^x$, where W is the $n \times n$ shift matrix

$$W_f = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

So we find the character table of $G=$Heis$(\mathsf{Z}/p^n\mathsf{Z})$ to be that of Table 1 below.

Table 1. Character Table of G = Heis$(\mathsf{Z}/p^n\mathsf{Z})$. The conjugacy classes of type $C^{(e)} = C^{(e)}_{x,y,v}$ are as in Lemma 1 with $p^{n-e} \,\|$g.c.d.(x,y). The representations $\Theta^{(f)} = \Theta^{(f)}_{r,s,t}$ are as in Proposition 1, with $f = 0,1,\ldots,n$, both r and t in $\mathsf{Z}/p^f\mathsf{Z}$, $s = p^f s_1$, and $s_1 \in (\mathsf{Z}/p^{n-f}\mathsf{Z})^*$. Write $\Psi(x) = \exp(2\pi i x/p^n)$, for $x \in \mathsf{Z}/p^n\mathsf{Z}$.

rep\class	$C^{(0)}$	$\cdots$	$C^{(e)}$	$\cdots$	$C^{(n)}$
|class|	1	$\cdots$	p^e	$\cdots$	p^n
# classes	p^n	$\cdots$	$p^{n+e}-p^{n+e-2}$	$\cdots$	$p^{2n}-p^{2n-2}$
$\Theta^{(n)}$	1	$\cdots$	$\Psi(tx+ry)$	$\cdots$	$\Psi(tx+ry)$
$\Theta^{(n-1)}$	$p\Psi(sz)$	$\cdots$	$p\Psi(tx+ry+sz)$	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\Theta^{(e)}$	$p^{n-e}\Psi(sz)$	$\cdots$	$p^{n-e}\Psi(tx+ry+sz)$	$\cdots$	0
$\Theta^{(e+1)}$	$p^{n-e-1}\Psi(sz)$	$\cdots$	0	$\cdots$	0
$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\Theta^{(0)}$	$p^n\Psi(sz)$	$\cdots$	0	$\cdots$	0

We will consider Cayley graphs with vertex set Heis$(\mathsf{Z}/p^n\mathsf{Z})$ and edge set $S = \{\ X^{\pm1}, A^{\pm1}\ \}$, where $X = (x,y,z)$ and $A = (a,b,c)$. We want connected graphs. Thus we need the following theorem.

Theorem 1 Suppose that $S = \{\ X^{\pm1}, A^{\pm1}\ \}$, where $X = (x,y,z)$ and $A = (a,b,c)$ are in Heis$(\mathsf{Z}/p^n\mathsf{Z})$. If $ay \not\equiv bx (\mathrm{mod}\, p)$ then the set S generates Heis$(\mathsf{Z}/p^n\mathsf{Z})$.

Proof. Let G be the subgroup of Heis$(\mathsf{Z}/p^n\mathsf{Z})$ generated by S. Note that $(a,b,c)(x,y,z) = (a+x, b+y, c+z+ay)$ and $(a,b,c)^{-1} = (-a,-b,-c+ab)$. Also we have $(a,b,c)^e(x,y,z)^f = (ae+fx, be+fy, *)$. Then for every $\begin{pmatrix} u \\ v \end{pmatrix} \in (\mathsf{Z}/p^n\mathsf{Z})^2$, the matrix equation

$$\begin{pmatrix} ae+xf \\ be+yf \end{pmatrix} = \begin{pmatrix} a & x \\ b & y \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

is solvable for some $\begin{pmatrix} e \\ f \end{pmatrix} \in (\mathsf{Z}/p^n\mathsf{Z})^2$, as the determinant of the 2×2 matrix, $ay-bx$, is a unit. So we know that G contains elements $g = (1,0,r)$ and $h = (0,1,s)$, for some r, s in $\mathsf{Z}/p^n\mathsf{Z}$. Now

$$gh = (1,1,r+s+1), \quad (gh)^{-1} = (-1,-1,-r-s), \quad hg = (1,1,r+s).$$

It follows that $(gh)^{-1}hg = (0,0,-1)$.

Thus G contains $(0,0,w)$, for all $w \in \mathsf{Z}/p^n\mathsf{Z}$. Now for every $(u,v,w) \in$Heis$(\mathsf{Z}/p^n\mathsf{Z})$ there is a w_0 such that $(u,v,w_0) \in G$. Then $(u,v,w) = (u,v,w_0)(0,0,w-w_0)$ which puts $(u,v,w) \in G$. This completes the proof that $G=$Heis$(\mathsf{Z}/p^n\mathsf{Z})$. ∎

Next we consider the Cayley graphs with vertex set Heis$(\mathsf{Z}/p^n\mathsf{Z})$ as in Theorem 1. We will show that for odd p they are all isomorphic. In order to do this, we need a Lemma.

Lemma 2 Consider the element $(x,y,z) \in$Heis$(\mathsf{Z}/p^n\mathsf{Z})$. Then, for all integers e, $(x,y,z)^e = \left(ex, ey, ez + \frac{e(e-1)xy}{2} \right)$.

Proof. This follows by induction on e. ∎

Corollary 1 Let $(x,y,z) \in$Heis$(\mathsf{Z}/p^n\mathsf{Z})$. Then we have the following facts.

Case 1) Assume p is an odd prime.

a) $(x,y,z)^{p^n} = (0,0,0)$.
b) If x or y is not divisible by p, then the order of (x,y,z) is p^n.
Case 2) Let $p = 2$.
a) $(x,y,z)^{2^{n+1}} = (0,0,0)$.
b) If both x and y are odd, the order of (x,y,z) is 2^{n+1}.
c) If one of x and y is odd and the other is even, then the order of (x,y,z) is 2^n.

Theorem 2 Suppose the prime $p \neq 2$. Then the Cayley graphs $X(G,S)$ with vertex set $G =$Heis$(\mathbb{Z}/p^n\mathbb{Z})$ and edge set $S = \{\ X^{\pm 1}, A^{\pm 1}\ \}$, where $X = (x,y,z)$ and $A = (a,b,c)$ with $ay \not\equiv bx (\mathrm{mod}\, p)$, are all isomorphic. When $p = 2$, there are at most 2 isomorphism classes of graphs.

Proof. First assume that p is not 2. Note that the center of Heis$(\mathbb{Z}/p^n\mathbb{Z})$ consists of elements $(0,0,z),\ z \in \mathbb{Z}/p^n\mathbb{Z}$. The order of both A and X is p^n. And $AX \neq XA$ implies A and X are not in the center. Look at $C = XAX^{-1}A^{-1}$. Then we have the relations.

$$A^{p^n} = X^{p^n} = C^{p^n} = I, \quad XC = CX, \quad XA = AXC. \tag{6}$$

It follows that

$$A^iX^jC^kA^{i'}X^{j'}C^{k'} = A^{i+i'}X^{j+j'}C^{k+k'+i'j}. \tag{7}$$

Note that $(i',j',k')(i,j,k) = (I + i', j + j', k + k' + i'j)$.

Define the map $F(A^iX^jC^k) = (i,j,k)^{-1} \in$Heis$(\mathbb{Z}/p^n\mathbb{Z})$, for all $i,j,k \in \mathbb{Z}$. Then F gives a graph isomorphism $F : X(G,S) \to X(G,S_0)$, where $S_0 = \{\ (\pm 1,0,0),(0,\pm 1,0)\ \}$. To see this, you need to know that every element of $G =$Heis$(\mathbb{Z}/p^n\mathbb{Z})$ can be written uniquely in the form $A^iX^jC^k$, where $0 \leq i,j,k < p^n$. Since A and X generate G, the relations (6) above show that every element of G can be written in this form and since there are exactly p^{3n} elements in G, the result is unique.

The proof for $p = 2$ and $n \geq 2$ is similar except that we have two cases. In the case that exactly two of a,b,x,y are odd, we obtain an isomorphism from $X(G,S)$ to $X(G,S_0)$, where $S_0 = \{\ (\pm 1,0,0),(0,\pm 1,0)\ \}$.

In the case that $p = 2,\ n \geq 2$, and exactly three of a,b,x,y are odd, we obtain an isomorphism from $X(G,S)$ to $X(G,S_0)$ with $S_0 = \{\ (\pm 1,0,0),(1,1,0)^{\pm 1}\ \}$. Without loss of generality, we can assume a and b are both odd and thus that the order of A is 2^{n+1}. Set $C = XAX^{-1}A^{-1} = (0,0,bx - ay)$. Since $bx - ay$ is odd, the order of C is 2^n. Then we have $A^{2^n} = C^{2^{n-1}}$. In this case it can occur that $i + i' \geq 2^n$. If so, we can find $m \in \{0,1\}$ so that $i + i' - 2^n m \leq 2^n$ and replace equation (7) with

$$A^iX^jC^kA^{i'}X^{j'}C^{k'} = A^{i+i'-2^n m}X^{j+j'}C^{k+k'+i'j+2^{n-1}m}. \tag{8}$$

This same equation holds with our choice of generators in S_0 in Case 2. From here the proof proceeds as before. ■

Definition 1 For all prime p, define the Cayley graph

$$\mathcal{H}(p^n) = X\left(\text{Heis}(\mathbb{Z}/p^n\mathbb{Z}), \{(\pm 1,0,0),(0,\pm 1,0)\}\right).$$

When $p = 2$, define a second Cayley graph

$$\mathcal{H}(2^n)' = X\left(\text{Heis}(\mathbb{Z}/2^n\mathbb{Z}), \left\{(1,1,0)^{\pm 1},(\pm 1,0,0)\right\}\right).$$

Next we study the spectra of the graphs $\mathcal{H}(p^n)$ and $\mathcal{H}(p^n)'$. Theorem 3 says that the spectra approach a continuous line segment $[-4,4]$ as p^n approaches infinity. To prove it we need a familiar result in spectral graph theory.

Proposition 2 Suppose $X = X(G,S)$ is a Cayley graph of a finite group G. Then the eigenvalues of the adjacency operator of X are the eigenvalues of the $d_\pi \times d_\pi$ matrices

$$M_\pi = \sum_{s \in S} \pi(s),$$

each taken with multiplicity d_π as π runs through $\widehat{G}$, a complete set of irreducible unitary representations of G.

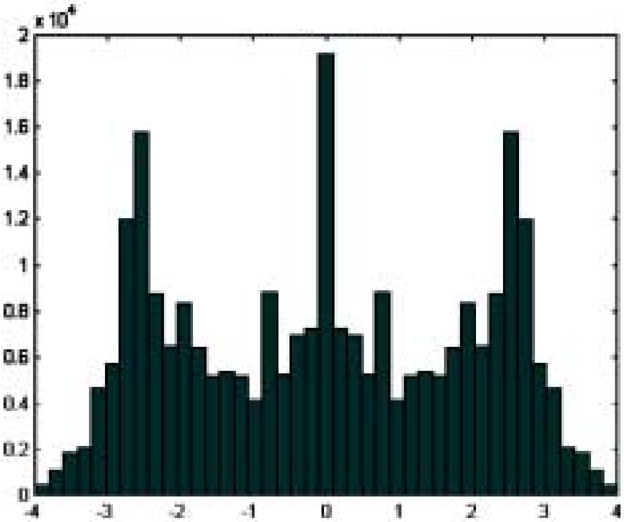

Figure 1: Histogram for spectrum of adjacency operator of $X\left(\text{ Heis}(\mathsf{Z}/64\mathsf{Z}),\left\{\ (\pm1,0,0),(0,\pm1,0)\ \right\}\ \right).$

Proof. See Terras [26], p. 257. ■

Examples.

It is not hard to use Propositions 1 and 2 and a home computer to find spectra of $\mathcal{H}(q)$ and $\mathcal{H}(q)'$ when q is less than or equal to the size of a matrix your computer can handle. Figures 1 and 2 show some histograms for the graphs $\mathcal{H}(q)$, when $q = 64$ and 81. Here the degree is 4 and the shape of the histogram is quite distinctive. It is the shape we see for any of these degree 4 Heisenberg graphs.

For comparison we include histograms of the eigenvalues for degree 6 Heisenberg graphs

$$X\left(\text{ Heis}(\mathsf{Z}/q\mathsf{Z}),\left\{\ (\pm1,0,0),(0,\pm1,0),(0,0,\pm1)\ \right\}\ \right).$$

Here the shape of the histogram is rather different. See Figures 3 and 4. We used Matlab to draw all the histograms.

Theorem 3 The spectra of the degree 4 Heisenberg graphs $\mathcal{H}(p^n)$ and $\mathcal{H}(p^n)'$ (see Definition 1) approach a continuous interval $[-4,4]$ as $p^n \to \infty$.

Proof. Use Proposition 2 and look only at the eigenvalues corresponding to the one dimensional representations of $G = \text{Heis}(\mathsf{Z}/p^n\mathsf{Z})$. This corresponds to the top line in Table 1. Let us assume that p is odd and our set S is the set $S_0 = \left\{\ (\pm1,0,0),(0,\pm1,0)\ \right\}$. Then we have an eigenvalue corresponding to $\Theta^{(f)}_{r,0,t}(x,y,z) = \exp\left(\ 2\pi i(tx+ry)/p^n\ \right)$, where $r,t \in \mathsf{Z}/p^n\mathsf{Z}$. Then the corresponding eigenvalue of the adjacency matrix of $\mathcal{H}(p^n)$ is

$$\lambda_{r,t} = 2\cos\left(\ \tfrac{2\pi t}{p^n}\ \right) + 2\cos\left(\ \tfrac{2\pi r}{p^n}\ \right). \qquad (9)$$

As r varies between 1 and $p^n - 1$, the graph of $\cos(2\pi r/p^n)$ approximates a continuous line between -1 and $+1$. The result follows.

In the case $p = 2$, we can make a similar argument for $\mathcal{H}(p^n)'$. ■

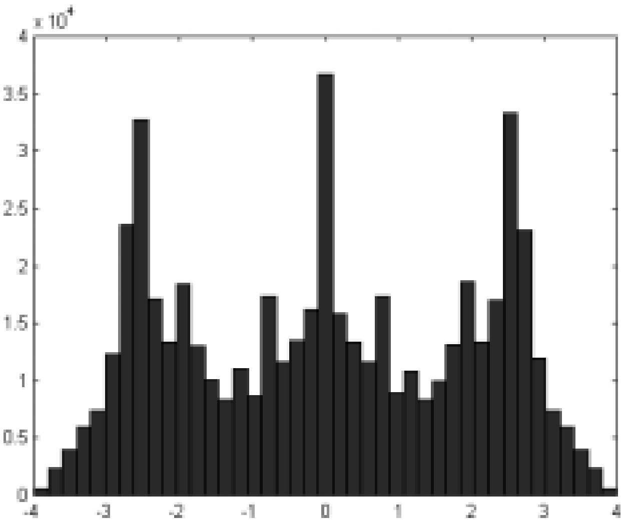

Figure 2: Histogram for spectrum of adjacency operator of $X\left(\text{ Heis}(\mathsf{Z}/81\mathsf{Z}), \left\{ \ (\pm 1, 0, 0), (0, \pm 1, 0) \ \right\} \ \right)$.

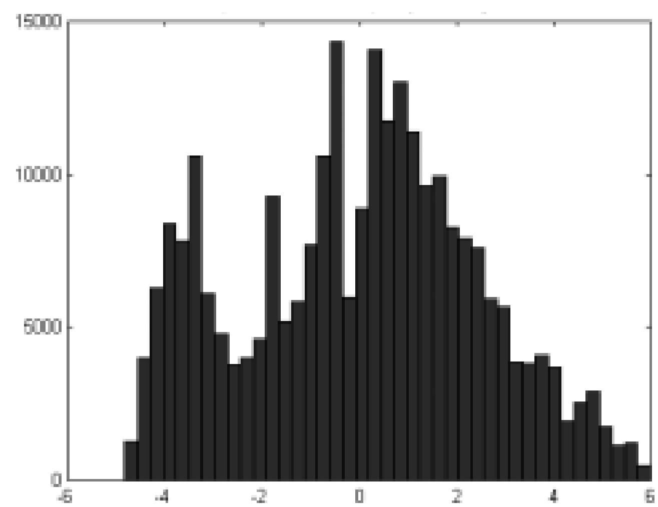

Figure 3: Histogram for spectrum of adjacency operator of $X\left(\text{ Heis}(\mathsf{Z}/64\mathsf{Z}), \left\{ \ (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1) \ \right\} \ \right)$.

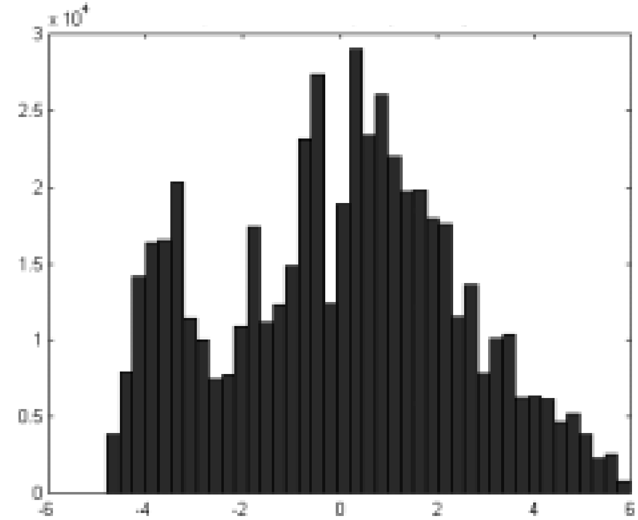

Figure 4: Histogram for spectrum of adjacency operator of $X\left(\text{ Heis}(\mathsf{Z}/81\mathsf{Z}), \left\{ \ (\pm1,0,0),(0,\pm1,0),(0,0,\pm1) \ \right\} \right)$.

3 Spectra via Coverings and Ihara-Selberg Zeta Functions

We say that Y is an **unramified finite covering** of a finite graph X if there is a covering map $\pi : Y \to X$ which is an onto graph map (i.e., taking adjacent vertices to adjacent vertices) such that for every $x \in X$ and for every $y \in \pi^{-1}(x)$, the set of points adjacent to y in Y is mapped by π one-to-one, onto the points in X which are adjacent to x. Note that when graphs have loops and multiple edges, one must be a bit more careful with this definition if one wants Galois theory to work properly. See Stark and Terras [24], p. 137. A d-sheeted covering is a **normal** covering iff there are d graph automorphisms $\sigma : Y \to Y$ such that $\pi(\sigma(y)) = \pi(y)$ for all $y \in Y$. These automorphisms form the Galois group $G(Y/X)$. See Stark and Terras [23],[24] for examples of normal and non-normal coverings and the factorization of their zeta functions.

Take a spanning tree T in X. View Y as $|G|$ sheets, where each sheet is a copy of T labeled by the elements of the Galois group G. So the points of Y are (x, g), with $x \in X$ and $g \in G$. Then an element $a \in G$ acts on the cover by $a(x, g) = (x, ag)$.

Suppose the graph X has m vertices. Define the $m \times m$ matrix $A(g)$ for $g \in G$ by defining the i, j entry to be

$$A(g)_{i,j} = \text{the number of edges in } Y \text{ between } (i, e) \text{ and } (j, g), \tag{10}$$

where e denotes the identity in G. Using these $m \times m$ matrices, we can find a block diagonalization of the adjacency matrix of Y as follows.

Proposition 3 If Y is a normal d-sheeted covering of X with Galois group G, then the adjacency matrix of Y can be block diagonalized where the blocks are of the form

$$M_\rho = \sum_{g \in G}^{\oplus} A(g) \otimes \rho(g),$$

each taken $d_\rho =$ degree of ρ times, as the representations ρ run through $\widehat{G}$. Here $A(g)$ is defined in formula (10).

Proof. The adjacency matrix A_Y of Y has the $(i,g),(j,h)$ entry for $i,j \in X$ and $g,h \in G$ given by

$$(A_Y)_{(i,a),(j,b)} = \text{the number of edges between } (i,a) \text{ and } (j,b). \tag{11}$$

and this is the same as the number of edges between (i,e) and $(j,a^{-1}b)$, if e is the identity of G.

Also define the $|G| \times |G|$ matrix $\sigma(g)$ indexed by elements $a,b \in G$:

$$(\sigma(g))_{a,b} = \begin{cases} 1, & \text{if } a^{-1}b = g, \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Note that σ is essentially the matrix of the right regular representation of G, since if δ_a is the vector with 1 in the a position and 0 everywhere else, we have $\sigma(g)\delta_a = \delta_{ag^{-1}}$.

It follows from (10),(11), and (12) that

$$A_Y = \sum_{g\in G} A(g) \otimes \sigma(g). \tag{13}$$

One of the fundamental theorems of representation theory (see Terras [26], p. 256) says that

$$\sigma(g) \cong \sum_{\rho\in\widehat{G}}^{\oplus} d_\rho \rho(g). \tag{14}$$

It follows that $A_Y \cong \sum_{\rho\in\widehat{G}}^{\oplus} d_\rho M_\rho$. This completes the proof of Proposition 3. ∎

Theorem 4 Assume p is odd. $\mathcal{H}(p^{n+1})$ is an unramified graph covering of $\mathcal{H}(p^n)$. Moreover it is a normal covering with abelian Galois group

$$\mathrm{Gal}(\mathcal{H}(p^{n+1})/\mathcal{H}(p^n)) \cong \Gamma \cong \left\{(a,b,c) \in \mathrm{Heis}(\mathsf{Z}/p^{n+1}\mathsf{Z}) \mid (a,b,c) \equiv 0 (\mathrm{mod}\, p^n)\right\}.$$

Proof. The projection $\pi : \mathcal{H}(p^{n+1}) \to \mathcal{H}(p^n)$ is just the reduction of the coordinates $\mathrm{mod}\, p^{n+1}$ to coordinates $\mathrm{mod}\, p^n$. Clearly this preserves adjacency. Moreover, given $g \in \mathcal{H}(p^n)$, if we take a point $g' \in \mathcal{H}(p^{n+1})$ in $\pi^{-1}g$, we see that the points in $\mathcal{H}(p^{n+1})$ adjacent to g' have the form $g's$, for $s \in S_0 = \{\ (\pm 1,0,0),(0,\pm 1,0)\ \}$. The points adjacent to g in $\mathcal{H}(p^n)$ are of the same form except computed $\mathrm{mod}\, p^n$. And π maps these adjacent points in $\mathcal{H}(p^{n+1})$ one-to-one, onto those in $\mathcal{H}(p^n)$.

If $(a,b,c) \in \Gamma$ defined in the statement of Theorem 4, we define the Galois group element

$$\gamma_{(a,b,c)}((x,y,z) \,\mathrm{mod}\, p^{n+1}) = (a,b,c)(x,y,z) \,\mathrm{mod}\, p^{n+1}.$$

It follows that $\pi \circ \gamma = \pi$, since $(a,b,c) \equiv 0(\mathrm{mod}\, p^n)$ and π reduces things $\mathrm{mod}\, p^n$. Moreover, it is easy to see that Γ is abelian since if (a,b,c) and (u,v,w) are both $\equiv 0(\mathrm{mod}\, p^n)$, then $(a,b,c)(u,v,w) = (a+u, b+v, c+w+av)$ and p^n divides both a and v so that $av \equiv 0(\mathrm{mod}\, p^{n+1})$. ∎

Corollary 2 The spectrum of $\mathcal{H}(p^n)$ is contained in the spectrum of $\mathcal{H}(p^{n+1})$.

Proof. Use Proposition 3 or see Stark and Terras [23], p. 131. ∎

Example. The last Theorem and Corollary also work if $p = 2$, except that then the graph at the bottom of the cover can be a multi-graph when $n = 1$, as in Figure 5. Consider the covering $\mathcal{H}(4)$ over $\mathcal{H}(2)$. Note that $\mathcal{H}(2)$ is a multigraph with 2 edges between any vertices that are adjacent, because $1 \equiv -1(\mathrm{mod}\, 2)$ and we want the graph to have degree 4. So the graph of $\mathcal{H}(2)$ is a cycle graph as in Figure 5. We label the vertices using the following table.

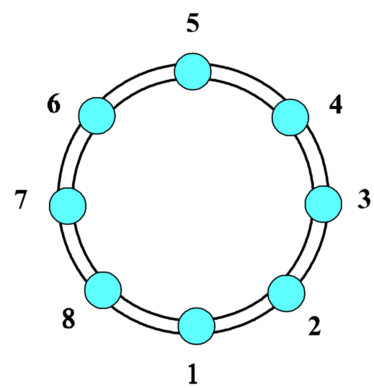

Figure 5: The Cayley Graph $\mathcal{H}(2) = X(\text{Heis}(\mathsf{Z}/2\mathsf{Z}), \{(\pm 1, 0, 0), (0, \pm 1, 0)\})$.

Table 2. Vertex Labeling for $\mathcal{H}(2)$.

label	1	2	3	4	5	6	7	8
vertex	$(0,0,0)$	$(1,0,0)$	$(1,1,1)$	$(0,1,1)$	$(0,0,1)$	$(1,0,1)$	$(1,1,0)$	$(0,1,0)$

We obtain a spanning tree for $\mathcal{H}(2)$ by cutting one of each pair of double edges and then cutting both edges between vertices 6 and 7. This really gives a line graph but we will draw it as a circle cut between vertices 6 and 7. So we draw the covering graph $\mathcal{H}(4)$ by placing 8 copies of the cut circle which is the spanning tree of $\mathcal{H}(2)$ and labeling each with a group element from $\text{Gal}(\mathcal{H}(4)/\mathcal{H}(2))$. We know that this can be identified with the subgroup of $\text{Heis}(\mathsf{Z}/4\mathsf{Z})$ consisting of (u, v, w) where u, v, w are all even. We label the Galois group elements using the following table.

Table 3. Galois Group Labeling for $\text{Gal}(\mathcal{H}(4)/\mathcal{H}(2))$. In this labeling, a not e is the identity of the group.

label	a	b	c	d	e	f	g	h
Galois group element	$(0,0,0)$	$(2,0,0)$	$(2,2,2)$	$(0,2,2)$	$(0,0,2)$	$(2,0,2)$	$(2,2,0)$	$(0,2,0)$

The covering graph $\mathcal{H}(4)/\mathcal{H}(2)$ has 8 sheets and each sheet is a copy of the spanning tree of $\mathcal{H}(2)$. So every point on $\mathcal{H}(4)$ has a label (n, v), where $1 \le n \le 8$ and $v \in \{a, b, c, d, e, f, g, h\}$. We will just write nv. See Figure 6 for a picture of the tree with connections between level a and the rest. You can use the action of the Galois group to find all the edges of $\mathcal{H}(4)$. It makes a pretty complicated figure. The following table shows which connections are made in Figure 6. This table allows one to compute the matrices $A(g), g \in G = \text{Gal}(\mathcal{H}(4)/\mathcal{H}(2))$.

Table 4. Table of Connections Between Sheet a in $\mathcal{H}(4)$ and the other sheets.

vertex	adjacent vertices in $\mathcal{H}(4)$
$1a$	$2b, 8h, 2a, 8a$
$2a$	$1b, 3d, 1a, 3a$
$3a$	$2d, 4f, 2a, 4a$
$4a$	$3f, 5h, 3a, 5a$
$5a$	$4h, 6b, 4a, 6a$
$6a$	$5b, 7e, 7h, 5a$
$7a$	$6e, 6h, 8f, 8a$
$8a$	$1h, 7f, 7a, 1a$

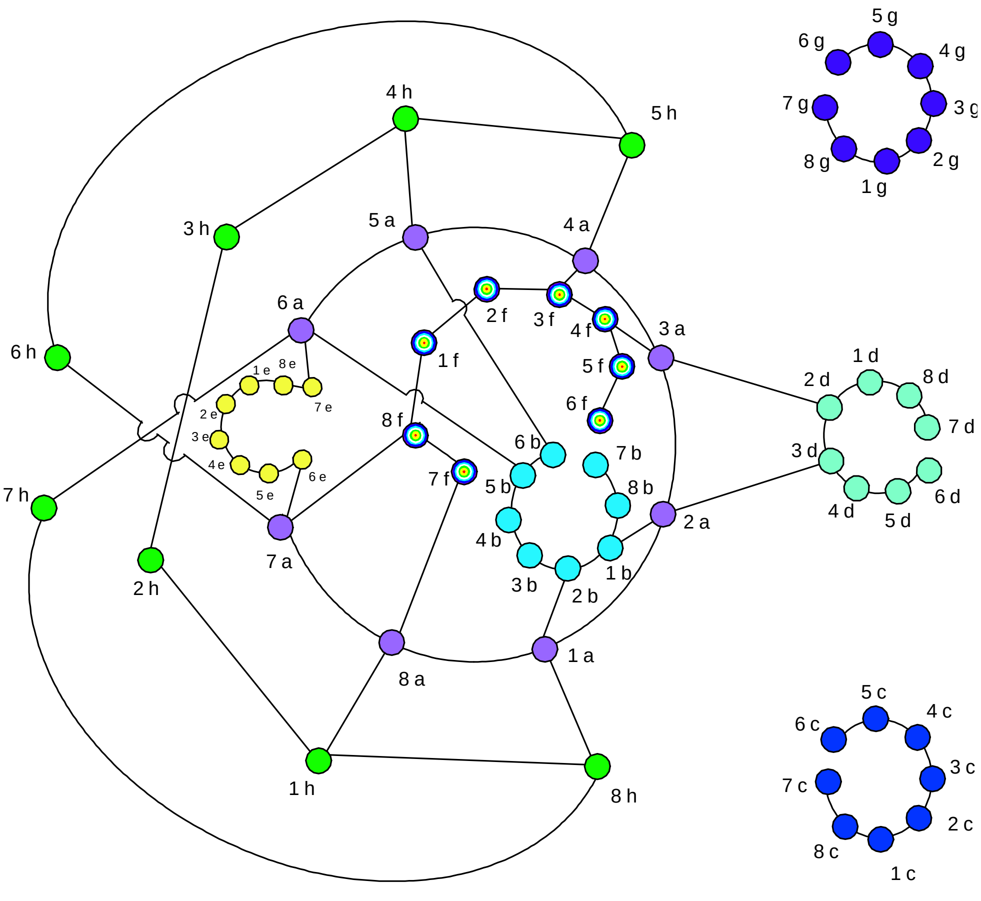

Figure 6: Connections Between Level a and the Rest of the Cayley Graph

$$\mathcal{H}(4) = X\left(\ \text{Heis}(\mathsf{Z}/4\mathsf{Z}), \left\{\ (\pm 1, 0, 0), (0, \pm 1, 0)\ \right\}\ \right).$$

The representations of the abelian Galois group have the form $\chi(a,b,c) = \exp\left(\frac{2\pi i(ra+sb+tc)}{4}\right)$, for $r, s, t (\bmod 2)$. Then one must compute the matrices $M_{\chi_{r,s,t}}$ appearing in Proposition 3. For example $M_{\chi_{0,0,0}}$ is the adjacency matrix of $\mathcal{H}(2)$ and

$$M_{\chi_{0,1,1}} = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{\chi_{1,0,0}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

The eigenvalues of the M_χ are to be found in the following table.

Table 5. Eigenvalues of $M_{r,s,t} = M_{\chi_{r,s,t}}$.

(r, s, t)	Eigenvalues of $M_{r,s,t}$
$(0,0,0)$	$-4, 0, 0, 4, -2\sqrt{2}, -2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}$
$(1,0,0)$ and $(0,1,0)$	$-2, -2, -2, -2, 2, 2, 2, 2$
$(1,1,0)$	$0, 0, 0, 0, 0, 0, 0, 0$
$(1,1,1), (0,1,1), (0,0,1),$ and $(1,0,1)$	$0, 0, 0, 0, -2\sqrt{2}, -2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}$

So we see that the spectrum of $\mathcal{H}(4)$ for $p = 2$ is given in table 6.
Table 6. Spectrum of $X(\text{Heis}(\mathsf{Z}/4\mathsf{Z}), \{(\pm 1, 0, 0), (0, \pm 1, 0)\}$.

eigenvalue	multiplicity
± 4	1
0	26
± 2	8
$\pm 2\sqrt{2}$	10

The **Artin L-function** associated to the representation ρ of $G = \text{Gal}(Y/X)$ can be defined by a product over prime cycles in X as

$$L(u, \rho, Y/X) = \prod_{[C] \text{ prime in X}} \det\left(I - \rho(Frob(\widetilde{C})u^{\nu(C)}\right)^{-1},$$

where $\widetilde{C}$ denotes any lift of C to Y and $Frob(\widetilde{C})$ denotes the Frobenius automorphism defined by

$$Frob(\widetilde{C}) = ji^{-1},$$

if $\widetilde{C}$ starts on Y-sheet labeled by $i \in G$ and ends on Y-sheet labeled by $j \in G$. As in Proposition 3, define

$$M_\rho = \sum_{g \in G} A(g) \otimes \rho(g). \tag{16}$$

Then, setting $Q_\rho = Q \otimes I_{d_\rho}$, with $d_\rho = d = \deg\rho$, we have the following analogue of formula (2):

$$L(u,\rho,Y/X)^{-1} = (1-u^2)^{(r-1)d_\rho} \det\left(I - M_\rho u + Q_\rho u^2\right). \tag{17}$$

See Stark and Terras [23] for an elementary proof and more information.

Formula (14) implies that the zeta function of Y factors as follows

$$\zeta_X(u) = \prod_{\rho\in\widehat{G}} L(u,\rho,Y/X)^{d_\rho}. \tag{18}$$

See Stark and Terras [24].

For our example, the Galois group is abelian and all degrees are 1. We obtain a factorization of the Ihara-Selberg zeta function of $\mathcal{H}(4)$ as a product of Artin L-functions of the Galois group of $\mathcal{H}(4)/\mathcal{H}(2)$. We use definition (16) and table 4 to compute the matrices $M_{\chi_{r,s,t}}$ as in formula (15). Then formula (17) gives the following list of L-functions. Here $Q = 3I_8$, $r = 9$.

Reciprocals of L-functions for $\mathcal{H}(4)/\mathcal{H}(2)$.

- 1) For $\chi = \chi_{0,0,0}$, A = adjacency matrix of $\mathcal{H}(2)$, and

$$\zeta_{\mathcal{H}(2)}(u)^{-1} = L(u,1)^{-1} = (1-u^2)^8(u-1)(u+1)(3u-1)(3u+1)\left(3u^2+1\right)^2\left(9u^4-2u^2+1\right)^2. \tag{19}$$

- 2) $L(u,\chi_{1,0,0})^{-1} = L(u,\chi_{0,1,0})^{-1} = (1-u^2)^8\left(3u^2+2u+1\right)^4\left(3u^2-2u+1\right)^4.$
- 3) $L(u,\chi_{1,1,1})^{-1} = L(u,\chi_{0,1,1})^{-1} = L(u,\chi_{0,0,1})^{-1} = L(u,\chi_{1,0,1})^{-1} = (1-u^2)^8\left(9u^4-2u^2+1\right)^2\left(3u^2+1\right)^4.$
- 4) When $\rho = \chi_{1,1,0}$ we find that $M_{\chi_{1,1,0}} = 0$, so that

$$L(u,\chi_{1,1,0})^{-1} = (1-u^2)^{(r-1)d}\det\left(I + Q_\rho u^2\right) = (1-u^2)^8\left(1+3u^2\right)^8.$$

It follows from these computations and (18) that the Ihara zeta function of $\mathcal{H}(4)$ is

$$\zeta_{\mathcal{H}(4)}(u)^{-1} = -(1-u^2)^{65}\left(9u^2-1\right)\left(3u^2+1\right)^{26}\left(9u^4-2u^2+1\right)^{10}\left(3u^2+2u+1\right)^8\left(3u^2-2u+1\right)^8. \tag{20}$$

4 Comparisons of Spectra and Butterflies

In Figures 7-9 we collect histograms of spectra of torus graphs

$$\mathcal{T}^{(n)}(q) = X((\mathsf{Z}/q\mathsf{Z})^n, \{\pm e_1, \pm e_2, \cdots, \pm e_n\}),$$

where e_i denotes the vector with 1 in the ith coordinate and 0 elsewhere. Because the torus groups $(\mathsf{Z}/q\mathsf{Z})^n$ are abelian, it is relatively easy to generate these figures. In fact, the eigenvalues of the adjacency matrix of $\mathcal{T}^{(n)}(q)$ are

$$\lambda_a = 2\left(\cos\left(\frac{2\pi i a_1 b_1}{q}\right) + \cos\left(\frac{2\pi i a_2 b_2}{q}\right) + \cdots + \cos\left(\frac{2\pi i a_n b_n}{q}\right)\right), \quad \text{for} \quad a,b \in (\mathsf{Z}/q\mathsf{Z})^n.$$

Note that, by equation (9), the spectrum of the degree 4 Heisenberg graph $\mathcal{H}(4)$ contains the spectrum of $\mathcal{T}^{(2)}(4)$. In fact, $\mathcal{H}(4)$ is actually a covering graph of $\mathcal{T}^{(2)}(4)$, via the covering map sending (x,y,z) to (x,y).

The histogram in Figure 7 is easily analyzed and seen to approach the limiting density $f(x) = \frac{1}{\pi\sqrt{1-(x/2)^2}}$. If you make the substitution $u = x^2/4$, you obtain the density for the arc sine law (see Feller [6]). It follows that the limiting density in Figure 8 is $f * f$ while that in Figure 9 is $f * f * f$. It is easy to recognize the three peak degree 4 $\mathcal{H}(q)$ histogram as in Figures 1 and 2 versus the one peak histogram for $\mathcal{T}^{(2)}(q)$ in Figure 8. Figure 9 represents the histogram of a degree 6 graph just as Figures 3 and 4 do, however Figure 9 is symmetric while Figures 3 and 4 are not. Moreover Figures 3 and 4 show a spectral gap which is not present in Figure 9.

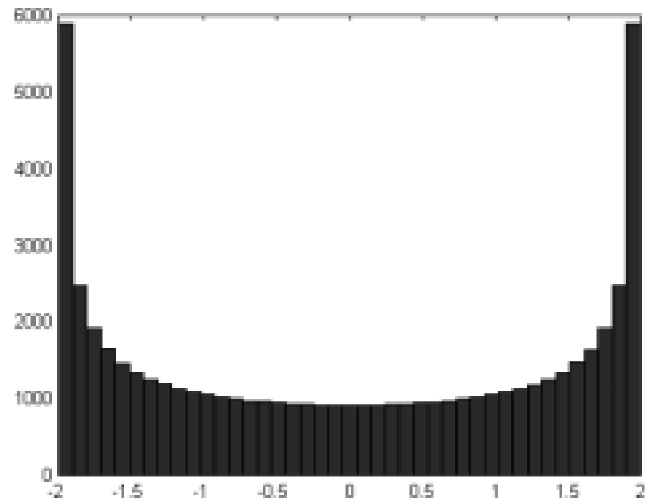

Figure 7. Histogram of the Spectrum of the Cayley Graph $X(\mathsf{Z}/99991\mathsf{Z}, \{\pm 1\})$.

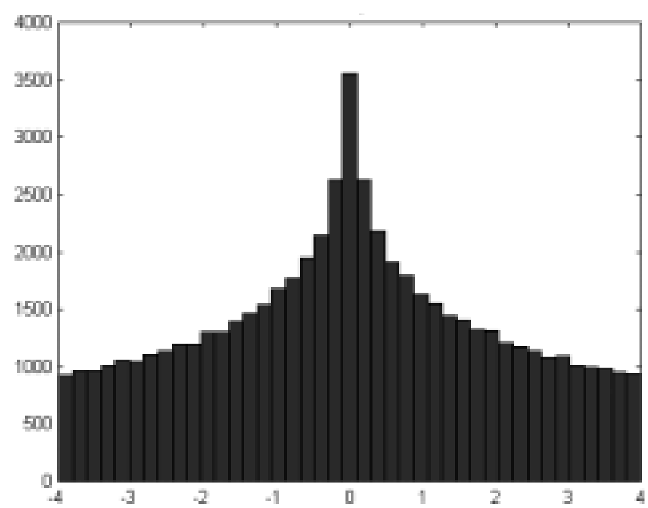

Figure 8. Histogram of the Spectrum of the Cayley Graph $X((\mathsf{Z}/128\mathsf{Z})^2, \{\pm e_1, \pm e_2\})$.

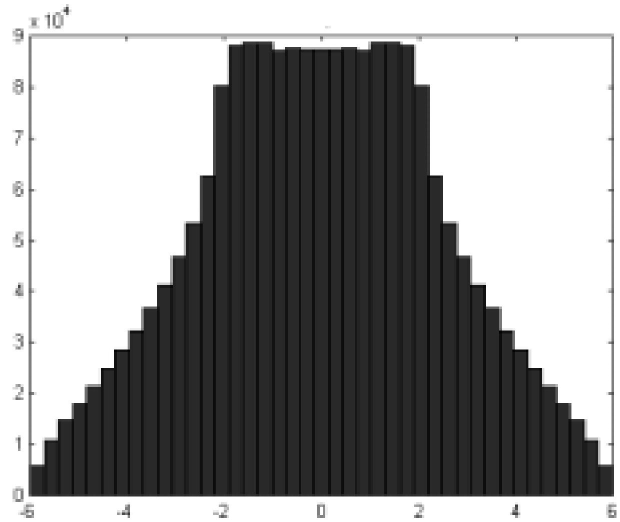

Figure 9. Histogram of the Spectrum of the Cayley Graph $X((\mathsf{Z}/128\mathsf{Z})^3, \{\pm e_1, \pm e_2, \pm e_3\})$.

We can easily compute the Selberg-Ihara zeta functions of the small torus graphs using covering graph theory. As in Theorem 4, the Galois group of $\mathcal{T}^{(n)}(p^{r+1})/\mathcal{T}^{(n)}(p^r)$ is

$$\Gamma \cong \left\{ x \in (\mathsf{Z}/p^{r+1}\mathsf{Z})^n \middle| \quad x \equiv 0 (\operatorname{mod} p^r) \right\}.$$

Since the 1-dimensional graphs are cycles, we know that

$$\zeta_{\mathcal{T}^{(1)}(q)}(u)^{-1} = (1-u^q)^2, \text{ for all } q.$$

In 2-dimensions, we consider only the smallest values of q (namely $q = 2$ and $q = 4$) and find that if $\Gamma = Gal(\mathcal{T}^{(2)}(4)/\mathcal{T}^{(2)}(2))$, the representations of Γ have the form $\chi_{r,s}(x,y) = \exp\left(\frac{2\pi i(rx+sy)}{4}\right)$, for $(x,y) \in \Gamma, (r,s) \in (\mathsf{Z}/2\mathsf{Z})^2$. Therefore $(x,y) \equiv 0(\operatorname{mod} 2)$. It follows that

$\zeta_{\mathcal{T}^{(2)}(4)}(u)^{-1} = \zeta_{\mathcal{T}^{(2)}(2)}(u)^{-1} L(u,\chi_{0,1})^{-1} L(u,\chi_{1,1})^{-1} L(u,\chi_{1,0})^{-1}$

$= -(1-u^2)^{17}(9u^2-1)(3u^2+1)^6(3u^2-2u+1)^4(3u^2+2u+1)^4.$

Here $\zeta_{\mathcal{T}^{(2)}(2)}(u)^{-1} = -(1-u^2)^5(9u^2-1)(3u^2+1)^2$.

From these results plus (19) and (20) we see that

$$\zeta_{\mathcal{H}(4)}(u)^{-1}/\zeta_{\mathcal{T}^{(2)}(4)}(u)^{-1} = (1-u^2)^{48}(3u^2+1)^{20}(3u^2-2u+1)^4(3u^2+2u+1)^4(9u^4-2u^2+1)^{10} \tag{21}$$

and

$$\zeta_{\mathcal{H}(2)}(u)^{-1}/\zeta_{\mathcal{T}^{(2)}(2)}(u)^{-1} = (1-u^2)^4(9u^4-2u^2+1)^2. \tag{22}$$

In Figures 10 and 11 we present Hofstadter butterfly-type pictures. Separate the part of the spectrum of the Cayley graph $\mathcal{H}_S(q)$ corresponding to the q-dimensional representations of Heis$(\mathsf{Z}/q\mathsf{Z})$ denoted $\pi_s = \Theta^{(0)}_{0,s,0}$, $s \in (\mathsf{Z}/q\mathsf{Z})^*$. Plot the part of the spectrum corresponding by Proposition 2 to π_s as points in the plane with y-coordinate s/q and x coordinate given by the various eigenvalues λ of the matrix

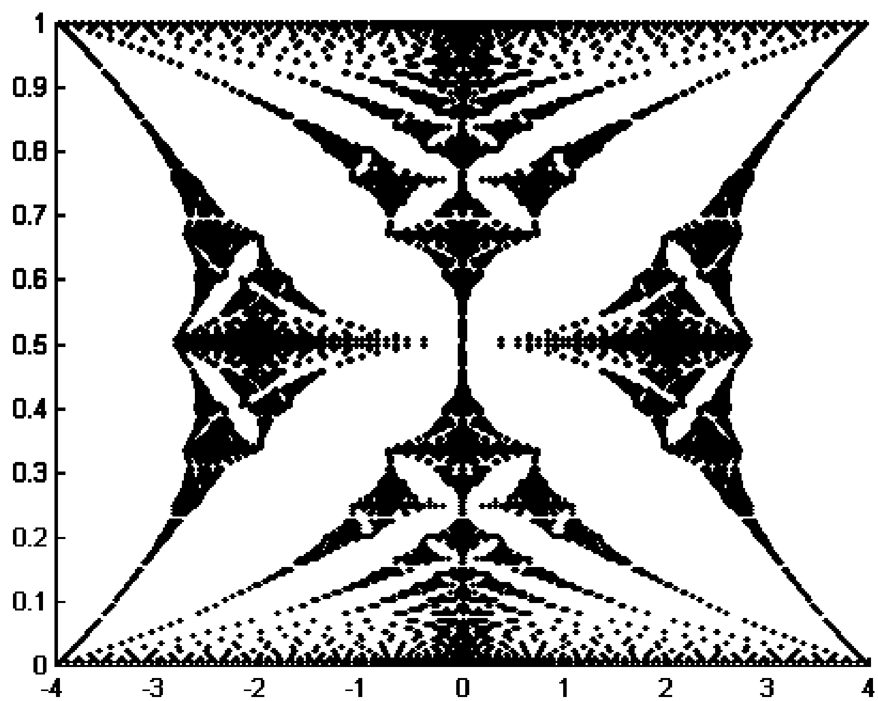

Figure 10. Hofstadter Butterfly Graph for the Spectrum of $X\big(\text{Heis}(\mathsf{Z}/13^2\mathsf{Z}), \{ (\pm1,0,0),(0,\pm1,0) \}\big)$.

$$M_{\pi_s} = \sum_{u \in S} \pi_s(u),$$

where S denotes the edge set for the Cayley graph $X(\text{Heis}(\mathsf{Z}/q\mathsf{Z}), S)$. Of course the eigenvalues λ lie in the interval $[-|S|, |S|]$.

Future Work. There are many other questions one can ask in this context.

Can one find the limiting density of the histograms for the graphs $\mathcal{H}_S(q)$? One expects to obtain the density function for the Cayley graphs $X(\text{Heis}(\mathsf{Z}), S)$, when $S = \{(\pm1,0,0),(0,\pm1,0)\}$ or $S = \{(\pm1,0,0),(0,\pm1,0),(0,0,\pm1)\}$. See Béguin, Valette, and Zuk [1].

Can such histograms be used to recognize groups involved in Cayley graphs? That is, can you see the shape of a group? This is an analogous questions to that of Mark Kac about hearing the shape of a drum (as the Dirichlet spectrum of the Laplace operator on a plane drum determines the fundamental frequencies of vibration). Here we wonder if one can somehow recognize groups from properties of the histograms of associated Cayley graphs with some sort of condition on the generating sets S. Instead of hearing the drum in its spectrum, we are trying to see it.

Can one attach hypergraphs to the Heisenberg group and compare with those found in Martínez [16]?

Can one generalize Proposition 3 to ramified graphs?

One should also consider the level spacing histograms. This means that you must order the eigenvalues of the adjacency operator $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$ and normalize so that the mean of the level spacings $\lambda_{i+1} - \lambda_i$ is 1. Then look at the histogram of the level spacings $\lambda_{i+1} - \lambda_i$. Does the result look like e^{-x} (Poisson) or some other distribution such as those arising in random matrix theory? We will not pursue this question here.

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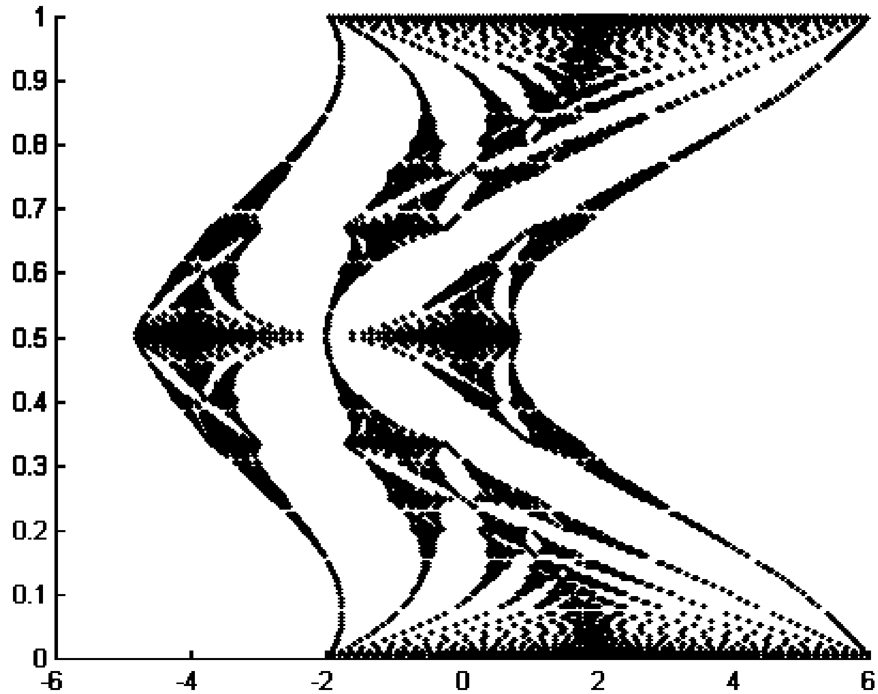

Figure 11. Hofstadter Butterfly Graph for the Spectrum of $X\left(\mathrm{Heis}(\mathsf{Z}/13^2\mathsf{Z}), \left\{ (\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1) \right\} \right)$.